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BACHELOR THESIS

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Well-Quasi-Ordering Certain Classes of Infinite Graphs

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Author's signature

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Abstract: This thesis studies well-quasi-orderings and better-quasi-orderings of graphs under various containment relations. We give a direct proof that out-trees are wqo by the homomorphism relation via transfinite induction on a hierarchy of trees that we introduce, and we investigate extensions of this approach to leaf-labeled trees. For graphs, we analyze structural parameters that yield natural wqo classes. In particular, we extend a theorem of Ding by proving that every class of (finite or infinite) graphs with bounded tree-depth is wqo by the induced subgraph relation, and we show that graph classes with bounded independence number are wqo by the subgraph relation. Lastly, we consider generalizations of well-quasi-orderings motivated by the large cardinal axiom known as Vopěnka's principle.

Keywords: well-quasi-ordering, better-quasi-ordering, infinite graphs

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Název práce: Dobrá kvaziuspořádání vybraných tříd nekonečných grafů

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Abstrakt: Tato práce se zabývá dobrými kvaziuspořádáními (wqo) a lepšími kvaziuspořádáními (bqo) v kontextu teorie grafů. Ukážeme, že stromy orientované směrem od kořene jsou wqo homomorfismy, a to tak, že představíme hierarchii těchto stromů a provedeme transfinitní indukci podél této hierarchie. Následně zvážíme, zda lze naši metodu aplikovat na stromy s barevnými listy. Co se týče grafů, studujeme parametry jako je stromová hloubka a velikost největší nezávislé množiny. Konkrétně ukážeme, že Dingova věta, která tvrdí, že třídy grafů s omezenou stromovou hloubkou jsou wqo indukovanými podgrafy, platí i pro nekonečné grafy. Dále ukážeme, že třídy grafů bez libovolně velkých nezávislých množin jsou wqo podgrafy. Nakonec, motivováni Vopěnkovým principem, zvážíme některá přirozená zobecnění dobrých kvaziuspořádání.

Klíčová slova: dobré kvaziuspořádání, lepší kvaziuspořádání, nekonečné grafy

Contents

Introduction

Well-quasi-orderings or *wqo* are a natural generalization of the notion of well-ordered sets to quasi-orderings. A binary relation $\preceq$ on a set Q is a *quasi-order* if it is reflexive and transitive (if it is also antisymmetric, then it is a partial order). A quasi-order is wqo if given any infinite sequence $x_1, x_2, \dots$ of elements of Q, there are indices $i < j$ such that $x_i \preceq x_j$. Well-quasi-orderings are an important tool in logic and computer science [SS12; SSW20], as they provide termination arguments in algorithms and decidability problems. Moreover, they allow monotone properties to be characterized by a finite set of forbidden obstructions (a property φ is *monotone* if whenever it holds for some x, then it also holds for all smaller $y \preceq x$). For example, planarity is a monotone property of graphs, and Kuratowski's theorem states that a graph is planar if and only if it contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$ as a subgraph.

According to Kruskal [Kru72], the concept of wqo has been rediscovered multiple times over the years. Its origins can be traced to a conjecture of Vázsonyi in the 1940s, which stated that finite trees are wqo by the topological minor relation. This conjecture is now commonly referred to as Kruskal's tree theorem, after it was proved by Kruskal [Kru60] and independently by Tarkowski [Tar60].

Nash-Williams [Nas65a] extended Kruskal's theorem and proved that all (finite or infinite) trees are wqo by the topological minor relation. In order to achieve this foundational result, he introduced the stronger notion of *better-quasi-orderings*, or *bqo*, and showed that rooted trees are bqo by the *homeomorphic embedding* relation, a variant of topological minors for rooted trees. Since every bqo is also a wqo, this immediately implies the wqo property for all trees.

A second corollary of this result is that out-trees (rooted, possibly infinite trees with edges oriented away from the root) are wqo by the weaker *homomorphism* relation. However, Nash-Williams' proof relies on the machinery of bqo theory and is highly intricate, providing little insight into the structure of the embeddings themselves. We present a relatively straightforward proof of this corollary by performing transfinite induction on a hierarchy of out-trees that we introduce. We also investigate whether our method can be utilized to prove a stronger statement about trees with labeled leaves.

Theorem 1. *Out-trees are wqo by the homomorphism relation.*

Another conjecture related to wqos, known as Wagner's conjecture, asserts that finite graphs are wqo by the minor relation. It was famously proved by Robertson and Seymour [RS04] in a series of around 20 papers and is widely regarded as one of the deepest results in 20th-century discrete mathematics. A core concept in their proof is *tree-width*, a parameter that measures how similar a given graph is to a tree; in particular, trees have tree-width 1. One of the first steps [RS90] towards this result was showing that graph classes with bounded tree-width are wqo by the minor relation (a graph parameter π is said to be *bounded* for some graph class $\mathcal{G}$ if there exists a constant k such that $\pi(G) < k$ for all $G \in \mathcal{G}$).

Minors can be generalized to infinite graphs, and a natural question arises: does the Robertson–Seymour theorem hold for infinite graphs as well? Nash-Williams' extension of Kruskal's theorem implies that it holds for infinite trees. Thomas [Tho88a] demonstrated that it fails for the class of all graphs, but he proved [Tho89] that it holds for any class of (finite or infinite) graphs with bounded tree-width.

Tree-depth is a graph parameter related to tree-width that measures how "star-like" a graph is. Importantly, the tree-width of a graph is always at most its tree-depth, so any graph class with bounded tree-depth also has bounded tree-width and is therefore wqo by the minor relation. Ding [Din92] further showed that, for finite graphs, bounded tree-depth classes are wqo by the induced subgraph relation. We extend Ding's result to infinite graphs, showing that the same statement holds in this broader setting.

Theorem 2. *Every class of (finite or infinite) graphs with bounded tree-depth is wqo by the induced subgraph relation.*

Furthermore, we prove another wqo theorem, this time for classes of graphs without large independent sets. We also consider its labeled variant, since both Kruskal's and Ding's theorems extend naturally to labeled graphs.

Theorem 3. *Every class of graphs with bounded independence number (the size of the largest independent set) is wqo by the subgraph relation.*

Other original results

- The theorem for out-trees with bqo-labeled leaves in Section 4.5.1.
- The counter-example for out-trees with wqo-labeled leaves in Section 4.5.1.
- The tree-depth compactness theorem in Section 5.3.1.
- The counter-example for the labeled version of Theorem 3 in Section 5.4.
- The theorem for bqo-labeled almost k-cliques in Section 5.4.

Outline of the Thesis

This thesis is structured as follows:

In Chapter 1, we provide definitions and statements needed throughout the thesis, namely a brief review of ordinal and cardinal numbers.

In Chapter 2, we give a basic treatment of well-quasi-ordering theory.

In Chapter 3, we motivate the definition of better-quasi-orderings and cover the basic constructions preserving the property of being better-quasi-ordered.

In Chapter 4, we study trees from both the graph-theoretic and order-theoretic perspectives and investigate their well-quasi-ordering properties. In particular, Section 4.5 is dedicated to Theorem 1.

In Chapter 5, we survey results about well-quasi-orderings of graphs by various containment relations. We prove Theorem 2 in Section 5.3.2 and Theorem 3 in Section 5.4.

In Chapter 6, we consider certain natural generalizations of well-quasi-orderings. We study the homomorphism order of graphs and introduce the notion of *class-wqo*, motivated by the large cardinal axiom known as Vopěnka's principle.

1 Preliminaries

We work within ZFC, the Zermelo–Fraenkel set theory augmented by the axiom of choice, the statement that every set x has a *choice function*: a function f with domain x such that for every nonempty set $t \in x$ we have $f(t) \in t$.

Recall that in ZFC we distinguish between sets and classes. If $\varphi(u, v_1, \dots, v_n)$ is a first-order formula and $p_1, \dots, p_n$ are set parameters, then the expression

$$\{u \mid \varphi(u, p_1, \dots, p_n)\}$$

is called a *class*. It serves as a meta-linguistic abbreviation for the collection of all sets x that satisfy the formula; more precisely, $x \in \{u \mid \varphi(u, p_1, \dots, p_n)\}$ abbreviates that $\varphi(x, p_1, \dots, p_n)$ holds. We view all sets as classes (since $z = \{x \mid x \in z\}$ for any set z), but not every class can be viewed as a set (consider the class of all sets). A class that is not a set is called a *proper class*. The major difference between sets and proper classes is that proper classes cannot be members of other classes or sets, while sets can. Furthermore, because classes are not formal objects of ZFC, its language does not allow us to quantify over them.

When we say that there is a 1-*to*-1 *correspondence* between two classes A and B, it means that there exists a bijection between them. The power set of a set X is denoted by $\mathcal{P}(X)$, the set of all subsets of X that have cardinality λ is denoted by $[X]^\lambda$, and the n-th Cartesian power of X is denoted by X^n. Concatenated expressions such as $a \in b \in c$ abbreviate that $a \in b \wedge b \in c$.

We use the terms "function," "map," and "mapping" interchangeably, and we employ the following notation for defining functions:

- $f \colon X \to Y$ denotes that $f \subseteq X \times Y$ is a function with domain X. If we say that $f \colon X \to Y$ is a bijection, it further entails that Y is the range of f.
- $f = g \circ h$ means that $f(x) = h(g(x))$ holds for all suitable x (note that we use the diagrammatic order for composition).

For a class A, we denote by $f[A]$ the class $\{y \mid (\exists x \in A)\,(x, y) \in f\}$. The *restriction* of f to A is the function $f \cap (A \times f[A])$ and it is denoted by $f \restriction A$. When we say that f maps A *into* a class B, it means that $f[A] \subseteq B$. Finally, an *indexed family of sets* $\langle X_i \mid i \in I\rangle$ is a map F with domain I, where X_i denotes the set $F(i)$.

1.1 Orders

We begin by defining quasi-orders, well-orders, and other related notions.

Definition 1.1. A binary relation $\leq$ on a class Q is a *quasi-order* if it is reflexive and transitive. If the ordering relation $\leq$ is clear from the context, we will often simply say that Q is a quasi-order. An antisymmetric quasi-order is called a *partial order*. A binary relation $<$ on a class Q is a *strict order* if it is irreflexive, transitive, and antisymmetric.

Every partial order $\leq$ on Q defines a strict order $<$ on Q and vice versa by

$$x < y \iff x \leq y \wedge x \neq y \quad \text{and} \quad x \leq y \iff x < y \vee x = y.$$

Therefore, we do not define properties for strict orders because they are implicitly defined by those of the corresponding non-strict order.

An *order-isomorphism* of partially ordered classes $(A, \leq_A)$ and $(B, \leq_B)$ is a bijective mapping $F\colon A \to B$ such that for all $x, y \in A$ we have

$$x \leq_A y \iff F(x) \leq_B F(y).$$

Definition 1.2. We say that x and y are *comparable* if $x \leq y$ or $y \leq x$. A partial order in which every pair of elements is comparable is called a *linear order*.

A *chain* of a quasi-order Q is a subset $Q' \subseteq Q$ in which each pair of elements is comparable. In particular, chains of partial orders are linearly ordered. An *antichain* is a subset in which no two distinct elements are comparable.

An element q of a quasi-order Q is *minimal* if whenever some $q' \in Q$ satisfies $q' \leq q$, then $q \leq q'$ also holds. It is a *minimum* or *least element* if $q \leq q'$ holds for all $q' \in Q$. We define a *maximal* element and *maximum* analogously.

Observation 1.3. *Every minimum is minimal. Partially ordered sets contain at most one minimum. Linear orders contain at most one minimal element, and if it exists, then it is also the minimum.*

An element p of a partial order P is an *upper bound* of a nonempty subset $P' \subseteq P$ if $p' \leq p$ for all $p' \in P'$. The least upper bound of P' is called its *supremum* (provided it exists). We similarly define the *infimum* of P'.

Definition 1.4. A quasi-order Q is *well-founded* if every nonempty subset $A \subseteq Q$ has a minimal element. A well-founded linear order is called a *well-order*.

Note that all these ordering relations are *hereditary*: if Q is a quasi-order, partial order, linear order, or well-order, then any $Q' \subseteq Q$ with the inherited ordering is also a quasi-order, partial order, linear order, or well-order, respectively.

1.2 Ordinal Numbers

We assume little familiarity with set theory and therefore briefly review the definitions and basic properties of ordinal and cardinal numbers, which are essential for working with infinite objects such as graphs. Readers already familiar with these notions may wish to skip this and the following section.

We use *von Neumann ordinals*, meaning that natural numbers are defined as

$$0 \coloneqq \varnothing,\ 1 \coloneqq \{0\},\ 2 \coloneqq \{0,1\},\ \ldots,\ n+1 \coloneqq \{0,1,\ldots,n\} = n \cup \{n\}.$$

The *set of all natural numbers* is denoted by ω. *Ordinal numbers* are a way to generalize natural numbers. Intuitively, they represent the *types of well-ordered sets*. Imagine we are trying to label the elements of a well-ordered set: we label the least element by 0, the next by 1, and so on. The *type* of the well-ordering is then the first label we did not have to use. But what if we run out of labels? Consider the following well-ordering of natural numbers:

$$1 \prec 2 \prec 3 \prec 4 \prec 5 \prec \cdots \prec 0$$

If we label the elements from left to right, all natural numbers are used for the segment $1 \prec 2 \prec \cdots$, leaving no label for 0. This is precisely why we need ordinal numbers: we assign the label ω to 0, and the type of this ordering is $\omega+1$. Similarly, the well-ordering

$$0 \prec 2 \prec 4 \prec 6 \prec \cdots \prec 1 \prec 3 \prec 5 \prec 7 \prec \cdots$$

has type $\omega + \omega$, because we use all naturals n to label the even numbers, and then all ordinals of the form $\omega + n$ to label the odd numbers.

Definition 1.5. A class X is *transitive* if for all $x \in X$ we have $x \subseteq X$, or equivalently, if for all x, y such that $y \in x \in X$ we have $y \in X$.

Theorem 1.6. *Every natural number and the set of all natural numbers ω are transitive and strictly well-ordered by the membership relation $\in$.*

From now on, we will denote the strictly well-ordered set $(\omega, \in)$ as $(\omega, <)$ and write $n < m$ instead of $n \in m$ when referring to natural numbers. More generally, ordinal numbers are precisely the sets with the properties described by the previous theorem.

Definition 1.7 (Ordinal number)**.** A set α is an *ordinal number* if it is transitive and strictly well-ordered by the membership relation $\in$. We denote the *class of all ordinal numbers* by On.

Theorem 1.8. *The class* On *is itself transitive and strictly well-ordered by $\in$. This implies that it is not a set; otherwise,* On $\in$ On. *Furthermore, if X is a transitive proper class (strictly) well-ordered by $\in$, then $X =$* On.

As for notation, we will usually denote ordinals using letters from the beginning of the Greek alphabet: $\alpha, \beta, \gamma, \delta, \ldots$ Furthermore, we compare ordinals using the symbol '$<$'. That is, we write $\beta < \alpha$ instead of $\beta \in \alpha$.

It can be shown that finite ordinals are exactly the natural numbers, and ω is the first infinite ordinal. Because ordinals are well-ordered, every set of ordinals $A \subseteq \mathrm{On}$ has a supremum $\sup A$, the least ordinal β such that $\beta \geq \alpha$ for all $\alpha \in A$.

Theorem 1.9 (About types of well-orders)**.** *Every well-ordered set $(W, \preceq)$ is order-isomorphic to a unique ordinal number $(\alpha, \leq)$ known as the* order type *of W.*

Transfinite induction Ordinal numbers allow us to extend the concept of mathematical induction to *transfinite* induction on all ordinals.

If α is an ordinal, then we call all $\beta < \alpha$ the *predecessors* of α. The *successor* of α is the ordinal $\alpha + 1 := \alpha \cup \{\alpha\}$. We say that α is the *direct predecessor* of $\alpha + 1$. It is easy to show that $\alpha + 1$ is the smallest ordinal larger than α.

Definition 1.10. An ordinal number α is called

(a) an *isolated* ordinal if $\alpha = 0$ or if α has a direct predecessor,

(b) a *limit* ordinal otherwise.

Isolated ordinals $\alpha > 0$ are also sometimes called *successor* ordinals.

Example. Examples of isolated ordinals include all natural numbers or the ordinals $\omega + 1$, $\omega \cdot 2 + 7$, and $\omega^\omega + 2$, which we will define later. Examples of limit ordinals include ω, $\omega + \omega$, or $\omega \cdot \omega$.

Theorem 1.11 (Transfinite Induction Principle)**.** *Let $\varphi(\alpha)$ be a property of ordinal numbers such that*

(i) *$\varphi(0)$ holds,* . . . *base case*

(ii) *whenever $\varphi(\alpha)$ holds, then $\varphi(\alpha + 1)$ holds as well,* . . . *successor stage*

(iii) *if α is a limit ordinal and $\varphi(\beta)$ holds for all $\beta < \alpha$, then $\varphi(\alpha)$ holds.*

Then $\varphi(\alpha)$ holds for every ordinal number α.

Proof. Suppose that φ fails for some ordinal γ, and let S be the set of all ordinals $\delta \leq \gamma$ for which φ fails. Since ordinals are well-ordered, the set S has a minimum element α: the first ordinal for which φ fails. From (i), $\alpha > 0$. If $\alpha = \beta + 1$ is isolated, then $\varphi(\alpha)$ holds by (ii) since $\varphi(\beta)$ holds. If α is limit, then $\varphi(\alpha)$ holds by (iii) since $\varphi(\beta)$ holds for all $\beta < \alpha$. Either way, we get a contradiction. □

One can similarly prove the following alternative formulation.

Theorem 1.12 (Transfinite Induction Principle II)**.** *Let $\varphi(\alpha)$ be a property of ordinal numbers such that for every ordinal α, the following is true:*

if $\varphi(\beta)$ holds for all $\beta < \alpha$, then $\varphi(\alpha)$ holds as well.

Then $\varphi(\alpha)$ holds for every ordinal number α.

Ordinal arithmetic The definition above may not provide much intuition for working with ordinal numbers. We therefore briefly discuss basic ordinal arithmetic. Although this material will not be used later in the thesis, it may help readers unfamiliar with ordinals to develop some intuition.[1]

The *lexicographical ordering* of a Cartesian product $A \times B$ is defined by $(a_1, b_1) \leq_L (a_2, b_2)$ if $a_1 < a_2$, or $a_1 = a_2$ and $b_1 \leq b_2$. It is easy to see that if A and B are well-ordered by $\leq$, then $A \times B$ is well-ordered by $\leq_L$.

Definition 1.13. Let α and β be ordinal numbers. We define ordinals

(a) $\alpha + \beta$ as the order type of the set $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$ when ordered lexicographically,

(b) $\alpha \cdot \beta$ as the order type of the set $\beta \times \alpha$ when ordered lexicographically.

Notice that our previous notation of denoting $\alpha \cup \{\alpha\}$ by $\alpha + 1$ is consistent with the above definition. We can imagine $\alpha + \beta$ as a pile of decreasing matchsticks labeled by α, followed by another pile of matchsticks labeled by β. Notice that we are using $\beta \times \alpha$ in the definition of $\alpha \cdot \beta$. The ordinal $\alpha \cdot \beta$ can be imagined as taking a pile of matchsticks labeled by β, and then replacing each stick with a copy of α (a pile of matchsticks labeled by α).

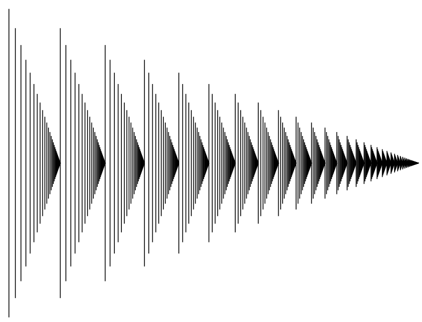

Figure 1.1: A representation of the ordinal $\omega \cdot \omega$. Each stick corresponds to an ordinal of the form $\omega \cdot m + n$ where m and n are natural numbers [GI15].

With this intuition, it should not be surprising that ordinal addition and multiplication are, in general, not commutative. It is easy to see that $1 + \omega = \omega$,

[1] I also recommend the following video by Vsauce, which illustrates how to construct larger ordinals from already existing ones: `https://www.youtube.com/watch?v=SrU9YDoXE88`.

but $\omega + 1 \neq \omega$. For multiplication, consider $2 \cdot \omega$, the order type of countably infinitely many copies of $\{0, 1\}$ stacked behind each other. This can be clearly labeled by ω, so $2 \cdot \omega = \omega$. But $\omega \cdot 2$ is the order type of two consecutive copies of ω. When we try to label them using ω, we use all $n \in \omega$ to label the first copy and need more ordinals for the second copy. Therefore $\omega \cdot 2 > \omega$.

Observation 1.14. *For any ordinals α, β, γ and natural $n \in \omega$ it holds that*

(a) $\alpha + 0 = \alpha = 0 + \alpha, \qquad \alpha \cdot 0 = 0 = 0 \cdot \alpha, \qquad \alpha \cdot 1 = \alpha = 1 \cdot \alpha,$

(b) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma, \qquad \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma,$

(c) $\alpha \cdot 2 = \alpha + \alpha, \qquad \alpha \cdot 3 = \alpha + \alpha + \alpha, \qquad \alpha \cdot (n + 1) = \alpha \cdot n + \alpha.$

Definition 1.15. For ordinal numbers α and β, we define α^β recursively as

(i) $\alpha^0 \coloneqq 1$,

(ii) $\alpha^{\beta+1} \coloneqq \alpha^\beta \cdot \alpha$,

(iii) if β is a limit ordinal, then $\alpha^\beta \coloneqq \sup\{\alpha^\gamma \mid 0 < \gamma < \beta\}$.

To get an intuition for ordinal powers, consider the ordinal $\omega^2 = \omega \cdot \omega$. It represents multiple copies of ω arranged in the same manner as ω. To construct $\omega^3 = (\omega \cdot \omega) \cdot \omega$, we take multiple copies of ω^2 and arrange them according to ω. If we repeat this process countably infinitely many times, we arrive at ω^ω. We can continue and arrive at larger and larger ordinals, such as $\omega^{(\omega^\omega)}$ or $\omega^{(\omega^{(\omega^\omega)})}$. Eventually, we construct the ordinal

$$\varepsilon_0 \coloneqq \sup\left\{\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \dots\right\},$$

which is closely related to Peano Arithmetic. It might be surprising that this ordinal is still countable and, in a certain sense, quite small, even for countable ordinals. For more details, see the survey in Section 3 of [Smo26].

1.3 Cardinal Numbers

Similar to how ordinal numbers represent the order types of well-ordered sets, *cardinal numbers* represent the *sizes* of well-ordered sets.

Definition 1.16. If x and y are sets, we define the following relations:

(a) $x \approx y$ if there exists a bijection $f\colon x \to y$, … x and y are *equinumerous*

(b) $x \preceq y$ if there exists an injection $f\colon x \to y$, … x is *subvalent* to y

(c) $x \prec y$ if $x \preceq y$ and $x \not\approx y$.

Theorem 1.17 (Cantor's Theorem)**.** *For every set x, we have $x \prec \mathcal{P}(x)$.*

Theorem 1.18 (Cantor–Bernstein Theorem)**.** $x \approx y \iff (x \preceq y \wedge y \preceq x)$.

Definition 1.19 (Cardinal number)**.** An ordinal κ is a *cardinal number* if

$$(\forall \alpha \in \mathrm{On})(\alpha < \kappa \implies \alpha \prec \kappa).$$

We denote the *class of all cardinal numbers* by Cn.

Every $n \in \omega$ and ω are cardinal numbers. If $\alpha \geq \omega$, then $\alpha + 1$ is not a cardinal number; thus, every infinite cardinal number is a limit ordinal number. But not every limit ordinal number is a cardinal number: for instance $\omega + \omega > \omega$, but $\omega + \omega \approx \omega$, so $\omega + \omega$ is not a cardinal number. By the same logic, if $\alpha > \omega$ is countable, then it is not a cardinal number. On the other hand, the first uncountable ordinal ω_1 is a cardinal number. As for notation, we will denote cardinal numbers using letters from the middle of the Greek alphabet: $\kappa, \lambda, \mu, \nu, \ldots$

If a set x is equinumerous with a cardinal number κ, then we define $|x| := \kappa$, and we say that κ is the *cardinality* of x. Observe that if α is an ordinal, then $|\alpha| \leq \alpha$, and that $|\alpha| = \alpha$ holds $\iff$ α is a cardinal number $\iff$ α is the first ordinal with cardinality $|\alpha|$.

Observation 1.20. *If x and y have their cardinalities defined, then*

(a) $x \approx y \iff |x| = |y|$,

(b) $x \approx |x|$.

Observation 1.21. *The cardinality $|x|$ is defined $\iff$ x can be well-ordered.*

Proof. If x can be well-ordered, then $|x|$ is the least order type of the well-orderings of x, as each of those orderings induces a bijection between x and the ordinal order type of the ordering. On the other hand, if $|x| = \kappa$ is defined, then we can well-order x by inheriting the order of κ. □

In this thesis, we always assume the axiom of choice, which is equivalent to the statement that all sets can be well-ordered. Hence, we can safely assume that every set has its cardinality defined and can thus be replaced by an equinumerous cardinal number.

Alephs Cardinal numbers are closed under taking suprema: If $\lambda = \sup \kappa_i$ were not a cardinal, then $|\lambda| < \lambda$. Since λ is the supremum, there is some κ_j such that $|\lambda| < \kappa_j \leq \lambda$. But this contradicts κ_j being a cardinal. One can also show that for every cardinal, there exists a larger cardinal. Therefore, the class of all cardinals Cn is a proper class; otherwise its supremum would be the largest cardinal.

From the fact that Cn is a closed proper class of ordinals, one can prove that there is a uniquely determined bijective increasing mapping $\aleph \colon \mathrm{On} \to \mathrm{Cn} \setminus \omega$ enumerating infinite cardinals that satisfies $\aleph(\alpha) = \sup\{\aleph(\beta) \mid \beta < \alpha\}$ for every limit ordinal α. Cantor introduced the symbol $\aleph$ ("aleph"), the first letter of the Hebrew alphabet, to denote this function; its values $\aleph(\alpha)$ are denoted by $\aleph_\alpha$.

Historically, ordinals and cardinals were not concrete sets but abstract concepts: ordinals described well-ordering types, while cardinals measured size. This distinction led to the development of two parallel notation systems, ω_α and $\aleph_\alpha$. Von Neumann's 1923 set-theoretic definition of ordinals unified these ideas by providing canonical representatives for ordinal types. Today, we often write ω_α when considering the cardinal $\aleph_\alpha$ viewed as an ordinal with its well-order. In particular, the first uncountable ordinal is denoted by ω_1.

The *successor* of a cardinal κ is the smallest cardinal larger than κ, and we denote it by κ^+. Furthermore, we say that κ is the *predecessor* of κ^+. Finally, $\lambda > 0$ is a *limit* cardinal if it has no predecessor.

Example. Clearly $\aleph_{\alpha+1} = \aleph_\alpha^+$ are successor cardinals, and $\aleph_0$ is a limit cardinal. It is easy to show that $\aleph_\alpha$ for $\alpha > 0$ is a limit cardinal $\iff$ α is a limit ordinal.

Cardinal arithmetic If κ and λ are cardinal numbers, we define cardinals

(a) $\kappa + \lambda := |(\{0\} \times \kappa) \cup (\{1\} \times \lambda)|$,

(b) $\kappa \cdot \lambda := |\lambda \times \kappa|$,

(c) $\kappa^\lambda := |\{f \mid f : \lambda \to \kappa\}|$.

In other words, $\kappa + \lambda$ and $\kappa \cdot \lambda$ are cardinal numbers that represent the size of the set on the right side of the equation, while ordinal addition and multiplication express the order type of the same set when ordered lexicographically. Observe that cardinal addition and multiplication are associative, commutative, and distributive. Ordinal addition and multiplication are associative, but in general, they are not commutative or right-distributive. Intuitively, this is because the ordinal operations "keep track" of the underlying orderings. Also note that $2^\lambda = |\mathcal{P}(\lambda)|$ as there is a clear 1-to-1 correspondence between functions $f : \lambda \to \{0, 1\}$ and subsets $A \subseteq \lambda$.

Theorem 1.22. *For every ordinal α, we have $|\aleph_\alpha \times \aleph_\alpha| = \aleph_\alpha$.*

From this, it is not difficult to show that if κ and λ are cardinals, and at least one of them is infinite, then $\kappa + \lambda = \max\{\kappa, \lambda\}$. If, in addition, they are nonzero, then $\kappa \cdot \lambda = \max\{\kappa, \lambda\}$. When applied to infinite sets, we get:

Corollary 1.23. *If A and B are infinite sets, then*

$$|A \cup B| = |A \times B| = \max\{|A|, |B|\}.$$

Furthermore, if $|A| > |B|$, then $|A \setminus B| = |A|$.

Lemma 1.24. *For any set S it holds that $|\bigcup S| \leq |S| \cdot \sup\{|A| \mid A \in S\}$.*

Corollary 1.25. *The union of an arbitrary collection of $\aleph_\alpha$ sets, each of cardinality at most $\aleph_\alpha$, has cardinality at most $\aleph_\alpha$.*

Regular cardinals The pigeonhole principle states that ω cannot be partitioned into a finite number of finite sets, or equivalently, that if $A \subseteq \omega$ is finite, then $\sup(A) < \omega$. Hence, we cannot "reach" $\aleph_0 = \omega$ from below via any subset of smaller cardinality. Infinite cardinals with this property are called *regular*.

Definition 1.26 (Cofinality)**.** A subset $A \subseteq \alpha$ of a limit ordinal α is *cofinal* in α if $\sup(A) = \alpha$ (the limit of the increasing sequence of elements of A is α). The *cofinality* of α is the "length" of the shortest increasing sequence with limit α:

$$\mathrm{cf}(\alpha) := \min\{\text{order type of } (A, \leq) \mid A \subseteq \alpha \land \sup(A) = \alpha\}.$$

It can be shown that $\mathrm{cf}(\alpha)$ is equal to the minimal size $|A|$ of a cofinal subset A of α, so cofinality is always an infinite cardinal. Therefore $\omega \leq \mathrm{cf}(\alpha) \leq |\alpha|$.

Definition 1.27 (Regular cardinal)**.** An infinite cardinal number κ is a *regular cardinal* if $\mathrm{cf}(\kappa) = \kappa$, and a *singular cardinal* otherwise.

Regular cardinals behave similarly to ω in that they are *almost* closed under taking suprema: as long as the length of the sequence is less than κ, the limit will never reach κ. We can equivalently express this via unions of smaller sets.

Theorem 1.28. *An infinite cardinal number κ is singular $\iff$ there exists a set X such that $\kappa = \bigcup X$, where $|X| < \kappa$ and $|x| < \kappa$ for all $x \in X$.*

It is not difficult to show that $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$, so cofinality is always a regular cardinal. We know that ω is regular, and it can be shown in ZFC that every infinite successor cardinal $\aleph_{\alpha+1}$ is also regular. Perhaps somewhat surprisingly, the existence of a regular limit cardinal other than ω cannot be proved from the axioms of ZFC because it would allow one to construct a model of ZFC inside ZFC, contradicting Gödel's second incompleteness theorem. These hypothetical cardinals were first suggested by Felix Hausdorff in 1908, and today we call them *weakly inaccessible*. If a weakly inaccessible cardinal κ further satisfies that for all cardinals $\lambda < \kappa$ we have $2^\lambda = |\mathcal{P}(\lambda)| < \kappa$, then κ is said to be *strongly inaccessible*, or often simply just *inaccessible*.

Large cardinals Many problems in set theory and infinite combinatorics lead to the question of whether cardinals with certain properties exist. When the existence of such cardinals cannot be proved in ZFC, they are called *large cardinals*. Statements asserting the existence of such cardinals are known as *large cardinal axioms*, and they form a well-studied hierarchy [Kan94]. The existence of weakly and strongly inaccessible cardinals are the weakest of these axioms.

Some theorems can be proved only under assumptions asserting the existence of appropriate large cardinals. One has to take a "leap of faith" when making such assumptions since ZFC cannot prove their consistency.

1.4 Graphs

We define basic graph theoretic terms to make it clear how they extend to infinite graphs. Note that graphs in this thesis have no loops or parallel edges.

Definition 1.29 (Graph)**.** A *graph* G is a pair (V, E) where V is an arbitrary nonempty set of *vertices* and $E \subseteq [V]^2$ is a set of undirected *edges*.

Given a graph G, we denote its vertices by $V(G)$ or V_G, and its edges by $E(G)$ or E_G. We often write $uv \in E_G$ instead of $\{u, v\} \in E_G$. By the size $|G|$ of a graph G, we mean the size of V_G. In particular, G is *finite* if V_G is finite. A *path* of length $k \geq 0$ in a graph G is a sequence of $k + 1$ distinct vertices

$$v_0, v_1, \ldots, v_k,$$

such that each consecutive pair $v_i v_{i+1}$ is an edge of the graph. We say that the path *connects* v_0 to v_k, and we call them the *endpoints* of the path; the other vertices of the path are called its *internal* vertices. We say that a graph is *connected* if there is a path between any pair of its vertices. The *degree* of a vertex is the number of its *neighbours*: the vertices connected to it via an edge. An edge e is said to be *incident* with a vertex v if $v \in e$.

We denote the path on n vertices by P_n, the cycle on n vertices by C_n, and the complete bipartite graph with partitions of sizes n and m by $K_{n,m}$. A graph G is *acyclic* if it does not contain C_n as a subgraph for any n.

Definition 1.30 (Tree)**.** A *tree* is a connected acyclic graph.

The vertices of a tree that have degree 1 are called *leaves*. Finite trees always have at least one leaf, but infinite trees might have no leaves. Given a tree T, we can choose one of its nodes to be the *root*, and we imagine the tree sprouting upwards from this root. Such a tree is said to be *rooted*.

Definition 1.31 (Isomorphism)**.** An *isomorphism* of graphs H and G is a bijection $f\colon V_H \to V_G$ such that for each pair of vertices $u, v \in V_H$ we have

$$uv \in E_H \iff f(u)f(v) \in E_G.$$

Graphs G and H are said to be *isomorphic*, written $G \cong H$, if there is an isomorphism between them. We identify[2] isomorphic graphs, as they are structurally indistinguishable.

Definition 1.32 (Subgraph)**.** A graph H is a *subgraph* of a graph G if there is an injection $f\colon V_H \to V_G$ that satisfies $uv \in E_H \implies f(u)f(v) \in E_G$.

Definition 1.33 (Induced subgraph)**.** A graph H is an *induced subgraph* of G if there is an injection $f\colon V_H \to V_G$ that satisfies $uv \in E_H \iff f(u)f(v) \in E_G$.

Observation 1.34. *The subgraph and induced subgraph relations are quasi-orders on the class of all graphs. Furthermore, they are partial orders on the class of all finite graphs (if we identify isomorphic graphs).*

It is an easy and interesting exercise to find two non-isomorphic infinite graphs that are both subgraphs or induced subgraphs of each other.

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} \coloneqq (V, [V]^2 \setminus E)$, also sometimes denoted by G^c. Since induced subgraphs take edges to edges, and non-edges to non-edges, we have that H is an induced subgraph of $G \iff \overline{H}$ is an induced subgraph of $\overline{G}$.

The subgraph of a graph $G = (V, E)$ *induced* by vertices $W \subseteq V$ is the graph $G[W] \coloneqq (W, E \cap [W]^2)$. If v is a vertex of G, then $G - v$ denotes the subgraph induced by $V \setminus \{v\}$. If e is an edge, resp. non-edge of G, then $G - e$, resp. $G + e$, denotes the graph $(V, E \setminus \{e\})$, resp. $(V, E \cup \{e\})$). If G and H are two vertex-disjoint graphs, then their *disjoint union* is the graph $G + H \coloneqq \big(V_G \cup V_H, E_G \cup E_H\big)$. Note that if G and H are not vertex disjoint, we can still accomplish this using a simple set theory trick by replacing V_G and V_H with $\{0\} \times V_G$ and $\{1\} \times V_H$, respectively, and modifying the edges accordingly.

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[2]Some interesting technical details are hidden here. If we work only with finite graphs, then we can, without loss of generality, assume that their vertices are natural numbers. In that case, there are only countably many non-isomorphic graphs, and each isomorphism class is countable; one may therefore formally work either with isomorphism classes or use the axiom of choice to select a canonical representative from each class. The case when we consider only graphs with size bounded by a cardinal κ is similar.

However, if we allow graphs of arbitrary infinite size, then both of these approaches fail. There is a proper class of non-isomorphic graphs (certainly at least one graph for each cardinal), so selecting a representative from every isomorphism class would require making a proper-class-sized choice, which is not possible in standard set theory. Additionally, each of the isomorphism classes is also a proper class (even if we assume that the vertices of graphs range only over ordinal numbers). Hence, the class of isomorphism classes does not exist (proper classes cannot be elements of other classes), so we cannot work with isomorphism classes either. We could restrict the definition of what counts as a graph even further to ensure that isomorphism classes are sets, but there is a much cleaner technique known as *Scott's trick*, which allows us to work with graphs in their most general form.

To each graph G, one assigns a set $\tau(G)$ known as its *type*, such that $G \cong H \iff \tau(G) = \tau(H)$. Knowing the type of a graph gives us all the information associated with its structure up to isomorphism, allowing us to formally work with types of graphs instead of graphs themselves. Briefly, this is done by defining a hierarchy of sets $V_0 \coloneqq \varnothing$, $V_{\alpha+1} \coloneqq \mathcal{P}(V_\alpha)$, and $V_\lambda \coloneqq \bigcup_{\alpha<\lambda} V_\alpha$ for limit λ, and showing that every set x appears at some stage V_α (this is equivalent to the axiom of foundation). Consequently, we can define the *rank* of a set x as $\varrho(x) \coloneqq \min\{\alpha \mid x \subseteq V_\alpha\}$. We then define $\tau(G)$ as the set $\{H \mid H \cong G \wedge (\forall H')\big(H' \cong G \Rightarrow \varrho(H) \leq \varrho(H')\big)\}$. Thus, the type of a graph G is the set of all isomorphic graphs with minimal possible rank α. This is indeed a set (rather than a proper class) because it is a subset of $V_{\alpha+1}$.

A *component* of G is a connected induced subgraph $G[W]$ such that for every $v \in V \setminus W$, the graph $G[W \cup \{v\}]$ is no longer connected. If G is connected, then it has only one component, namely G. If G is not connected, then it is the disjoint union of its components.

Definition 1.35 (Minor)**.** A graph H is a *minor* of a graph G if there exists a family $\langle V_v \mid v \in V(H) \rangle$ of disjoint subsets of $V(G)$ such that each V_v induces a nonempty connected subgraph of G, and for each $uv \in E_H$ there are $x \in V_u$ and $y \in V_v$ such that $xy \in E_G$.

Definition 1.36 (Topological minor)**.** A graph H is a *topological minor* of G if there exists an injection $f\colon V_H \to V_G$ and a family $\langle P_{uv} \mid uv \in E_H \rangle$ of internally vertex-disjoint paths in G, where each path P_{uv} connects $f(u)$ to $f(v)$, and no internal vertex of any P_{uv} belongs to $f[V_H]$. The mapping f together with the paths $\langle P_{uv} \mid uv \in E_H \rangle$ is called a *homeomorphic embedding* of H in G.

Observation 1.37. *If H is a topological minor of G, then H is a minor of G.*

A *subdivision* of a graph G is formed by replacing some of its edges with paths. Clearly, H is a topological minor of $G \iff$ a subdivision of H is a subgraph of G. The *contraction* of an edge $uv \in E_G$ is formed by removing that edge and merging u and v into a single vertex w, reconnecting all edges previously incident with either u or v to w. It is easy to show that if G is finite, then H is a minor of $G \iff$ a graph isomorphic to H can be obtained from G by deleting edges and vertices and by contracting edges.

Cliques For a nonzero cardinal number κ, we denote by $K_\kappa \coloneqq (\kappa, [\kappa]^2)$ the *complete graph* on κ vertices and by $E_\kappa \coloneqq (\kappa, \varnothing)$ the *empty graph* on κ vertices. A *clique* in a graph G is a subgraph isomorphic to a complete graph, and an *independent set* in G is an induced subgraph isomorphic to an empty graph. The *clique number* and *independence number* of a graph G are defined as follows:

$$\begin{aligned} \omega(G) &\coloneqq \sup\{|C| \mid C \text{ is a clique in } G\}, \\ \alpha(G) &\coloneqq \sup\{|I| \mid I \text{ is an independent set in } G\}. \end{aligned}$$

If $\omega(G)$, resp. $\alpha(G)$ is finite, then this supremum is also a maximum. However, a general infinite graph G with infinite $\omega(G)$ or $\alpha(G)$ is not guaranteed to contain a clique of size $\omega(G)$ or an independent set of size $\alpha(G)$, respectively.

1.5 Ramsey's Theorem

We conclude the preliminaries with two classical theorems of Ramsey [Ram30], which will be invoked at several points throughout the thesis.

Theorem 1.38 (Infinite Ramsey's Theorem)**.** *For every number of colors r, dimension k, and for every r-coloring $\chi\colon [\omega]^k \to r$ of k-sets of ω, there exists an infinite subset $A \subseteq \omega$ such that $\chi \restriction [A]^k$ is constant (every k-set of A has the same color). We say that A is* homogeneous *for χ.*

We also include a proof of this beautiful theorem.

Proof. We proceed by induction on k. The base case $k = 1$ is the pigeonhole principle. Suppose $k \geq 2$ and χ is a coloring of k-sets of ω. For $t \in \omega$ define a coloring of $(k-1)$-sets of $\omega \setminus \{t\}$ as

$$\chi_t(A) \coloneqq \chi(A \cup \{t\}).$$

Let $t_0 \coloneqq 0$ and consider the coloring χ_{t_0} of $[\omega \setminus \{t_0\}]^{k-1}$. Use the induction hypothesis for $k-1$ to obtain an infinite $A_0 \subseteq \omega \setminus \{t_0\}$ homogeneous for χ_{t_0}. Notice that every k-set of $\{t_0\} \cup A_0$ that contains t_0 has the same color c_0 with respect to χ.

We repeat this process. Let $t_1 \coloneqq \min(A_0)$ and consider the coloring χ_{t_1} of $[A_0 \setminus \{t_1\}]^{k-1}$. By using the induction hypothesis, we obtain an infinite $A_1 \subseteq A_0 \setminus \{t_1\}$ homogeneous for χ_{t_1}. Note that every k-set of $\{t_0\} \cup \{t_1\} \cup A_1$ that contains t_1 has the same color c_1, which might differ from c_0.

Repeating this process indefinitely, we construct an infinite sequence $t_0, t_1, \ldots$ such that for every t_n it holds that every k-set of $\{t_i \mid i < \omega\}$ containing t_n has the same color c_n. Since there are only finitely many colors, by the pigeonhole principle, there exists an infinite subsequence $t_{i_0}, t_{i_1}, t_{i_2}, \ldots$ such that all t_{i_j} correspond to the same color. Hence, the set $\{t_{i_j} \mid j < \omega\}$ is homogeneous for χ. □

Ramsey's theorem implies that every infinite graph either contains an infinite independent set or an infinite clique. Indeed, take a countable subset of vertices and color pairs of them red if they are connected by an edge, and blue otherwise. By Ramsey's theorem, there is an infinite homogeneous subset of vertices.

Weakly compact cardinals In particular, every graph on ω vertices contains a clique of size ω or an independent set of size ω. Note that ZFC cannot prove the existence of another cardinal with this property — a cardinal $\kappa > \omega$ such that every graph on κ vertices contains a clique of size κ or an independent set of size κ. Such a cardinal would be called *weakly compact*, and it is another example of a large cardinal; this time much stronger than an inaccessible cardinal, but still quite low in the large cardinal hierarchy.

Theorem 1.39 (Finite Ramsey's Theorem)**.** *For every size n, number of colors r, and dimension k, there exists N such that for every r-coloring χ of $[N]^k$, there exists a subset $A \subseteq N$ of size n such that $\chi \restriction [A]^k$ is constant (every $a \in [A]^k$ has the same color).*

The finite version of Ramsey's theorem implies that large finite graphs cannot have both small clique number and small independence number.

2 Introduction to WQO Theory

In this chapter, we survey basic results about *well-quasi-orderings* or *wqo*. We reserve the letter Q to always denote a quasi-ordered class. Whenever a subclass Q' of Q is defined, we assume that Q' inherits the order from Q. If $\leq$ and $\leq^+$ are both quasi-orders on Q and $\leq \subseteq \leq^+$, then we say that $\leq^+$ *extends* $\leq$. For elements q and q' of Q, we define relations

(i) $q \equiv q'$ if $q \leq q'$ and $q' \leq q$,

(ii) $q < q'$ if $q \leq q'$ but $q \not\equiv q'$ (or equivalently $q' \not\leq q$),

(iii) $q \geq q'$ if $q' \leq q$, and $q > q'$ if $q' < q$.

Clearly, $\equiv$ is an equivalence relation on Q and it partitions Q into disjoint equivalence classes. If these classes are sets, we can consider the partially ordered class $Q/\equiv$ we obtain from Q by identifying $\equiv$-equivalent elements. If Q is a partial order, then $q \equiv q'$ implies $q = q'$, so $Q/\equiv$ is the same as Q (up to formalism).

2.1 Definition and Basic Properties of WQOs

In this section, we define well-quasi-orderings and provide multiple equivalent characterizations. We then explain the notion of the *maximal order type* of a wqo, and show how wqos can be used to finitely test monotone properties.

A *Q-sequence* is a mapping $f\colon A \to Q$ where $A \in [\omega]^\omega$ is an infinite set of natural numbers whose elements are called *indices*. Observe that every Q-sequence $f\colon A \to Q$ corresponds to a unique sequence $f'\colon \omega \to Q$. Thus we may regard Q-sequences simply as sequences $x_0, x_1, x_2, \dots$ (or $x_1, x_2, x_3, \dots$) where each $x_i \in Q$, and we will freely switch between these viewpoints. A *subsequence* of a Q-sequence $f\colon A \to Q$ is any Q-sequence that is the restriction of f to a subset of A. A Q-sequence $f\colon A \to Q$ is called *good* if there are indices $i < j$ in A such that $f(i) \leq f(j)$, and is called *bad* otherwise.

Definition 2.1 (Well-quasi-ordering)**.** Q is *wqo* if every Q-sequence is good.

Definition 2.2. A Q-sequence $x_1, x_2, x_3, \dots$ is called an *infinite decreasing chain* if $x_1 > x_2 > x_3 > \cdots$. It is said to be *non-decreasing* if $x_1 \leq x_2 \leq x_3 \leq \cdots$.

Lemma 2.3. *Q is well-founded $\iff$ Q admits no infinite decreasing chains.*

Proof. "$\Rightarrow$" The set of elements forming an infinite decreasing chain has no minimal element. "$\Leftarrow$" Suppose a nonempty subset $A \subseteq Q$ has no minimal element. Then for every $x \in A$ there is $x' \in A$ such that $x' \leq x$ but $x \not\leq x'$, so $x' < x$. This allows us to construct an infinite decreasing chain. □

Remark. The direction "$\Leftarrow$" actually cannot be proved in bare ZF, as we need some kind of choice principle to *choose* x_{i+1} from all $x' \prec x_i$ for infinitely many i.

Theorem 2.4. *The following conditions are equivalent.*

(i) *Q is wqo.*

(ii) *Q admits no infinite decreasing chains and contains no infinite antichains.*

(iii) *Every Q-sequence $x_1, x_2, x_3, \dots$ contains an infinite non-decreasing subsequence $x_{i_1} \le x_{i_2} \le x_{i_3} \le \cdots$ for some indices $i_1 < i_2 < i_3 < \cdots$.*

(iv) *Every nonempty subset of Q contains at least one, but only finitely many (non-equivalent) minimal elements. In particular, every wqo is well-founded.*

If Q is a set, then we get another condition.

(v) *Every linear extension $\le_L$ of $\le$ on $Q/\equiv$ is a well-ordering.*

Proof. (i)⇒(ii) An infinite decreasing chain would be bad, and any sequence formed by distinct elements of an infinite antichain would also be bad.

(ii)⇒(iii) By Ramsey's theorem: color a pair $\{i, j\} \in [\omega]^2$ for $i < j$ red if $x_i \le x_j$, blue if $x_i > x_j$, and white if x_i and x_j are incomparable. Ramsey's theorem claims that there is an infinite homogeneous subset, and (ii) implies that it cannot be blue or white, so it has to be red. The elements of this subset are indices of the sought-after infinite non-decreasing subsequence.

(iii)⇒(i) This one is obvious.

(ii)⇒(iv) Lemma 2.3 implies that Q is well-founded. Furthermore, if a subset of Q contained infinitely many non-equivalent minimal elements, then they would form an infinite antichain in Q.

(iv)⇒(ii) Since Q is well-founded, it admits no infinite decreasing chains. And if there were an infinite antichain, all its elements would be minimal.

(iv)⇒(v) Given a nonempty subset $\mathcal{A}$ of equivalence classes $[q]$ of $Q/\equiv$, consider the union of these classes $\bigcup \mathcal{A} \subseteq Q$, and its (non-equivalent) minimal elements $q_1, \dots, q_n$. The minimum $[q]$ of $\mathcal{A}$ we are looking for is $\min_{\le_L} [q_i]$.

(v)⇒(ii) We use a theorem of Szpilrajn [Szp30] which states that, in ZFC,[1] every partially ordered set has a linear extension. Suppose for contradiction that there is an infinite decreasing chain $x_1 > x_2 > x_3 > \cdots$ in Q. It induces an infinite decreasing chain $[x_1] > [x_2] > [x_3] > \cdots$ in $Q/\equiv$, and every linear extension $\le_L$ of $\le$ contains it as well, so it cannot be a well-ordering.

Similarly, an infinite antichain $A = \{a_i \mid i \in \omega\}$ in Q induces an infinite antichain $\mathcal{A} = \{[a_i] \mid i \in \omega\}$ in $Q/\equiv$. We can define a strict linear order $<_\mathcal{A}$ on $\mathcal{A}$ as $[a_i] <_\mathcal{A} [a_j]$ if $i > j$. Consider the transitive closure $\le'$ of $\le \cup \le_\mathcal{A}$. It is easy to verify that $\le'$ is a partial order on $Q/\equiv$ (cycles cannot form because elements of $\mathcal{A}$ are incomparable in $\le$), so by Szpilrajn's theorem, $\le'$ has a linear extension $\le_L$, in which $\mathcal{A}$ is an infinite decreasing chain. Hence, $\le_L$ is not a well-ordering. □

Transfinite sequences We show that part (iii) of the previous theorem generalizes to transfinite sequences whose length is a regular cardinal. A *Q-sequence of length α* is a mapping $f \colon A \to Q$ where $A \subseteq \mathrm{On}$ is a set of ordinals with order type α. Note that f determines a unique sequence $f' \colon \alpha \to Q$ that we may denote by $\langle q_\delta \mid \delta < \alpha \rangle$ where each $q_\delta \in Q$. A *subsequence* of $f \colon A \to Q$ is a restriction of f to a subset $B \subseteq A$. We should note that the length β of a subsequence can never exceed the length α of the original sequence because whenever $B \subseteq A$, the order type of B is at most the order type of A.

Proposition 2.5. *If Q is wqo, then every Q-sequence whose length is a regular cardinal contains a non-decreasing subsequence of the same length.*

[1] Briefly, we consider the ordered set $(E, \subseteq)$ of all extensions of the given partial order, and we show that the union of a chain in E is again a partial order. Zorn's lemma gives us a $\subseteq$-maximal partial order, and we verify that it is linear.

Proof (cf. Lemma 4.5 of [KT90]). Let κ be a regular cardinal and let $\langle q_\alpha \mid \alpha < \kappa\rangle$ be a Q-sequence of length κ. We need to show that there is an increasing sequence of indices $\langle \alpha_\beta \mid \beta < \kappa\rangle$ such that $q_{\alpha_\beta} \leq q_{\alpha_{\beta'}}$ for all $\beta < \beta' < \kappa$. For a cofinal subset A of κ we call an ordinal $\delta \in A$ *terminal* for A if the set

$$A(\delta) \coloneqq \{\alpha \in A \mid q_\delta \not\leq q_\alpha\}$$

of indices of elements q_α that do not extend q_δ is cofinal in A. We claim that there is a cofinal subset of κ without a terminal element. Suppose not, put $A_0 = \kappa$, let δ_0 be terminal for A_0, and define $A_1 \coloneqq A(\delta_0)$. We can repeat this and inductively define δ_n as a terminal element for A_n and $A_{n+1} \coloneqq A(\delta_n)$. Then $q_{\delta_0}, q_{\delta_1}, \ldots$ is a bad sequence in Q, a contradiction.

So let A be a cofinal subset of κ with no terminal element. We put $\alpha_0 \coloneqq \min(A)$ and for $\beta < \kappa$ define inductively

$$\alpha_\beta \coloneqq \sup\{\sup A(\alpha_{\beta'}) + 1 \mid \beta' < \beta\}.$$

Because A has no terminal element we have $\sup A(\alpha_{\beta'}) + 1 < \kappa$, and since κ is regular we also have $\alpha_\beta < \kappa$. Clearly $\langle \alpha_\beta \mid \beta < \kappa\rangle$ is as desired. □

Maximal order types Every well-ordered set is order-isomorphic to a unique ordinal number, called its order type. Part (v) of the previous theorem motivates the following definition.

Definition 2.6. The *maximal order type* of a wqo $(Q, \leq)$ is the supremum of the order types of all the possible linearizations of $Q/\equiv$. We denote it by $o(Q, \leq)$.

In their seminal paper [JP77], de Jongh and Parikh showed that this supremum is in fact a maximum: there exists a linearization $\preceq$ of $Q/\equiv$ that attains the maximal order type $o(Q, \leq)$. This naturally leads to the problem of determining the maximal order types of various well-quasi-orderings. We will not pursue this direction further, but we refer the interested reader to Schmidt's thesis [Sch20] for a comprehensive survey.

Finite characterizations of monotone properties Given a quasi-ordered set $(Q, \leq)$, we say that a property $\varphi(x)$ is *monotone* if it is downward closed with respect to $\leq$; that is, if whenever $\varphi(x)$ holds and $y \leq x$, then $\varphi(y)$ holds as well. We say that a property $\varphi(x)$ is *finitely testable* if there exists a finite set $\mathcal{F}$ of "forbidden" elements called *obstructions* such that

$$\varphi(x) \text{ holds} \iff (\forall F \in \mathcal{F}) : F \not\leq x.$$

Theorem 2.7. *If Q is wqo, then every monotone property of Q is finitely testable.*

Proof. Let φ be a monotone property of Q, and let $\overline{Q}$ be the set of all $x \in Q$ for which φ does not hold. If $\overline{Q} = \varnothing$, then take $\mathcal{F} \coloneqq \varnothing$. Otherwise, $\overline{Q}$ contains at least one, but only finitely many non-equivalent minimal elements $F_1, \ldots, F_n$. We claim that $\varphi(x) \iff (\forall i)\, F_i \not\leq x$.

"$\Rightarrow$" Suppose $F_j \leq x$ for some j. Then $\varphi(F_j)$ holds since φ is monotone and $\varphi(x)$ holds, contradicting how we chose F_j.

"$\Leftarrow$" Suppose $\varphi(x)$ does not hold; then $x \in \overline{Q}$. Consider the set $Y \coloneqq \{y \in \overline{Q} \mid y \leq x\}$. Since Q is wqo, Y has a minimal element $F \leq x$, and it is easy to see that F is minimal in $\overline{Q}$ as well, so F is equivalent to some F_j. We have $F_j \equiv F \leq x$ and thus $F_j \leq x$, a contradiction. □

In particular, the subgraph, induced subgraph, minor, and topological minor relations are all quasi-orders on the class of all graphs. This means that if we find an interesting graph class $\mathcal{G}$ that is wqo by one of these relations, we can characterize any monotone property of $\mathcal{G}$ using a finite set of forbidden obstructions.

For example, Robertson and Seymour [RS04] famously showed that the class of all finite graphs is wqo by the minor relation, and planarity is a minor-monotone property. Hence, there should be a finite characterization of planar graphs. And indeed, a well-known theorem of Wagner states that a finite graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.

2.2 Elementary Constructions of WQOs

A central theme in wqo theory is the construction of more complicated well-quasi-orders from simpler ones. As a starting point, we consider quasi-orders that are already known to be wqo, namely finite sets and well-ordered classes.

Observation 2.8. *Let Q be a quasi-ordered class.*

(a) *If Q is finite, then $(Q, =)$ is wqo.* *. . . from the pigeonhole principle*

(b) *If Q is well-ordered, then Q is wqo.* *. . . from Theorem 2.4* (ii) *or* (iv)

In particular, any ordinal α and the class of all ordinals On *are wqo.*

Order-reflecting maps Suppose that Q is wqo, and that we want to show that a different quasi-order P is also wqo. One of the simplest ways of doing this is to find an *order-reflecting map* of P into Q.

Definition 2.9. If $h\colon P \to Q$ is an injection such that $h(x) \leq_Q h(y)$ implies $x \leq_P y$ for all $x, y \in P$, then h is called an *order-reflecting map* of P into Q. If there exists an order-reflecting map of P into Q, we write $P \hookrightarrow Q$.

Observation 2.10. *If Q is wqo and $P \hookrightarrow Q$, then P is also wqo.*

Proof. Let $h\colon P \to Q$ be an order-reflecting map, and let $f\colon A \to P$ be a P-sequence. Then $f \circ h\colon A \to Q$ forms a Q-sequence, and since Q is wqo, there are indices $i < j$ in A such that $h(f(i)) \leq_Q h(f(j))$, and thus also $f(i) \leq_P f(j)$. □

In particular, if $(Q, \leq)$ is wqo and $\leq^+$ extends $\leq$, then $(Q, \leq^+)$ is wqo.

Combining WQOs A common construction in wqo theory is *enriching* a wqo Q by combining it with another wqo P. The simplest way to do this is to take the disjoint union[2] $P \dot\sqcup Q$ ordered by the rule that $x \leq y \iff x \leq_P y$ or $x \leq_Q y$.

Observation 2.11. *If P and Q are wqos, then $P \dot\sqcup Q$ is wqo.*

Proof. By the pigeonhole principle, every infinite sequence in $P \dot\sqcup Q$ has an infinite subsequence consisting only of elements from one of the original wqos. □

Perhaps more interesting is the *product quasi-ordering* of two wqos P and Q. Consider the Cartesian product $P \times Q$ ordered by the rule that

$$(p, q) \leq_\times (p', q') \iff p \leq_P p' \wedge q \leq_Q q'.$$

A classical theorem of Dickson [Dic13], known as *Dickson's lemma*, states that if both P and Q are wqos, then their product quasi-ordering is again wqo.

[2]Formally, $P \dot\sqcup Q$ is the class $(\{0\} \times P) \cup (\{1\} \times Q)$, and we modify $\leq_P$ and $\leq_Q$ accordingly.

Theorem 2.12 (Dickson's lemma)**.** *If P and Q are wqos, then $P \times Q$ is wqo.*

Proof. Let $(p_i, q_i)_{i<\omega}$ be a sequence in $P \times Q$. Since P is wqo, the induced sequence $(p_i)_{i<\omega}$ contains an infinite non-decreasing subsequence $(p_i)_{i\in A}$ for some infinite set $A \subseteq \omega$. The pairs $(p_i, q_i)_{i\in A}$ are thus non-decreasing in the first coordinate. Since Q is wqo, there exist $i < j$ in A such that $q_i \leq_Q q_j$. We have $(p_i, q_i) \leq_\times (p_j, q_j)$, so $P \times Q$ is wqo. □

Dickson's lemma can easily be extended to ordering $Q_1 \times \cdots \times Q_n$ by induction. Consequently, n-tuples of natural numbers, ordered coordinate-wise, are wqo.

2.3 Higman's Finite Sequence Theorem

Dickson's lemma allows us to compare sequences of fixed finite length, but what about sequences of unbounded or even transfinite length? Given an ordinal α, denote by Q^α the class of all *α-sequences* of elements of Q; that is, functions $f\colon \alpha \to Q$. Then $Q^{<\alpha} := \bigcup_{\beta<\alpha} Q^\beta$ denotes the class of all sequences shorter than α, and $Q^{\mathrm{On}} := \bigcup_{\beta\in\mathrm{On}} Q^\beta$ denotes the class of all sequences of arbitrary transfinite length. In particular, $Q^{<\omega}$ denotes the class of all finite sequences of elements of Q. We may think of them as finite words over the alphabet Q.

For a given quasi-order Q, the class $Q^{<\alpha}$ is ordered by the rule that

$$(a_\xi)_{\xi<\beta_1} \sqsubseteq (b_\xi)_{\xi<\beta_2}$$

if there is a strictly increasing mapping $f\colon \beta_1 \to \beta_2$ such that $a_\xi \leq b_{f(\xi)}$ for all $\xi < \beta_1$. In particular, if A is a finite alphabet (ordered by equality), and u and v are finite words over A, then $u \sqsubseteq v \iff u$ is a subword of v.

If one disregards the order and multiplicities of elements in such sequences, the comparison reduces to one between subsets. Accordingly, we equip the class of all subsets[3] $\mathcal{P}(Q)$ of Q with the quasi-order defined by the rule that

$$X \sqsubseteq_1 Y$$

if there exists an injection $f\colon X \to Y$ such that $x \leq f(x)$ for all $x \in X$. If we further omit the rule that f has to be injective, we obtain the order $\sqsubseteq_m$ on $\mathcal{P}(Q)$.

Higman [Hig52] showed that if Q is wqo, then the class of all finite sequences of elements of Q is again wqo. This result is now known as *Higman's lemma*, or as *Higman's finite sequence theorem.* Note that it immediately implies that the class of all finite subsets of Q is also wqo. Nash-Williams [Nas65b] later generalized Higman's result and showed that if Q is wqo, then the subclass of Q^{On} consisting of all transfinite sequences with *finite range* is again wqo (a sequence has finite range if only finitely many elements of Q appear in it).

We will prove Higman's lemma shortly, but first we emphasize the finite character of these results. A fundamental limitation of wqos is that they are not, in general, preserved under unrestricted infinite constructions. In particular, Rado [Rad54] showed that Higman's lemma already fails for sequences of length ω. Nash-Williams later discovered a condition stronger than wqo that is preserved under passage from Q to $\mathcal{P}(Q)$ and even Q^{On}, and he introduced the term *better-quasi-ordering* for such orderings. We will study them in the next chapter.

Theorem 2.13 (Higman)**.** *If Q is wqo, then $Q^{<\omega}$ is wqo*

[3]If Q is a set, then $\mathcal{P}(Q)$ is its power set.

Corollary 2.14. *If Q is wqo, then the class of all finite subsets of Q is wqo.*

In particular, word embedding over a finite alphabet is always wqo. The proof we present uses an important technique introduced by Nash-Williams in [Nas63]. We assume that there is a bad sequence in $Q^{<\omega}$ and construct a *minimal bad sequence*. We then make the structures in this sequence "smaller," and show that these smaller structures must be wqo; otherwise, the bad sequence we found would not be minimal. From the wqo property of this smaller sequence, we then obtain the wqo property for the minimal bad sequence, demonstrating that it actually was never bad in the first place.

Proof of Theorem 2.13. We will treat finite sequences as finite words w over the alphabet Q. For contradiction, suppose that there is a bad sequence in $Q^{<\omega}$. We construct a *minimal bad sequence* $w_0, w_1, w_2, \dots, w_i, \dots$ as follows:

Let w_0 be a string of minimal length that appears as the first term in a bad sequence. Let w_1 be a string of minimal length that appears as the second term in a bad sequence that starts with w_0. In general, let w_i for $i > 0$ be a string of minimal length that appears as the i-th term (starting at $i = 0$) in a bad sequence that starts with $w_0, w_1, \dots, w_{i-1}$.

Notice that the strings w_i are all nonempty because the empty sequence embeds into any other sequence. For every i, write w_i as $a_i w_i'$, where a_i is the first letter of w_i and w_i' is the rest of the word.

Since Q is wqo, the sequence $(a_i)_{i<\omega}$ contains an infinite non-decreasing subsequence $a_{f(0)} \leq a_{f(1)} \leq \cdots$ for some increasing function $f \colon \omega \to \omega$. We claim that the sequence $w'_{f(0)}, w'_{f(1)}, \dots$ is good: that there are indices $i < j$ such that $w'_{f(i)} \sqsubseteq w'_{f(j)}$. Notice that this immediately implies $w_{f(i)} \sqsubseteq w_{f(j)}$, so the minimal bad sequence we chose was, in fact, not bad; a contradiction.

To prove the claim, suppose for contradiction that $w'_{f(0)}, w'_{f(1)}, \dots$ is bad and observe that then the sequence $w_0, w_1, \dots, w_{f(0)-1}, w'_{f(0)}, w'_{f(1)}, \dots$ is also bad: if we had $w_j \sqsubseteq w'_{f(k)}$ for some $j < f(0) \leq f(k)$, then also $w_j \sqsubseteq w_{f(k)}$, contradicting the badness of our minimal bad sequence. But notice that $|w'_{f(0)}| < |w_{f(0)}|$, which contradicts the minimality of $w_{f(0)}$. □

3 Introduction to BQO Theory

Nash-Williams introduced *better-quasi-orderings* or *bqo* in [Nas65a] to prove that the class of all (finite or infinite) trees is wqo by the topological minor relation. While the arguments of wqo theory tend to break down under infinite constructions, better-quasi-orderings provide a stronger and more robust framework that survives many such constructions. It turns out that the wqos encountered in practice are, in fact, bqos, as we hope to illustrate in the rest of this thesis.

3.1 Limitations of WQOs

We demonstrate the failure of Higman's lemma for infinite sequences using Rado's counter-example [Rad54].

Proposition 3.1 (Rado)**.** *There exists a wqo set Q such that Q^ω is not wqo.*

Proof. Let $Q = \{(i, j) \,|\, i < j < \omega\}$ be ordered by $(i_1, j_1) \preceq (i_2, j_2)$ if $i_1 = i_2$ and $j_1 \leq j_2$, or $j_1 < i_2$. We first verify that Q is wqo. Let $f \colon \omega \to Q$ be a Q-sequence and denote $f(k) = (i_k, j_k)$. By Dickson's lemma, $\omega \times \omega$ is wqo, so there is an increasing function $h \colon \omega \to \omega$ such that $i_{h(0)} \leq i_{h(1)} \leq \cdots$ and $j_{h(0)} \leq j_{h(1)} \leq \cdots$. If there are $k < l$ such that $i_{h(k)} = i_{h(l)}$, then $(i_{h(k)}, j_{h(k)}) \preceq (i_{h(l)}, j_{h(l)})$, and we are done. Otherwise, we have $i_{h(0)} < i_{h(1)} < \cdots$, and so there is n such that $j_{h(0)} < i_{h(n)}$. Therefore $(i_{h(0)}, j_{h(0)}) \preceq (i_{h(n)}, j_{h(n)})$.

We now construct a bad sequence of Q^ω (a sequence of infinite sequences). For each $t \in \omega$, define a sequence $f_t \colon \omega \to Q$ by $f_t(n) := (t, t+n+1)$. If we now assume that Q^ω is wqo, then there are $t < s$ such that $f_t \sqsubseteq f_s$. Then $f_t(s-t) \preceq f_s(n)$ for some n, and $(t, s+1) \preceq (s, s+n+1)$, which is false. □

Observe that the exact same counter-example works for subsets as well: the set $[Q]^\omega \subseteq \mathcal{P}(Q)$ of all countably infinite subsets of Q is neither wqo by $\sqsubseteq_1$ nor by $\sqsubseteq_m$. According to Laver [Lav71], Nash-Williams discovered the definition of better-quasi-orderings by working from this counter-example.

3.2 Motivation for BQOs

We motivate the definition of better-quasi-orderings following Laver [Lav71].

Given a quasi-order Q, we define $\mathcal{P}^0(Q) := Q$, $\mathcal{P}^{\alpha+1}(Q) := \mathcal{P}(\mathcal{P}^\alpha(Q))$, and $\mathcal{P}^\lambda(Q) := \bigcup_{\alpha<\lambda} \mathcal{P}^\alpha(Q)$ for limit ordinals λ, where $\mathcal{P}(Q)$ denotes the class of all subsets of Q. In particular, $\mathcal{P}^1(Q) = \mathcal{P}(Q)$. Note that this definition of limit stages works up to formalism, since we should identify "equivalent" sets $X \in \mathcal{P}^\beta(Q), \{X\} \in \mathcal{P}^{\beta+1}(Q), \{\{X\}\} \in \mathcal{P}^{\beta+2}(Q), \ldots$ that were introduced in the previous successor stages. With this in mind, by $X \in \mathcal{P}^\lambda(Q)$ we refer to the first of the equivalent occurrences of X. To avoid a trivial notational problem, assume that $q \notin \mathcal{P}^{\alpha+1}(Q)$ holds for any $q \in Q$, $\alpha \in \mathrm{On}$.

For an ordinal γ, we inductively define a quasi-order $\sqsubseteq_a$ on $\mathcal{P}^\gamma(Q)$ as follows: If $\gamma = 0$, we let $\sqsubseteq_a$ be the standard order $\leq$ on Q. If γ is a successor ordinal and $X, Y \in \mathcal{P}^\gamma(Q)$, we let $X \sqsubseteq_a Y$ if $(\forall X' \in X)(\exists Y' \in Y)$ such that $X' \sqsubseteq_a Y'$. In particular, for $\gamma = 1$ we get the order $\sqsubseteq_m$ on $\mathcal{P}(Q)$ defined in the previous chapter. If γ is limit, the definition is by induction on $\alpha, \beta < \gamma$. For $X \in \mathcal{P}^\alpha(Q)$ and $Y \in \mathcal{P}^\beta(Q)$, we let $X \sqsubseteq_a Y$ if

(i) $\alpha = 0$, $\beta = 0$, and $X \le Y$ as elements of Q, or

(ii) $\alpha = 0$, $\beta > 0$, and $\exists Y' \in Y$ such that $X \sqsubseteq_a Y'$, or

(iii) $\alpha > 0$, $\beta > 0$, and $(\forall X' \in X)(\exists Y' \in Y)$ such that $X' \sqsubseteq_a Y'$.

The definition for Q to be bqo is combinatorial, but Nash-Williams [Nas65a] remarks that it is equivalent to $\mathcal{P}^{\alpha}(Q)$ being wqo for every ordinal number α, and Laver [Lav71] states that it is equivalent to just $\mathcal{P}^{\omega_1}(Q)$ being wqo (see [Peq17] for a proof). Finally, Pouzet [Pou72] proved that Q is bqo $\iff$ $Q^{<\omega_1}$ is wqo.

If $X \in Q$, then we say that X is an *atom*. We pause for a lemma.

Lemma 3.2. *If $X \in \mathcal{P}^{\gamma}(Q)$ is not an atom and $X' \in X$, then $X' \sqsubseteq_a X$.*

Proof. We proceed by transfinite induction on the stage α in which X' first appeared. If X' is an atom, then this follows from (ii) since $X' \sqsubseteq_a X' \in X$. Suppose that $X' \in \mathcal{P}^{\alpha}(Q)$ and that we have already verified the claim for all $\beta < \alpha$. Then $X'' \sqsubseteq_a X'$ for every $X'' \in X'$, and thus also $X' \sqsubseteq_a X$ since (iii) is satisfied by choosing $Y' = X' \in X$. □

Assume a quasi-order Q is given such that $\mathcal{P}^{\gamma}(Q)$ is not wqo. Accordingly, there is a sequence $X_0, X_1, X_2, \dots$ of members of $\mathcal{P}^{\gamma}(Q)$ such that whenever $i < j$, then $X_i \not\sqsubseteq_a X_j$. Put $I_0 \coloneqq \{i \mid X_i \notin Q\}$. Suppose that $j \in I_0$, $i < j$, and notice that $X_i \not\sqsubseteq_a X'_j$ for all $X'_j \in X_j$; otherwise, the previous lemma and transitivity would imply $X_i \sqsubseteq_a X_j$. If also $i \in I_0$, then there is $X'_i \in X_i$ such that $X'_i \not\sqsubseteq_a X'_j$ for all $X'_j \in X_j$. Choose such a X'_i for every $j > i$ and denote it by $X_{i,j}$ (if $j \notin I_0$, choose $X_{i,j} \in X_i$ arbitrarily). Observe now that whenever $j \in I_0$ and $i < j < k$, then $X_i \not\sqsubseteq_a X_{j,k}$, and if $i \in I_0$, then $X_{i,j} \not\sqsubseteq_a X_{j,k}$.

Put $I_1 \coloneqq \{(i,j) \mid X_{i,j} \notin Q\}$. Suppose that $(j,k) \in I_1$, $i < j$, and notice that $X_i \not\sqsubseteq_a X'_{j,k}$ and $X_{i,j} \not\sqsubseteq_a X'_{j,k}$ (if $X_{i,j}$ is defined) for all $X'_{j,k} \in X_{j,k}$. If also $(i,j) \in I_1$, then, because $X_{i,j} \not\sqsubseteq_a X_{j,k}$, there is $X'_{i,j} \in X_{i,j}$ such that $X'_{i,j} \not\sqsubseteq_a X'_{j,k}$ for all $X'_{j,k} \in X_{j,k}$. Choose such a $X'_{i,j}$ for every $k > j$ and denote it by $X_{i,j,k}$ (if $(j,k) \notin I_1$, choose $X_{i,j,k} \in X_{i,j}$ arbitrarily). Observe that whenever $(j,k) \in I_1$ and $i < j < k < l$, then $X_i \not\sqsubseteq_a X_{j,k,l}$, and if $i \in I_0$, then $X_{i,j} \not\sqsubseteq_a X_{j,k,l}$, and if $(i,j) \in I_1$, then $X_{i,j,k} \not\sqsubseteq_a X_{j,k,l}$.

Continuing this process indefinitely, we construct sets $X_{a_1,a_2,\dots,a_n} \in Q$ (we will always eventually reach Q since the axiom of foundation asserts that $\in$ is well-founded on the class of all sets) where $a_1 < a_2 < \dots < a_n$, such that whenever $a_1 < a_2 < \dots < a_n < \dots < a_m$ and the corresponding sets are defined, then $X_{a_1,a_2,\dots,a_n} \not\sqsubseteq_a X_{a_2,\dots,a_m}$. In light of this discussion, the definitions of Nash-Williams [Nas65a] given in the following section should appear more natural.

3.3 Definition and Basic Properties of BQOs

For a set X, denote by $A_\omega(X)$ the set of all infinite strictly increasing sequences $f\colon \omega \to X$ ("A" as in "ascending"), and by $[X]^{<\omega}$ the set of all nonempty finite strictly increasing sequences of elements of X. Note that this notation is justified as $[X]^{<\omega}$ is, up to formalism, the set of all nonempty finite subsets of X. Further note that for infinite X we have $\left|[X]^{<\omega}\right| = |X|$. An *initial segment* of a sequence $f\colon \lambda \to X$ where $\lambda \le \omega$ is a restriction $f \restriction n$ for any $n \le \lambda$. We define a relation $\triangleleft$ on $[\omega]^{<\omega}$ as follows: $s \triangleleft t \iff$ there is a sequence $u = u_1 \dots u_n \dots u_m \in [\omega]^{<\omega}$ such that $s = u_1 \dots u_n$ and $t = u_2 \dots u_m$ where $m > n \ge 1$. Observe that the relation $\triangleleft$ is irreflexive ($s \not\triangleleft s$) and anti-transitive ($s \triangleleft t \triangleleft u \implies s \not\triangleleft u$).

For a sequence s, we denote by $\bar{s}$ its range; for a set of sequences $B \subseteq [\omega]^{<\omega}$, we denote by $\overline{B} \subseteq \omega$ the set $\bigcup_{s \in B} \bar{s}$. Note that if B is infinite, then its underlying set $\overline{B}$ must also be infinite. A *block* is an infinite subset $B \subseteq [\omega]^{<\omega}$ such that each $u \in A_\omega(\overline{B})$ has an initial segment $s \in B$. Now let Q be a quasi-order. A *Q-pattern* is any mapping from a block into Q. Recall that a Q-sequence $f\colon A \to Q$ is good if there are $i, j \in A$ such that $i < j$ and $f(i) \le f(j)$. A Q-pattern $f\colon B \to Q$ is *good* if there are $s, t \in B$ such that $s \triangleleft t$ and $f(s) \le f(t)$, and is *bad* otherwise.

Definition 3.3 (Better-quasi-ordering)**.** Q is *bqo* if every Q-pattern is good.[1]

Observe that the property of being wqo amounts to *some* Q-patterns being good: consider the block $B_1 \subseteq [\omega]^{<\omega}$ consisting of all sequences of length 1 (so $B_1 \cong \omega$ up to formalism). A subset $B_1' \subseteq B_1$ is a block $\iff$ $B_1' \cong \overline{B_1'}$ is infinite, so there is a clear 1-to-1 correspondence between Q-sequences and Q-patterns of the form $f\colon B_1' \to Q$. Finally, notice that a Q-pattern $f\colon B_1' \to Q$ is good $\iff$ the corresponding Q-sequence $f'\colon \overline{B_1'} \to Q$ is good, because for sequences of length 1 we have $(i) \triangleleft (j) \iff i < j$. Therefore Q is wqo $\iff$ every Q-pattern $f\colon B \to Q$ where $B \subseteq B_1$ is good. To summarize:

Observation 3.4. *If Q is bqo, then Q is wqo.*

Arrays and barriers In order to make bqos easier to work with, we consider a special type of blocks: For $s, t \in [\omega]^{<\omega}$ write $s \subset t$ if s is a subsequence of t (if $\bar{s} \subseteq \bar{t}$). A block B is called a *barrier* if $(B, \subset)$ is an antichain. A *Q-array* is any mapping from a barrier into Q; thus a Q-array is a special kind of Q-pattern.

Lemma 3.5. *Every block B contains a barrier $B' \subseteq B$.*

Proof. See Lemma 20 of [Nas65a]. □

Corollary 3.6. *Q is bqo $\iff$ every Q-array is good.*

Proof. Clearly "⇒" holds. To show "⇐" suppose for contradiction that Q is not bqo. Accordingly, there is a bad Q-pattern $f\colon B \to Q$, and by Lemma 3.5 there is a barrier $B' \subseteq B$. Notice that the Q-array $f \restriction B'$ is bad, a contradiction. □

Next, we verify that the definition of bqo is feasible: that every barrier (and thus every block) contains many pairs of elements s and t such that $s \triangleleft t$.

Lemma 3.7. *If B is a barrier and $s \in B$, then there is $t \in B$ such that $s \triangleleft t$.*

Proof. Let u be any element of $A_\omega(\overline{B})$ with s as an initial segment. Denote by ${}_*u$ the sequence we obtain from u by omitting its first term. Since B is a block, ${}_*u$ has an initial segment $t \in B$. Since B is a barrier, t cannot be an initial segment of ${}_*s$ (because then t would be a subsequence of s) and therefore $s \triangleleft t$. □

Lemma 3.8. *If B is a barrier and $X \subseteq \overline{B}$ is infinite, then $B \cap [X]^{<\omega}$ is a barrier.*

[1]The definition of bqo used in this thesis is the original one due to Nash-Williams [Nas65a]. Modern treatments of bqo theory often use an equivalent topological characterization due to Simpson [Sim85a], which defines bqos in terms of Borel functions $[\omega]^\omega \to Q$. This definition is often preferable because it allows one to use the tools of descriptive set theory (namely the Galvin–Prikry theorem), simplifying many proofs. We present the original definition here because it reflects the historical development of the subject and clarifies how the notion of bqo arose naturally from the study of wqos. The reader interested in the topology-based approach to bqo theory is referred to Section 3 of [Tho89]. A modern and well-motivated treatment of bqo theory can be found in [Peq17].

Proof. See Lemma 14 of [Nas65a]. □

Lemma 3.9 (Special case of the Nash-Williams Partition Theorem)**.** *If a barrier B is finitely colored, there exists an infinite subset $X \subseteq \overline{B}$ such that $B \cap [X]^{<\omega}$ is monochromatic. Hence B contains a monochromatic sub-barrier $B' \subseteq B$.*

Remark. By *finitely coloring* a set B, we mean partitioning it into finitely many nonempty disjoint subsets $B_1, \ldots, B_n$ such that $B = B_1 \cup \cdots \cup B_n$. We then say that a subset $A \subseteq B$ is *monochromatic* if $A \subseteq B_i$ for some $i \in \{1, \ldots, n\}$.

Proof. See Lemma 4 of [Nas65b]. □

A Q-array $f\colon B \to Q$ is *perfect* if $f(s) \leq f(t)$ holds for every pair $s \triangleleft t$ in B. A *sub-array* of a Q-array $f\colon B \to Q$ is any Q-array that is the restriction of f to a subset of B. The following is the bqo analogue of Theorem 2.4 (iii).

Lemma 3.10. *If Q is bqo, then every Q-array contains a perfect sub-array.*

Proof. Let $f\colon B \to Q$ be a Q-array. For every pair $s \triangleleft t$ in B, let $u_{s,t}$ witness this. Notice that B being a barrier implies that if $u_{s,t} = u_{s',t'}$, then $s = s'$ and $t = t'$. Let B_2 be the set consisting of the sequences $u_{s,t}$ for all pairs $s \triangleleft t$ from B, and note that B_2 is a barrier by Lemma 15 of [Nas65a]. Color $u_{s,t} \in B_2$ red if $f(s) \leq f(t)$, and blue otherwise. By Lemma 3.9, there is an infinite subset $X \subseteq \overline{B_2} \subseteq \overline{B}$ such that $B_2' := B_2 \cap [X]^{<\omega}$ is monochromatic. By Lemma 3.8, $B' := B \cap [X]^{<\omega}$ is a barrier. Observe that whenever $s, t \in B'$ and $s \triangleleft t$, then $u_{s,t} \in B_2'$. Thus if B_2' were blue, then the Q-array $f \restriction B'$ would be bad, but Q is bqo. Hence B_2' is red, and the Q-array $f \restriction B'$ is perfect. □

3.4 Constructions Preserving BQOs

All wqo-preserving constructions from the previous chapter also preserve bqos. Moreover, while infinite constructions such as subsets and sequences might break wqos, as seen in Rado's counterexample, bqos survive these operations.

Recall that $P \hookrightarrow Q$ means that there is an order-reflecting map of P into Q.

Observation 3.11. *If Q is bqo and $P \hookrightarrow Q$, then P is also bqo.*

Proof. Essentially the same as the proof of Observation 2.10. □

In particular, if $(Q, \leq)$ is bqo and $\leq^+$ extends $\leq$, then $(Q, \leq^+)$ is bqo.

Proposition 3.12. *Let Q be a quasi-ordered class.*

(a) *If Q is finite, then $(Q, =)$ is bqo.*

(b) *If Q is well-ordered, then Q is bqo.*

Hence the property of being bqo sits between the notions of being well-ordered and being wqo.

Proof. (a) Let $f\colon B \to Q$ be a Q-array. Since Q is finite, f defines a finite coloring of B, and by Lemma 3.9, there exists a monochromatic sub-barrier $B' \subseteq B$. By Lemma 3.7, there are $s, t \in B'$ such that $s \triangleleft t$. Notice that $f(s) = f(t)$ holds because B' is monochromatic, hence f is good.

(b) Let $f\colon B \to Q$ be a Q-array. Repeated application of Lemma 3.7 yields an infinite sequence $s_1 \triangleleft s_2 \triangleleft \cdots$ of elements of B. Since Q is well-ordered, it cannot be the case that $f(s_1) > f(s_2) > \cdots$, and thus f is good. □

We collect the most important constructions that preserve the property of being bqo into a single theorem.

Theorem 3.13 (Nash-Williams)**.** *Let P and Q be quasi-ordered classes.*

(i) P, Q *bqo* $\implies$ $P \dot{\sqcup} Q$ *bqo.* *...disjoint union of bqos*

(ii) P, Q *bqo* $\implies$ $P \times Q$ *bqo.* *...Dickson's lemma for bqos*

(iii) Q *bqo* $\implies$ $Q^{<\omega}$ *bqo.* *...Higman's lemma for bqos*

(iv) Q *bqo* $\implies$ $\mathcal{P}(Q)$ *bqo.* *...under both* $\sqsubseteq_m$ *and* $\sqsubseteq_1$

(v) Q *bqo* $\implies$ Q^{On} *bqo.* *...Nash-Williams' transfinite sequence theorem*

Proof. We will prove only (i) and (ii). Clearly (iii) and (iv) both follow from (v). See [Nas65a] and [Nas68] for the original proofs of (iv) and (v), respectively, and see Theorem 3.16 of [Tho89] for a proof of (v) using the topological definition of bqos, which is much shorter than Nash-Williams' original proof.

(i) Let $f\colon B \to P \dot{\sqcup} Q$ be a $(P \dot{\sqcup} Q)$-array. Color $s \in B$ red if $f(s) \in P$ and blue if $f(s) \in Q$. Lemma 3.9 implies that B contains a monochromatic sub-barrier $B' \subseteq B$. Without loss of generality assume that B' is red. Hence $f \restriction B'$ is a P-array, and since P is bqo, there are $s \lhd t$ in B' such that $f(s) \leq f(t)$, proving that f is good.

(ii) Let $f\colon B \to P \times Q$ be a $(P \times Q)$-array. For $s \in B$, denote by (p_s, q_s) the element $f(s)$, and by $f_P\colon B \to P$ the P-array defined by $f_P(s) \coloneqq p_s$. Because P is bqo, f_P contains a perfect sub-array $f'_P = f_P \restriction B'$ for some $B' \subseteq B$ from Lemma 3.10. Let $f'_Q\colon B' \to Q$ be the Q-array defined by $f'_Q(s) \coloneqq q_s$. Because Q is bqo, there are $s \lhd t$ in B' such that $f'_Q(s) \leq_Q f'_Q(t)$. Because f'_P is perfect, we also have $f'_P(s) \leq_P f'_P(t)$, and thus $f(s) \leq f(t)$, proving that f is good. □

Remark. Traditionally, (iii) is obtained as a consequence of (v), the proof of which uses the very powerful and general techniques of bqo theory [Nas68; Tho89]. Pakhomov and Soldà [PS25] recently managed to prove (iii) while avoiding these techniques, relying only on an explicit characterization of the underlying orders.

A special case of Nash-Williams' transfinite sequence theorem is that better-quasi-ordered classes satisfy Higman's lemma for arbitrary countable sequences; that is, if Q is bqo, then $Q^{<\omega_1}$ is wqo. Pouzet [Pou72] proved that this property is, in fact, a sufficient condition for Q to be bqo.

Theorem 3.14 (Pouzet)**.** *Let Q be an arbitrary quasi order. Then*

$$Q \text{ is bqo} \iff Q^{<\omega_1} \text{ is wqo}.$$

Proof. See Theorem 2.8 of [Mar94]. □

4 Well-Quasi-Ordering Trees

In this chapter, we study well-quasi-orderings of trees in both the graph-theoretic and order-theoretic sense. We prove Kruskal's tree theorem and mention its strengthenings by Friedman, Nash–Williams, and Laver. We then turn our attention to tree-homomorphisms, prove Theorem 1, and consider a natural extension for trees with labeled leaves. Finally, we survey results on well-quasi-ordering trees of height greater than ω.

4.1 Order-Theoretic Trees

In graph theory, (possibly infinite) trees are acyclic connected graphs. We will study *order-theoretic trees*, a more general concept, as we will soon see that graph trees correspond to order-theoretic trees of height at most ω.

Definition 4.1 (Tree)**.** An *order-theoretic tree* is a well-founded partially ordered set $(T, <_T)$ such that for every $x \in T$, the set

$$(\leftarrow, x) \coloneqq \{y \in T \mid y \leq_T x\}$$

is a chain, and there is a unique minimal element called the *root* of T.

The elements $x \in T$ of a tree are called the *nodes* of T. In graph theory terms, $y <_T x$ means that y lies on the unique path $(\leftarrow, x)$ from x to the root, but note that this "path" might not be a connected graph in the usual sense. Observe that every chain in a tree is well-ordered since it is well-founded and linearly ordered. A *subtree* of a tree $(T, <_T)$ is a subset $S \subseteq T$ with the inherited order. We will often simply write T instead of $(T, <_T)$ and $<$ instead of $<_T$ if the order is clear from the context. Two trees T and S are *isomorphic* if they are order-isomorphic as partially ordered sets. Similar to graphs, we identify isomorphic trees.

The subtree *sprouting* from $x \in T$ is the subtree $T_x \coloneqq \{x' \in T \mid x \leq x'\}$. A node y is called a *predecessor* of x if $y < x$, a *successor* of x if $x < y$, and an *immediate successor* of x if it is a $\leq$-minimal successor of x. By nodes *below* and *above* x, we mean the set of predecessors and successors of x, respectively. A node x is called a *leaf* if it has no immediate successors. The *branching factor* of x is the cardinality of the set of its immediate successors. If every node has finite branching factor, then we say that the tree is *finitely branching*.

The *height* or *level* of x is the ordinal type of the well-ordered set $(\leftarrow, x)$, and we denote it by $|x|_T$. A tree is divided into *levels*, starting with level zero. For an ordinal number α, the *α-th level* of T is the set

$$T(\alpha) \coloneqq \{x \in T \mid |x|_T = \alpha\}.$$

The *height* of a tree T is $\mathrm{H}(T) \coloneqq \sup\{|x|_T + 1 \mid x \in T\}$, or equivalently, the least ordinal α for which $T(\alpha) = \varnothing$.

The *meet* $x \wedge_T y$ of x and y is the infimum of the set $\{x, y\}$ with respect to the tree order. If T is clear from the context, we will simply write $x \wedge y$. Note that meets in trees of height greater than ω might not exist.

A *branch* of a tree is a $\subseteq$-maximal chain. It is easy to show using Zorn's lemma that every chain can be extended into a branch. The *length* of a branch is its ordinal type with respect to $\leq_T$. It is easy to see that every branch has length at most $\mathrm{H}(T)$. A *cofinal branch* is a branch of length $\mathrm{H}(T)$.

Example. To build intuition, consider the following examples of trees:

- There is a 1-to-1 correspondence between rooted graph-trees and order-theoretic trees of height at most ω. However, trees of height more than ω have some "limit points" along their branches, meaning that the tree is not connected in the usual graph sense. This, however, does not mean that such trees are not useful in graph theory [Pit24].
- Every ordinal $(\alpha, <)$ is a tree of height α with a single branch.
- Given a set A and an ordinal α, the set $A^{<\alpha}$ of all sequences of elements from A of length less than α, ordered by inclusion, is called the *full A-ary tree* of height α. The immediate successors of a node $f\colon \beta \to A$ are the nodes $f_a\colon \beta + 1 \to A$ where $f_a \restriction \beta = f$ and $f_a(\beta) = a$ for each $a \in A$.
- In particular, $2^{<\omega}$ is the full binary tree of height ω.

Finding cofinal branches A central topic in the study of order-theoretic trees is determining when a tree must contain a cofinal branch. It is easy to see that every tree of finite height contains a cofinal branch, and the following well-known theorem of Kőnig provides a sufficient condition for trees of height ω.

Theorem 4.2 (Kőnig's lemma)**.** *Every finitely branching tree of height ω contains a cofinal branch.*

Note that the condition of being finitely branching is equivalent to having finite levels. For any infinite cardinal κ, it is an easy exercise to construct a tree of height κ with no cofinal branch and with levels of cardinality at most $\mathrm{cf}(\kappa)$.

Theorem 4.3. *If $\lambda < \mathrm{cf}(\kappa)$, then every tree of height κ with levels of size strictly less than λ contains a cofinal branch.*

A classical result of Aronszajn shows that if the levels are not *strictly* smaller, then the cofinal branch might not exist.

Theorem 4.4 (Aronszajn)**.** *There is a tree of height ω_1 with countable levels that contains no cofinal branch.*

An infinite cardinal κ is said to have the *tree property* if every tree of height κ with levels of size strictly less than κ contains a cofinal branch. Kőnig's lemma essentially says that ω has the tree property, and Aronszajn's theorem says that ω_1 does not. The tree property is very rare: ZFC cannot prove the existence of any cardinal other than ω that has it, and one can show that the property of being weakly compact, which we mentioned in Section 1.5, is equivalent to being strongly inaccessible and having the tree property.

See [BŠ01], [Jec03], or any other standard set theory book for more details.

4.2 Kruskal's Tree Theorem

Kruskal's theorem states that finite graph-trees are wqo by the topological minor relation. This was first conjectured by Vázsonyi in the 1940s and independently proved by Kruskal [Kru60] and Tarkowski [Tar60] in 1960. Nash-Williams [Nas63] later simplified the proof by introducing the minimal bad sequence method we used when proving Higman's lemma.

In order to prove Kruskal's theorem, we will show something more general, namely that if Q is wqo, then the class of all finite Q-labeled order-theoretic trees is wqo by the *homeomorphic embedding* relation. We will also consider a finite version of Kruskal's theorem due to Friedman [Sim85b], and briefly mention the proof-theoretic importance of these theorems.

Suppose now that Q is a quasi-ordered class.

Definition 4.5. A *Q-labeled tree* is a tree T endowed with a function $\ell_T\colon T \to Q$.

We denote the class of all Q-labeled trees by $\mathsf{T}(Q)$, the class of all trees with height at most ω by $\mathsf{T}_\omega(Q)$, the class of all trees without an infinite branch by $\mathsf{T}_{<\omega}(Q)$, and the class of all finite trees by $\mathsf{T}_\mathsf{f}(Q)$. Note that

$$\mathsf{T}_\mathsf{f}(Q) \subseteq \mathsf{T}_{<\omega}(Q) \subseteq \mathsf{T}_\omega(Q) \subseteq \mathsf{T}(Q).$$

We emphasize that $\mathsf{T}_{<\omega}(Q)$ does *not* denote the class of trees with finite height. The unlabeled equivalents of the classes defined above are denoted by

$$\mathsf{T}_\mathsf{f} \subseteq \mathsf{T}_{<\omega} \subseteq \mathsf{T}_\omega \subseteq \mathsf{T}.$$

Finally, we denote the set of all leaves of a tree T by $\mathcal{L}(T)$.

Definition 4.6 (Homeomorphic embedding)**.** We define the *homeomorphic embedding* order $\preceq$ on $\mathsf{T}(Q)$ as follows. For $S, T \in \mathsf{T}(Q)$, we write $S \preceq T$ if there exists a *homeomorphic embedding* of S into T; that is, an injection $\varphi\colon S \to T$ such that for all $x, y \in S$, we have

(i) $\ell_S(x) \leq \ell_T(\varphi(x))$, . . . φ *respects* labels

(ii) if $x \wedge y$ exists, then $\varphi(x) \wedge \varphi(y)$ exists, and $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$,

Since φ is injective and $x \leq y$ is equivalent to $x \wedge y = x$, condition (ii) entails that $x <_S y \iff \varphi(x) <_T \varphi(y)$. If φ further has the property that

$$x \in \mathcal{L}(S) \implies \varphi(x) \in \mathcal{L}(T),$$

we say that φ is *leaf-preserving.* If a leaf-preserving homeomorphic embedding of S into T exists, we write $S \preceq_l T$. The corresponding orders for unlabeled trees are obtained by omitting condition (i).

Theorem 4.7 (Kruskal's Tree Theorem)**.** *If Q is wqo, then $\mathsf{T}_\mathsf{f}(Q)$ is wqo by leaf-preserving homeomorphic embeddings $\preceq_l$, and thus also by $\preceq$.*

Remark. If we further require that whenever y is an immediate successor of x, then $\varphi(y)$ is an immediate successor of $\varphi(x)$, even the class of unlabeled finite trees fails to be wqo. Indeed, it is an easy exercise to find an infinite antichain.

Observe that if $Q = \{\bullet\}$ contains a single label, then $\preceq$ reduces to a natural extension of topological minors to rooted graph-trees, yielding the desired result:

Corollary 4.8. *The class of all finite trees is wqo by topological minors.*

Proof. Given a sequence of trees, choose a rooting for each of them, find the rooted topological minor, and forget the rooting. □

Note that Higman's lemma further implies that forests are wqo by the topological minor relation because forests can be seen as sets of trees.

Proof of Theorem 4.7. Assume for contradiction that $\mathsf{T_f}(Q)$ is not wqo and construct a minimal bad sequence $T_0, T_1, T_2, \ldots$ as follows:

Let T_0 be a tree of minimal size $|T_0|$ that appears as the first term in a bad sequence. For $i > 0$ let T_i be a tree of minimal size that appears as the i-th term (starting at $i = 0$) in a bad sequence that starts with $T_0, T_1, \ldots, T_{i-1}$. Note that we have $|T_i| \geq 2$ for all but finitely many i, as otherwise there would be an infinite subsequence of single-node trees, allowing us to define a bad sequence in Q. Let n be such that $|T_i| \geq 2$ for all $i \geq n$.

For $i \geq n$ denote by r_i the root of T_i and by $\mathcal{B}_i$ the set of subtrees of T_i that sprout from the immediate successors of r_i. We claim that the set $\mathcal{B} := \bigcup_{i<\omega} \mathcal{B}_i$ is wqo by $\preceq_l$. Suppose for contradiction that $B_0, B_1, \ldots$ is a bad sequence in $\mathcal{B}$ and for each j select $f(j)$ arbitrarily so that $B_j \in \mathcal{B}_{f(j)}$. Let k be such that $f(k)$ is minimal and consider the sequence

$$T_0, T_1, T_2, \ldots, T_{f(k)-1}, B_k, B_{k+1}, \ldots$$

Observe that this is a bad sequence, contradicting the minimality of $T_{f(k)}$ since $|B_k| < |T_{f(k)}|$. Indeed, if there were a good pair $T_i \preceq_l B_j$ for some $i < f(k)$ and $j \geq k$, then we would have a good pair $T_i \preceq_l T_{f(j)}$ in the minimal bad sequence because the way we chose k implies that $i < f(k) \leq f(j)$. Hence $\mathcal{B}$ is wqo.

Because Q is wqo, there is an infinite sequence of indices $i_1 < i_2 < \cdots$ such that $\ell_{T_{i_1}}(r_{i_1}) \leq \ell_{T_{i_2}}(r_{i_2}) \leq \cdots$. Since $\mathcal{B}$ is wqo, Higman's lemma implies that $\mathcal{B}^{<\omega}$ is wqo, and so there are $j < k$ such that $\mathcal{B}_{i_j} \sqsubseteq_1 \mathcal{B}_{i_k}$, which together with $\ell_{T_{i_j}}(r_{i_j}) \leq \ell_{T_{i_k}}(r_{i_k})$ gives $T_{i_j} \preceq_l T_{i_k}$, contradicting the badness of the minimal bad sequence. □

Remark. The same proof shows that finite Q-labeled *structured trees* (trees in which the immediate successors of every node are well-ordered) are wqo as well. Simply use $\mathcal{B}_{i_j} \sqsubseteq \mathcal{B}_{i_k}$ instead of $\mathcal{B}_{i_j} \sqsubseteq_1 \mathcal{B}_{i_k}$ at the very end.

A finite version of Kruskal's theorem Harvey Friedman introduced certain "finite miniaturizations" of Kruskal's theorem in an unpublished manuscript. We now consider one of them. A full account of Friedman's work and its implications is given by Simpson [Sim85b] and Smith [Smi85].

Theorem 4.9 (Friedman)**.** *If Q is finite, then for every $k \geq 1$, there is $n \geq 2$ such that, for any sequence $T_1, \ldots, T_n$ of trees in $\mathsf{T_f}(Q)$ satisfying $|T_m| \leq k(m+1)$, there are indices $i < j \leq n$ such that $T_i \preceq T_j$.*

Proof (cf. [Gal91]). The proof uses a compactness argument that should feel familiar to readers with some background in Ramsey theory. Suppose for contradiction that there is some $k \geq 1$ such that for all $n \geq 2$, there is some bad sequence $T_1, \ldots, T_n$. Notice that Q being finite, together with the size restriction $|T_m| \leq k(m+1)$, implies that there are only finitely many (bad) sequences of length n. Hence, the set of all finite bad sequences (for all $n \geq 2$) can be arranged into an infinite tree $\mathcal{T}$ as follows: the root of $\mathcal{T}$ is the empty sequence, and the nodes of $\mathcal{T}$ (other than the root) are finite bad sequences ordered by the relation of being an initial segment. From our previous remark it follows that $\mathcal{T}$ is finitely branching, and $\mathcal{T}$ is infinite because a bad sequence exists for every $n \geq 2$. By Kőnig's lemma, $\mathcal{T}$ contains a cofinal branch that corresponds to an infinite bad sequence $T_1, T_2, \ldots,$ which contradicts Kruskal's tree theorem. □

Proof-theoretic side note To rigorously measure the logical strength of mathematical theorems and formal systems, proof theorists employ two complementary frameworks: *reverse mathematics* and *ordinal analysis*. Reverse mathematics investigates the precise axiomatic assumptions necessary to prove specific mathematical theorems; rather than deducing theorems from axioms, it works backward from a known theorem to isolate the minimal subsystem of second-order arithmetic required for its proof. In contrast, ordinal analysis measures the overarching logical capacity of these formal systems by finding their *proof-theoretic ordinals*.

Both Kruskal's theorem and its finite versions are remarkable from a proof-theoretic point of view because they are not provable in relatively strong logical systems. Informally, the *proof-theoretic ordinal* of a logical system[1] T is (typically) a very large countable ordinal $||T||$ defined as the supremum of all ordinals α such that T can prove the well-foundedness of α (or equivalently, establish transfinite induction up to α). Going from transfinite induction up to α to transfinite induction up to $\alpha + 1$ requires only 1 logical step, so $||T||$ is always a limit ordinal, and T can no longer justify transfinite induction up to $||T||$ itself.

Gentzen [Gen36] famously proved that the proof-theoretic ordinal of Peano Arithmetic (PA) is the ordinal ε_0, which we previously mentioned in Section 1.2. Since then, various purely finitary combinatorial statements that are true, but unprovable in PA have been discovered [PH77; KP82]. This should be surprising since PA is equivalent[2] to the theory of finite sets, meaning that accepting the axiom of infinity changes what we can prove about finite sets. The reason why PA cannot prove these statements is precisely that their proofs inherently require transfinite induction up to (or beyond) ε_0. These results serve as concrete, natural manifestations of Gödel's first incompleteness theorem. Theorem 4.9 is an example of such a statement: it is clearly true, involves no infinities, but it is possible to show [Smi85] that it is unprovable in PA (and in fact not even in much stronger systems such as the system ATR_0 that we mention next). One might object that our proof did not explicitly invoke transfinite induction. However, we relied on something substantially stronger — Kruskal's tree theorem, which, due to its infinite nature, cannot even be formulated in the language of PA.

Regarding the logical strength of Kruskal's theorem (for unlabeled trees), Friedman showed that it cannot be proved within the system ATR_0. Informally, ATR_0 corresponds to *predicative mathematics*, which avoids the use of "circular" (impredicative) definitions that define an object S by quantifying over a collection T already containing S. For example, the definition of the minimal bad sequence in the proof of Kruskal's theorem presented above is impredicative. Friedman's result demonstrates that Kruskal's theorem is a natural example of a mathematical truth whose proof strictly requires impredicative reasoning. The proof-theoretic ordinal of ATR_0 is the famous *Feferman–Schütte ordinal* Γ_0, an ordinal much larger than ε_0, whose definition is, at least in my opinion, a very interesting journey [Smo26].

Notice that the statement of Theorem 4.9 can be generalized in the following form. If $f : \omega \times \omega \to \omega$ is a function, then let $S(f)$ be the statement: For every $k \geq 1$, there is $n \geq 2$ such that for any sequence $T_1, \dots, T_n$ of trees satisfying $|T_m| \leq f(m, k)$, there are indices $i < j \leq n$ such that $T_i \preceq T_j$. Observe that all of these statements are true, as the proof of Theorem 4.9 only uses the fact that the

[1] In this context, a logical system typically refers to a subsystem of second-order arithmetic.

[2] A theorem is provable in PA if and only if an "equivalent" theorem is provable in the theory of finite sets; that is, ZFC with the axiom of infinity replaced by the axiom "there are no limit ordinals." From this, one can prove that all sets are finite. See [Sla07] for more details, an intuitive explanation of why this should hold, and for more references.

size of the trees in the sequence is bounded. If we let $|Q| = 1$, then Theorem 4.9 reduces to $S(k + km)$. Friedman showed that $S(k + m)$ is still unprovable in $\mathsf{ATR_0}$. Loebl and Matoušek [LM87] considered functions of smaller growth rate and proved that $S(k + 4\log_2 m)$ is unprovable in PA. This is an almost optimal result because PA can prove $S(k + \frac{1}{2}\log_2 m)$.

These independence results also tell us something about how fast the smallest number n that satisfies $S(f)$ grows depending on k. A *fundamental sequence* of a countable limit ordinal α is an increasing sequence $\alpha[0] < \alpha[1] < \dots$ such that $\alpha = \sup\{\alpha[n] \mid n < \omega\}$. Once we choose assignments of fundamental sequences up to some large countable ordinal λ, we can define a *hierarchy of fast-growing functions* $F_\alpha\colon \omega \to \omega$ for $\alpha < \lambda$ as follows:

(i) $F_0(n) \coloneqq n + 1$,

(ii) $F_{\alpha+1}(n) \coloneqq F_\alpha(F_\alpha(\dots F_\alpha(n)\dots))$, where F_α is composed n times,

(iii) $F_\alpha(n) \coloneqq F_{\alpha[n]}(n)$ for limit ordinals α.

For example, $F_1(n) = 2n$, $F_2(n) \geq 2^n$, $F_3(n) \geq 2 \uparrow\uparrow n$, where $a \uparrow\uparrow n$ denotes the exponential tower with base a of height n, and $F_\omega(n) > A(n, n)$, where A denotes the Ackermann function. Note that the precise values of F_α for $\alpha \geq \omega$ depend on the chosen fundamental sequences, but there are some standard assignments; for example, up to ε_0 using Cantor normal forms, or up to Γ_0 using Veblen functions. We say that a function $f\colon \omega \to \omega$ *dominates* a function $g\colon \omega \to \omega$ if for all sufficiently large n we have $f(n) > g(n)$. There is a fundamental connection between the fast-growing hierarchy and proof-theoretic ordinals [FW98].

Theorem 4.10. *Let T be an arithmetic theory and $f\colon \omega \to \omega$ be a recursive*[3] *function. Then T can prove that the value $f(k)$ is defined for every input $k \iff f$ is eventually dominated by some F_α for $\alpha < ||T||$.*

Therefore, since $\mathsf{ATR_0}$ cannot prove $S(k + m)$, it means that the minimal number n_k required to satisfy $S(k + m)$ for k grows faster than all F_α for $\alpha < \Gamma_0$.

See [Sim09] or [Sti18] for a comprehensive treatment of reverse mathematics, and [Rat06] for an introduction to ordinal analysis. The connection between Kruskal's theorem and the ordinal Γ_0 is studied in detail in [Gal91]. For a more set-theoretic perspective on this ordinal, see Section 3.3 of [Smo26].

4.3 Friedman's Gap Condition

Denote by $\mathcal{T}_n \coloneqq \mathsf{T_f}(n)$ the class of all finite n-labeled trees (that is, the label set is $n = \{0, 1, \dots, n-1\}$), and consider the following strengthening of homeomorphic embeddings. For two trees $S, T \in \mathcal{T}_n$, let $S \preceq_g T$ if there exists a homeomorphic embedding φ of S into T such that for all $x, y \in S$ we have:

(i) $\ell_S(x) = \ell_T(\varphi(x))$, so φ not only respects but *preserves* labels,

(ii) if y is an immediate successor of x and $\varphi(x) <_T t <_T \varphi(y)$, then

$$\ell_T(t) \geq \ell_T(\varphi(y)) = \ell_S(y),$$

(iii) if r is the root of S and $t <_T \varphi(r)$, then $\ell_T(t) \geq \ell_T(\varphi(r)) = \ell_S(r)$.

[3] Informally, a function $f\colon \omega \to \omega$ is *recursive* if there is an algorithm that for every input $n \in \omega$ halts and outputs the value $f(n)$.

Parts (ii) and (iii) are known as the *gap condition*. While the gap condition may seem arbitrary, it is quite important due to its close connection to the graph minor theorem of Robertson and Seymour [FRS87]. The following statement is known as *extended Kruskal's theorem* and is due to Harvey Friedman [Sim85b].

Theorem 4.11 (Extended Kruskal's Theorem). *For every natural number n, the class $\mathcal{T}_n$ is wqo by $\preceq_g$.*

Remark. This no longer holds when we replace n with an infinite ordinal α, since the sequence $[0], [1], [2], \dots$ of single node trees whose roots are labeled by natural numbers is bad. However, if we change (i) to $\ell_S(x) \leq \ell_T(\varphi(x))$, then $\mathcal{T}_\alpha$ is wqo for every ordinal α, as conjectured by Friedman and proved by Kříž [Kří89].

There is also a more general version of Friedman's theorem that allows labeling leaves using a larger set of labels. For $n \in \omega$ and a quasi-order Q disjoint from n, denote by $\mathcal{T}_n(Q)$ the class of all finite $(n \cup Q)$-labeled trees such that the labels from Q may only occur on leaves. More formally, if $T \in \mathcal{T}_n(Q)$ and $\ell(x) \in Q$ for some $x \in T$, then x is a leaf of T. Note that the root of a single-node tree is considered to be a leaf under our definition. In particular, $\mathcal{T}_n(\varnothing) = \mathcal{T}_n$. We naturally extend the relation $\preceq_g$ to $\mathcal{T}_n(Q)$ by requiring (i) and (ii) to hold whenever $\ell_S(x), \ell_S(y) \in n$, and (iii) to hold if $\ell_S(r) \in n$. In addition, we enforce an extra condition that

(iv) if $\ell_S(x) \in Q$, then $\ell_T(\varphi(x)) \in Q$ and $\ell_S(x) \leq_Q \ell_T(\varphi(x))$.

Theorem 4.12 (Friedman). *If Q is wqo, then $\mathcal{T}_n(Q)$ is wqo by $\preceq_g$ for every n.*

We emphasize the tremendous proof-theoretic strength of this theorem. Friedman showed that it is unprovable in the system $\Pi^1_1\text{-}\mathsf{CA}_0$, which is the strongest of the "big five" axiomatic systems that reverse mathematics studies and is (much) stronger than ATR_0. See [Fre22] for a modern treatment of Friedman's theorem.

4.4 Nash-Williams' Theorem for Infinite Trees

After proving the tree theorem, Kruskal conjectured that it should hold for infinite trees as well. Nash-Williams [Nas65a] settled this conjecture a few years later by inventing better-quasi-orderings and showing that infinite trees are in fact bqo.

Theorem 4.13 (Nash-Williams). *The class T_ω of all order-theoretic trees of height at most ω is bqo by the homeomorphic embedding relation $\preceq$.*

Proof. See [Küh01] for a modern treatment of the proof. □

Corollary 4.14. *The class of all (finite or infinite) trees is wqo by the topological minor relation.*

Laver later proved a natural generalization of Theorem 4.13 for labeled trees, which he then famously used to resolve Fraïssé's conjecture: that countable linear orders are wqo by the order embeddability relation (an order L is *embeddable* into an order M if there exists an order-preserving injection from L into M). We discuss this in more detail in Section 4.6.

Theorem 4.15 (Laver [Lav71]). *If Q is bqo, then $\mathsf{T}_\omega(Q)$ is bqo by $\preceq$.*

4.5 WQO of Out-Trees by Homomorphisms

In this section, we prove Theorem 1. An *oriented graph* $\overrightarrow{G}$ is obtained from a graph G by replacing each edge $\{u, v\} \in E_G$ with one of the ordered pairs (u, v) or (v, u). Given two oriented graphs H and G, we say that there is a *homomorphism* of H into G if there exists a (not necessarily injective) mapping $f\colon V_H \to V_G$ satisfying

$$(u, v) \in E_H \implies (f(u), f(v)) \in E_G.$$

An *out-tree* is an oriented (possibly infinite) tree obtained by rooting a tree and orienting each edge away from the root. The main result of this section is the following:

Theorem 4.16. *The class of all out-trees is wqo by the homomorphism relation.*

Remark. This extends to *out-forests* (disjoint unions of out-trees) as well. Indeed, an out-forest can be encoded as an out-tree by adding a new vertex and connecting it to the roots of all components of the forest. This newly added vertex becomes the root of the tree. It is easy to see that a homomorphism between these coding trees induces a homomorphism of the original forests.

Let us reformulate Theorem 4.16 in terms of order-theoretic trees.

Definition 4.17. Let $S, T \in \mathsf{T}_\omega$. An *embedding* of S into T is a map $\varphi\colon S \to T$ that satisfies $x <_S y \implies \varphi(x) <_T \varphi(y)$. An embedding φ is a

(1) *level-embedding* if $|x|_S = |\varphi(x)|_T$ holds for all $x \in S$. In particular, if y is an immediate successor of x, then $\varphi(y)$ is an immediate successor of $\varphi(x)$.

(2) *leaf-embedding* if whenever x is a leaf of S, then $\varphi(x)$ is a leaf of T.

Definition 4.18 (Tree-homomorphism)**.** The *tree-homomorphism* order $\leq_h$ on T_ω can be characterized in either of the two following equivalent ways:

(a) $S \leq_h T \iff$ there exists an embedding of S into T, or

(b) $S \leq_h T \iff$ there exists a level-embedding of S into T.

When restricted to $\mathsf{T}_{<\omega}$, we obtain a third equivalent characterization:

(c) $S \leq_h T \iff$ there exists a leaf-embedding of S into T.

Observation 4.19. *The above characterizations are indeed equivalent.*

Proof. Clearly (b) and (c) both imply (a).

(a)⇒(b) Given an embedding $\varphi\colon S \to T$, define a level-embedding φ' by mapping each $x \in S$ to the unique element of $\{y \in T \mid y \leq_T \varphi(x)\}$ at height $|x|_S$. In graph theory terms, if every edge is mapped to a path instead of an edge, we can contract these paths to obtain edges.

(a)⇒(c) When restricted to $\mathsf{T}_{<\omega}$, each branch contains a leaf. Hence, if $\varphi\colon S \to T$ is an embedding, then for all $x \in S$ there is some $y \in \mathcal{L}(T)$ such that $\varphi(x) \leq_T y$. Invoke the axiom of choice and select such y_x for every leaf x. To obtain a leaf-embedding, remap every leaf $x \in \mathcal{L}(S)$ to y_x. □

It is easy to see that the relation $\leq_h$ is reflexive and transitive, and in view of characterization (b), it suffices to establish the following equivalent result in order to prove Theorem 4.16.

Theorem 4.20. T_ω *is wqo by the tree-homomorphism relation* $\leq_h$.

Since every homeomorphic embedding is an embedding, Theorem 4.20 can be seen as an easy consequence of Theorem 4.13, whose proof relies on the heavy machinery of bqo theory. We will prove Theorem 4.20 while avoiding bqo theory entirely.

Proof of the Theorem

The outline of the proof is as follows. We first observe that the theorem trivially holds for sequences of trees containing a cofinal branch, leaving sequences of trees of height ω with no cofinal branch as the only interesting case (every tree of finite height contains a cofinal branch). We then introduce a hierarchy for these trees and prove the wqo property by transfinite induction on this hierarchy.

Lemma 4.21. *Let* $S, T \in \mathsf{T}_\omega$.

(i) *If* $\mathrm{H}(S) < \omega$ *and* $\mathrm{H}(S) \leq \mathrm{H}(T)$, *then* $S \leq_h T$.

(ii) *If* $\mathrm{H}(S) \leq \omega$, *and* T *contains an infinite branch, then* $S \leq_h T$.

Proof. Both of the claims follow from the fact that T contains a branch B of height at least $\mathrm{H}(S)$. Define a level-embedding φ of S into T by mapping every $x \in S$ to the unique element of B at height $|x|_S$. □

Once we establish the following lemma, we are finished.

Lemma 4.22. $\mathsf{T}_{<\omega}$ *is wqo by* $\leq_h$.

Proof of Theorem 4.20 using Lemma 4.22. Let $f\colon \omega \to \mathsf{T}_\omega$ be a sequence of trees of height at most ω. If there are indices $i < j$ such that T_i and T_j contain an infinite branch, then $T_i \leq_h T_j$ by Lemma 4.21. Otherwise there is an infinite subset $A \subseteq \omega$ such that T_i contains no infinite branch for all $i \in A$. Lemma 4.22 implies that there are $i < j$ in A such that $T_i \leq_h T_j$, hence f is good. □

From now on, until the end of this section, by "tree" we shall mean a tree without an infinite branch. Denote by $\mathrm{r}(T)$ the root of T, and by $\mathcal{B}(T)$ the set of all *branch-trees* of T: subtrees of T that sprout from the immediate successors of $\mathrm{r}(T)$. We say that a tree T is a *bush* if $\mathrm{H}(T) = \omega$ and $\mathrm{H}(B) < \omega$ for all $B \in \mathcal{B}(T)$. A canonical example of a bush is a tree we will denote by W_1, whose branch-trees are isomorphic to the natural numbers ($n \in \omega$ with the usual order is a tree of height n). More precisely, there is a bijection $f\colon \mathcal{B}(W_1) \to \omega$ such that every $B \in \mathcal{B}(W_1)$ is isomorphic to $f(B)$. In this case, $\mathrm{r}(W_1)$ has branching factor ω and all other nodes of W_1 are either leaves or have branching factor 1, but the branching factor of nodes in a bush can be any cardinal number, as long as all branch-trees have finite height.

Lemma 4.23. *If* S *and* T *are bushes, then* $S \leq_h T$.

Proof. Observe that for every $B \in \mathcal{B}(S)$, there is $B' \in \mathcal{B}(T)$ such that $\mathrm{H}(B) \leq \mathrm{H}(B')$, and by Lemma 4.21 there is an embedding φ_B of B into B'. We construct an embedding of S into T by mapping $\mathrm{r}(S)$ to $\mathrm{r}(T)$ and mapping the nodes of $B \in \mathcal{B}(S)$ into T according to φ_B. □

Given trees S and T, by *replacing a leaf $x \in \mathcal{L}(T)$ with S*, we mean creating a tree $T' := (T \setminus \{x\}) \cup S$ the order of which is induced by $\leq_T$ and $\leq_S$, with the addition that for every $y \in T$ that was below x, we have $y \leq_{T'} z$ for all $z \in S$. Here we without loss of generality assume that S and $T \setminus \{x\}$ are disjoint; if not, we can fix this via a set theory trick we mentioned in Section 2.2.

Definition 4.24 (Bushy ordinal)**.** We say that an ordinal number α is *bushy* if either $\alpha = 1$ or α is a limit ordinal.

Definition 4.25. For a tree T, we define its *type* $\mathrm{tp}(T) \in \mathrm{On}$ inductively.

(i) $\mathrm{tp}(T) = 0$ if T has finite height (but $|T|$ might be infinite).

Now let α be a bushy ordinal.

(ii) $\mathrm{tp}(T) = \alpha$ if the type of T has not yet been defined and $\mathrm{tp}(B) < \alpha$ for all $B \in \mathcal{B}(T)$. We say that T is an *α-bush.*

(iii) $\mathrm{tp}(T) = \alpha + 1$ if the type of T has not yet been defined, and T can be obtained from a tree of type at most α by replacing some of its leaves with (possibly different) trees of type at most α.

(iv) $\mathrm{tp}(T) = \alpha + n + 1$ for $1 \leq n < \omega$ if the type of T has not yet been defined, and T can be obtained from a tree of type $\alpha + n$ by replacing some of its leaves with (possibly different) trees of type at most α.

Remark. Every successor ordinal β can be written as $\beta = \alpha + n$ for some limit ordinal α and $n \in \omega$. Suppose not; let $\beta_0 = \beta$, and for $i > 0$ let β_i be the ordinal satisfying $\beta_{i-1} = \beta_i + 1$. Then $\langle \beta_i \mid i \in \omega \rangle$ is an infinite strictly decreasing sequence of ordinals, which is impossible.

Observation 4.26. *T is a bush $\iff$ T is a 1-bush.*

The ordinal $\mathrm{tp}(T)$ represents the "minimum number of steps" needed to construct T from a finite-height tree by adding bushes and α-bushes, where we require $\mathrm{tp}(T) \geq \alpha$ in order to use α-bushes.

We first note that a tree of type α exists for each ordinal α. Let W_0 be any finite tree, and for $1 \leq n < \omega$, denote by W_{n+1} the tree we obtain from W_n by replacing all of its leaves with copies of W_1 (which has been defined previously). Clearly, $\mathrm{tp}(W_n) = n$ for all $n \in \omega$. Denote by W_ω a tree whose branch-trees are isomorphic to W_n for $n \in \omega$. In general, suppose that α is a limit ordinal and W_β have already been defined for all $\beta < \alpha$. Then let W_α be a tree whose branch-trees are isomorphic to W_β for $\beta < \alpha$, and let $W_{\alpha+n+1}$ for $n \in \omega$ be the tree we obtain from $W_{\alpha+n}$ by replacing all of its leaves with copies of W_α. From the following lemma, it is easy to see that $\mathrm{tp}(W_\gamma) = \gamma$ for all ordinals γ.

Definition 4.27 (Induced subtree)**.** A subtree $S \subseteq T$ is an *induced subtree* of T if, whenever $x \in T$ satisfies $\mathrm{r}(S) <_T x <_T y$ for some $y \in S$, then $x \in S$.

Lemma 4.28. *If S is an induced subtree of T, then* $\mathrm{tp}(S) \leq \mathrm{tp}(T)$*.*

Proof. We proceed by transfinite induction on $\mathrm{tp}(T)$. If $\mathrm{tp}(T) = 0$, then $\mathrm{H}(S) < \omega$, so $\mathrm{tp}(S) = 0$. Suppose that $\mathrm{tp}(T) = 1$. If $\mathrm{H}(S) < \omega$, then $\mathrm{tp}(S) = 0$, and if $\mathrm{H}(S) = \omega$, then necessarily $\mathrm{r}(S) = \mathrm{r}(T)$. Notice that S is a bush, so $\mathrm{tp}(S) = 1$.

Suppose that $\mathrm{tp}(T) = \alpha$ is a limit ordinal and assume that we have already proved the claim for all $\beta < \alpha$. If $\mathrm{r}(S) \in B$ for some $B \in \mathcal{B}(T)$, then from the induction hypothesis $\mathrm{tp}(S) \leq \mathrm{tp}(B) < \mathrm{tp}(T)$. Suppose that $\mathrm{r}(S) = \mathrm{r}(T)$ and assume $\mathrm{tp}(S) \neq \beta$ for any $\beta < \alpha$. Observe that every branch-tree B of S is an induced subtree of some branch-tree B' of T, and by the induction hypothesis $\mathrm{tp}(B) \leq \mathrm{tp}(B') < \mathrm{tp}(T) = \alpha$. By definition, S is an α-bush, so $\mathrm{tp}(S) = \alpha$.

Finally, assume that $\mathrm{tp}(T) = \alpha + n + 1$, where $n \in \omega$ and α is a bushy ordinal. This means that T can be obtained from a tree T' of type at most $\alpha + n$ by replacing some of its leaves with trees of type at most α. Denote these leaves by $L \subseteq T'$, and without loss of generality, assume that if $x \in L$ was replaced by a tree $R^{(x)}$, then $\mathrm{r}(R^{(x)}) = x$. Hence $R^{(x)} = T_x$ (the subtree of T sprouting from x). Let $S' = T' \cap S$. If $S' = \varnothing$, then S is an induced subtree of some T_x for $x \in L$. Since $\mathrm{tp}(T_x) \leq \alpha < \mathrm{tp}(T)$, we can apply the induction hypothesis to obtain $\mathrm{tp}(S) \leq \mathrm{tp}(T_x) \leq \alpha$. If $S' \neq \varnothing$, then S' is an induced subtree of T', so by the induction hypothesis, $\mathrm{tp}(S') \leq \mathrm{tp}(T') \leq \alpha + n$. If $S' = S$, we are done. Suppose not and observe that if x is a leaf of S', but not a leaf in S, then $x \in L$ and S_x is an induced subtree of T_x. By the induction hypothesis we have $\mathrm{tp}(S_x) \leq \mathrm{tp}(T_x) \leq \alpha$. Notice that we can obtain S from S' by replacing each leaf x of S' that is not a leaf in S with S_x, so $\mathrm{tp}(S) \leq \mathrm{tp}(S') + 1 \leq \alpha + n + 1$. □

Lemma 4.29. *If S is a subtree of T and $T \setminus S \subseteq \mathcal{L}(T)$, then* $\mathrm{tp}(S) = \mathrm{tp}(T)$.

Proof. By transfinite induction on $\mathrm{tp}(S)$. Observe that the claim holds when $\mathrm{tp}(S) \leq 1$. Suppose that $\mathrm{tp}(S) = \alpha$ is a limit ordinal, and assume we have already proved the claim for all $\beta < \alpha$. From Lemma 4.28 we know that $\mathrm{tp}(T) \geq \alpha$. For $B \in \mathcal{B}(T)$, denote by $B_S \in \mathcal{B}(S)$ the tree $B \cap S$. Notice that $B \setminus B_S \subseteq \mathcal{L}(B)$. Since $\mathrm{tp}(B_S) < \alpha$, by the induction hypothesis, we have $\mathrm{tp}(B) = \mathrm{tp}(B_S) < \alpha$. By definition, T is an α-bush, and the claim holds.

Finally, suppose that $\mathrm{tp}(S) = \alpha + n + 1$, where $n \in \omega$ and α is a bushy ordinal. Hence S can be obtained from a tree S' of type at most $\alpha + n$ by replacing some of its leaves $L \subseteq \mathcal{L}(S')$ with trees of type at most α; without loss of generality, assume that $x \in L$ is replaced by S_x. Define

$$T' \coloneqq T \setminus \bigcup \{T_x \setminus \{x\} \mid x \in L\}$$

and observe that $T' \setminus S' \subseteq \mathcal{L}(T')$, so $\mathrm{tp}(T') = \mathrm{tp}(S') \leq \alpha + n$. Furthermore, $T_x \setminus S_x \subseteq \mathcal{L}(T_x)$ for all $x \in L$, thus $\mathrm{tp}(T_x) = \mathrm{tp}(S_x) \leq \alpha$. Clearly, we can obtain T from T' by replacing each leaf $x \in L$ with T_x, so $\mathrm{tp}(T) \leq \alpha + n + 1$. Lemma 4.28 gives us $\mathrm{tp}(T) \geq \alpha + n + 1$. Therefore $\mathrm{tp}(T) = \mathrm{tp}(S)$. □

From repeated application of Lemma 4.29, it follows that it is impossible to decrease the type of T by iteratively deleting its leaves any finite number of times.

Lemma 4.30. *Let T be a tree and α a limit ordinal.*

(i) $\mathrm{tp}(T) \leq \sup\{\mathrm{tp}(B) \mid B \in \mathcal{B}(T)\} + 1$.

(ii) $\mathrm{tp}(T) = \alpha \iff \sup\{\mathrm{tp}(B) \mid B \in \mathcal{B}(T)\} = \alpha$.

Proof. By transfinite induction on $\beta \coloneqq \sup\{\mathrm{tp}(B) \mid B \in \mathcal{B}(T)\}$. It is easy to see that if $\beta = 0$, then $\mathrm{tp}(T) \leq 1$, and if $\beta = 1$, then $\mathrm{tp}(T) = 2$. Suppose now that β is a successor ordinal of the form $\beta = \gamma + n + 1$, where γ is bushy and $n \in \omega$. Let $\mathcal{B}_\beta$ be the set of branch-trees of T of type β and notice that $\mathcal{B}_\beta \neq \varnothing$. Each $B \in \mathcal{B}_\beta$ can be obtained from a tree B' of type at most $\gamma + n$ by replacing some of its leaves with trees of type at most γ. Denote by T' the tree we obtain from T by replacing each $B \in \mathcal{B}_\beta$ with B'. Observe that

$$\sup\{\mathrm{tp}(B) \mid B \in \mathcal{B}(T')\} \leq \gamma + n < \beta,$$

so $\text{tp}(T') \le \gamma + n + 1$ by the induction hypothesis. Since T can be obtained from T' by replacing some of its leaves with trees of type at most γ, we conclude that $\text{tp}(T) \le \text{tp}(T') + 1 = \beta + 1$.

Lastly, suppose that β is a limit ordinal; we will show that $\text{tp}(T) = \beta$. Observe that $\text{tp}(T) \ne \gamma$ for any $\gamma < \beta$ because there always is some $B \in \mathcal{B}(T)$ such that $\text{tp}(B) > \gamma$, and by Lemma 4.28 we have $\text{tp}(T) \ge \text{tp}(B)$. By definition, $\text{tp}(T) = \beta$. This finishes the proof of (i), and the "$\Leftarrow$" direction of (ii).

To show "$\Rightarrow$," let T be an α-bush and assume for contradiction that $\beta \coloneqq \sup\{\text{tp}(B) \mid B \in \mathcal{B}(T)\} < \alpha$. Then, by (i), we have $\alpha = \text{tp}(T) \le \beta + 1$, a contradiction with α being a limit ordinal. □

Lemma 4.31. *Every tree T has its type defined.*

Proof. Assume for contradiction that $\text{tp}(T)$ is not defined, and call a node $x \in T$ *bad* if $\text{tp}(T_x)$ is not defined. Since $T = T_{\text{r}(T)}$, the root of T is bad. Observe that Lemma 4.30 (i) implies that whenever a node $x \in T$ is bad, it has an immediate successor x' that is also bad; otherwise, we could bound $\text{tp}(T_x)$ from above via the types of the subtrees sprouting from the immediate successors of x. This allows us to define an infinite branch by $x_0 \coloneqq \text{r}(T)$ and $x_{i+1} \coloneqq x_i'$ for $i \ge 0$, contradicting the fact that T contains no infinite branches. □

Definition 4.32 (Collapse)**.** Let α be a bushy ordinal. The α-*collapse* of a tree T is the subtree of T defined as

$$\text{col}_\alpha(T) \coloneqq \{x \in T \mid \text{tp}(T_x) \ge \alpha\}.$$

Notice that Lemma 4.28 implies that $\text{col}_\alpha(T)$ is an induced subtree of T with the property that whenever $x \in \text{col}_\alpha(T)$, then $y \in \text{col}_\alpha(T)$ for all $y <_T x$.

Observation 4.33. *If x is a leaf of $\text{col}_\alpha(T)$, then T_x is an α-bush.*

Lemma 4.34. *Let $\text{tp}(T) = \alpha + n + 1$ where α is a bushy ordinal and $n \in \omega$.*

(i) $\text{tp}(\text{col}_\alpha(T)) \le \text{tp}(T) - 1$,

(ii) $n \ge 1 \implies \text{tp}(\text{col}_\alpha(T)) = \text{tp}(T) - 1$.

Proof. (i) We know that T can be obtained from a tree $T' \subseteq T$ of type at most $\alpha + n$ by replacing some of the leaves of T' with trees of type at most α. Note that Lemma 4.28 implies that $\text{col}_\alpha(T) \subseteq T'$, and thus also $\text{tp}(\text{col}_\alpha(T)) \le \alpha + n$.

(ii) Denote $S \coloneqq \text{col}_\alpha(T)$ and suppose for contradiction that $\text{tp}(S) \le \alpha + n - 1$. Let $S' \subseteq T$ be the tree obtained from S by adding to it the "roots of short deleted sprouts." More formally, for each $x \in S \setminus \mathcal{L}(S)$, we add all $y \in T \setminus S$ that are immediate successors of x in T. Note that every $y \in S' \setminus S$ satisfies $\text{tp}(T_y) < \alpha$. By Lemma 4.29, we have $\text{tp}(S') = \text{tp}(S) \le \alpha + n - 1$. By our previous remark and Observation 4.33 we have that whenever $x \in \mathcal{L}(S')$, then $\text{tp}(T_x) \le \alpha$. Notice that if we now replace each leaf x of S' by T_x, then we obtain T. Therefore $\text{tp}(T) \le \text{tp}(S') + 1 \le \alpha + n$, a contradiction. □

Lemma 4.35. *Let S and T be arbitrary trees and α be a bushy ordinal.*

(i) $\text{tp}(S) < \text{tp}(T) \implies S \le_h T$.

(ii) *If S and T are both α-bushes, then $S \le_h T$.*

(iii) *If $\text{col}_\alpha(S) \ne \varnothing$ and $\text{col}_\alpha(S) \le_h \text{col}_\alpha(T)$, then $S \le_h T$.*

Proof. We prove (i) and (ii) simultaneously by induction on $\mathrm{tp}(T)$. Note that we already have (i) for $\mathrm{tp}(T) = 1$ from Lemma 4.21, and (ii) for $\alpha = 1$ from Lemma 4.23. Suppose that α is a limit ordinal and that we already have (i) for all $\mathrm{tp}(T) < \alpha$. We claim that if $\mathrm{tp}(T) = \alpha$ and $\mathrm{tp}(S) \leq \alpha$, then $S \leq_h T$, showing both (i) and (ii) for $\mathrm{tp}(T) = \alpha$. Lemma 4.28 implies that $\mathrm{tp}(B) < \alpha$ for all $B \in \mathcal{B}(S)$, and by Lemma 4.30 (ii) there is $B' \in \mathcal{B}(T)$ such that $\mathrm{tp}(B) < \mathrm{tp}(B') < \alpha$. By the induction hypothesis, there exist embeddings φ_B of B into B' for each $B \in \mathcal{B}(S)$. We define an embedding φ of S into T by mapping $\mathrm{r}(S)$ to $\mathrm{r}(T)$ and copying the embeddings φ_B for $x \in B$.

Next, suppose that $\mathrm{tp}(T) = \alpha + n + 1$ and $\mathrm{tp}(S) \leq \alpha + n$, where α is a bushy ordinal and $n \in \omega$. We claim that there is $x \in T$ such that $\mathrm{tp}(T_x) = \alpha$. Suppose not: that for all $x \in T$ either $\mathrm{tp}(T_x) > \alpha$ or $\mathrm{tp}(T_x) < \alpha$. It is then easy to see that there is $z \in T$ such that $\mathrm{tp}(T_z) > \alpha$ and $\mathrm{tp}(B) < \alpha$ for all $B \in \mathcal{B}(T)$, since otherwise we could construct an infinite branch. But note that T_z satisfies the definition of an α-bush, so $\mathrm{tp}(T_z) = \alpha$, a contradiction.

Hence let $x \in T$ be such that $\mathrm{tp}(T_x) = \alpha$. If $\mathrm{tp}(S) < \alpha$, then by the induction hypothesis for (i) we have $S \leq_h T_x$, and if $\mathrm{tp}(S) = \alpha$, then the same holds by the induction hypothesis for (ii). Either way we conclude that $S \leq_h T$.

Therefore, assume that $\mathrm{tp}(S) > \alpha$, so necessarily $n \geq 1$. Define $C_S := \mathrm{col}_\alpha(S)$ and $C_T := \mathrm{col}_\alpha(T)$. By Lemma 4.34 we have $\mathrm{tp}(C_S) < \mathrm{tp}(C_T) = \alpha + n$. We can thus invoke the induction hypothesis to obtain a leaf-embedding φ of C_S into C_T. We will extend it to an embedding φ' of S into T. Let $x \in \mathcal{L}(C_S)$. By Observation 4.33, the subtree sprouting from x in S and the subtree sprouting from $\varphi(x) \in \mathcal{L}(T)$ in T are both α-bushes. By (ii), there is an embedding φ_x of S_x into $T_{\varphi(x)}$, and we use this embedding to extend φ to the nodes above x in S. Finally, let $x \in C_S \setminus \mathcal{L}(C_S)$ and let $y \in \mathcal{L}(C_S)$ be a leaf above x. If $z \in S \setminus C_S$ is an immediate successor of x, then $\mathrm{tp}(S_z) < \alpha$, and thus there is an embedding φ_z of S_z into $T_{\varphi(y)}$, and we let $\varphi' \restriction S_z = \varphi_z$. It is easy to verify that φ' is indeed an embedding of S into T, finishing the proof of (i) and (ii).

The construction described above demonstrates that (iii) holds as well. □

Lemma 4.36. *For every ordinal α, the class $\mathcal{T}(\alpha) := \{T \in \mathsf{T}_{<\omega} \mid \mathrm{tp}(T) \leq \alpha\}$ is wqo by the tree-homomorphism relation $\leq_h$.*

Proof. By transfinite induction on α. Let $T_0, T_1, T_2, \dots$ be a sequence of trees of type at most α; we want to find $i < j$ such that $T_i \leq_h T_j$. If $\alpha = 0$, then consider the sequence $\mathrm{H}(T_0), \mathrm{H}(T_1), \mathrm{H}(T_2), \dots$ Since ω is wqo, there are indices $i < j$ such that $\mathrm{H}(T_i) \leq \mathrm{H}(T_j)$ and thus $T_i \leq_h T_j$ by Lemma 4.21 (i). Suppose $\alpha > 0$ and assume that we have already proved the claim for all $\beta < \alpha$. If there are indices $i < j$ such that $\mathrm{tp}(T_i) < \mathrm{tp}(T_j)$, then $T_i \leq_h T_j$ by Lemma 4.35 (i). Hence assume that $\mathrm{tp}(T_0) \geq \mathrm{tp}(T_1) \geq \cdots$. Now, if there is an infinite subset of indices $A \subseteq \omega$ such that $\mathrm{tp}(T_i) < \alpha$ for all $i \in A$, then

$$\beta := \sup\{\mathrm{tp}(T_i) \mid i \in A\} = \mathrm{tp}(T_{\min A}) < \alpha.$$

Therefore $T_i \in \mathcal{T}(\beta)$ for all $i \in A$, and by the induction hypothesis for β we find indices $i < j$ in A such that $T_i \leq_h T_j$.

We can thus without loss of generality assume that $\mathrm{tp}(T_i) = \alpha$ for all $i \in \omega$. If α is a bushy ordinal, then $T_i \leq_h T_j$ for all $i < j$ from Lemma 4.35 (ii) and the claim trivially holds. Therefore, suppose that α is a successor ordinal of the form $\alpha = \gamma + n + 1$, where γ is bushy and $n \in \omega$. Consider the sequence of (nonempty) collapsed trees $T'_0, T'_1, T'_2, \dots$ where $T'_i = \mathrm{col}_\gamma(T_i)$. Notice that for all i we have

$T'_i \in \mathcal{T}(\gamma+n)$ by Lemma 4.34, and so the induction hypothesis yields indices $i < j$ such that $T'_i \leq_h T'_j$. By Lemma 4.35 (iii) we have that also $T_i \leq_h T_j$. □

We can now finally prove that $\mathsf{T}_{<\omega}$ is wqo by $\leq_h$, finishing the proof.

Proof of Lemma 4.22. Let $T_0, T_1, T_2, \ldots$ be a sequence of trees without infinite branches. Lemma 4.31 implies that there is a sequence of ordinals $\alpha_0, \alpha_1, \alpha_2, \ldots$ such that α_i is the type of T_i. If we let $\alpha := \sup_i \alpha_i$, then clearly $T_i \in \mathcal{T}(\alpha)$ for all $i \in \omega$, and by Lemma 4.36 there are indices $i < j$ such that $T_i \leq_h T_j$. □

4.5.1 Extension to Leaf-Labeled Trees

In this section, we consider a natural extension of the tree-homomorphism relation to trees that have their leaves labeled from a quasi-ordered class Q.

Definition 4.37. A *leaf-labeled tree* over Q is a tree T endowed with a function $\ell_T \colon \mathcal{L}(T) \to Q$.

Denote the class of all leaf-labeled trees over Q that do not contain an infinite branch by $\mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$, and the subclass of finite trees by $\mathsf{T}^{\mathcal{L}}_{\mathsf{f}}(Q)$.

Definition 4.38. Let $S, T \in \mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$. A *leaf-embedding* of S into T is a map $\varphi \colon S \to T$ such that

(i) if $x <_S y$, then $\varphi(x) <_T \varphi(y)$, and

(ii) if $x \in \mathcal{L}(S)$, then $\varphi(x) \in \mathcal{L}(T)$ and $\ell_S(x) \leq \ell_T(\varphi(x))$.

A *level-embedding* of S into T is a map $\varphi \colon S \to T$ such that

(i) if $x <_S y$, then $\varphi(x) <_T \varphi(y)$,

(ii) $|x|_S = |\varphi(x)|_T$ holds for all $x \in S$, and

(iii) if $x \in \mathcal{L}(S)$, then there is $y \in \mathcal{L}(T)$ such that $\varphi(x) \leq_T y$ and $\ell_S(x) \leq \ell_T(y)$.

Definition 4.39. The *tree-homomorphism* order $\leq_h$ on $\mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$ can be characterized in either of the two following equivalent ways:

(a) $S \leq_h T \iff$ there exists a leaf-embedding of S into T,

(b) $S \leq_h T \iff$ there exists a level-embedding of S into T.

The reader can verify that these characterizations are indeed equivalent using similar arguments to those we used to justify Definition 4.18.

Observation 4.40. *If Q is wqo, then $\mathsf{T}^{\mathcal{L}}_{\mathsf{f}}(Q)$ is wqo.*

Proof. Trivial consequence of Kruskal's theorem for $\preceq_l$, because leaf-preserving homeomorphic embeddings are a special type of leaf-embeddings. □

Remark. Chopra and Pakhomov [CP26] recently calculated the maximal order type (see Definition 2.6) of $\mathsf{T}^{\mathcal{L}}_{\mathsf{f}}(Q)$ as a function of the maximal order type of Q. They then found a connection between $\mathsf{T}^{\mathcal{L}}_{\mathsf{f}}(Q)$ and the subclass of $Q^{<\omega^\omega}$ consisting of all transfinite sequences with finite range of length less than ω^ω, and used the connection to calculate its maximal order type as well.

Theorem 4.41. *If Q is bqo, then $\mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$ is bqo.*

Proof. This is a consequence of Laver's generalization of Nash-Williams' theorem for infinite trees. Let $f\colon B \to \mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$ be a $\mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$-pattern, and let $Q' \subseteq Q$ be the subclass of labels used for labeling the trees in this pattern. Notice that Q' is a set: if we let $\kappa := \sup_i |T_i|$, then $|Q'| \leq |B| \cdot \kappa$. Invoke the axiom of choice to obtain a choice function g on $\mathcal{P}(Q')$, and define a full labeling of the leaf-labeled trees T from f to construct a $\mathsf{T}_{<\omega}(Q)$-pattern f' of labeled trees as follows.

For a leaf $x \in \mathcal{L}(T)$, copy the label $\ell(x)$. If the set of immediate successors of x is $A_x \neq \varnothing$ and their labels are already defined, put

$$\ell(x) := g(\{\ell(y) \mid y \in A_x\}).$$

Hence for every $x \in T$ there is a leaf $y \in \mathcal{L}(T)$ such that $x \leq y$ and $\ell(x) = \ell(y)$. Note that this definition is valid: if there were some $x \in T$ with $\ell(x)$ undefined, then x would have an immediate successor x_1 whose label is also undefined. By repeating this, we could construct an infinite branch, a contradiction.

Denote the labeled tree we obtain from T by $\overline{T}$ and observe that if $\overline{S} \preceq \overline{T}$, then $S \leq_h T$. Indeed, given a homeomorphic embedding $\varphi\colon \overline{S} \to \overline{T}$, we can define a level-embedding $\varphi'\colon S \to T$ by "contracting the paths induced by φ in T to edges." Formally, for $x \in S$ we let $\varphi'(x)$ be the unique element of

$$\{y \in T \mid y \leq_T \varphi(x)\}$$

at height $|x|_S$. Now if $x \in \mathcal{L}(S)$, there is a leaf $y \in \mathcal{L}(T)$ such that $\varphi(x) \leq_T y$ and $\ell_S(x) \leq \ell_T(\varphi(x)) = \ell_T(y)$. Since $\varphi'(x) \leq_T \varphi(x)$, we have $\varphi'(x) \leq_T y$, so φ' has the desired property.

Consider the $\mathsf{T}_{<\omega}(Q)$-pattern $f'\colon B \to \mathsf{T}_{<\omega}(Q)$ defined by $f'(s) := \overline{f(s)}$. By Theorem 4.15 there are $s, t \in B$ such that $s \lhd t$ and $f'(s) \preceq f'(t)$, and thus also $f(s) \leq_h f(t)$, so f is good. □

One might attempt to adapt the proof of Theorem 4.20 to derive Theorem 4.41 while avoiding Laver's generalization of Nash-Williams' theorem for infinite trees. Although this appears feasible, it would require a labeled statement that would serve as the base case for the induction in Lemma 4.36. Such a result for trees of bounded height can be deduced from the $\sqsubseteq_m$ version of Theorem 3.13 (iv), and its proof is implicit in the proof of Theorem 5.33. However, extending it to trees of unbounded height would likely require the use of the *minimal bad array lemma*,[4] the exact conceptual breakthrough Nash-Williams used to prove his theorem about infinite trees. Invoking it here would undermine our aim to bypass the powerful machinery behind Laver's theorem, defeating the point.

Another difficulty arises from the fact that we rely on Lemma 4.35, which fails for leaf-labeled trees. Most arguments in the proof of Lemma 4.36 would break down due to this. One argument that can be recovered is when we consider a sequence of trees $T_0, T_1, T_2, \dots$ of type $\gamma + n + 1$ and form a sequence of collapsed trees $T'_0, T'_1, T'_2, \dots$ that have smaller type. (If we were proving it for bqos, this would be an array or a pattern.) Denote by $\mathcal{T}_\alpha(Q)$ the class of all leaf-labeled trees over Q that have type at most α. From the induction hypothesis, we know that $\mathcal{T}_\gamma(Q)$ is bqo and thus $\mathcal{T}_{\gamma+n}(\mathcal{T}_\gamma(Q))$ is also bqo. We can add "roots of short deleted sprouts" (see the proof of Lemma 4.34) and define a labeling of $\mathcal{L}(T'_i)$ by letting $\ell_{T'_i}(x) \in \mathcal{T}_\gamma(Q)$ be the subtree of T_i sprouting from x. We then find a good pair $T'_i \leq_h T'_j$ with respect to this new labeling and observe that $T_i \leq_h T_j$.

[4]The minimal bad array lemma is the bqo theory counterpart of minimal bad sequence arguments for wqos. It is essentially due to Nash-Williams [Nas65a], though it was first stated explicitly by Laver [Lav78]. See [Tho89] for a precise formulation and proof.

Note that it is *not* possible to adapt the proof of Theorem 4.20 to show that if a quasi-order Q is wqo, then $\mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$ is wqo, as this statement is false.

Proposition 4.42. *There exists a wqo set Q such that $\mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$ is not wqo.*

Proof. From Rado's counter-example (see Section 3.1), there exists a wqo set Q such that $\mathcal{P}(Q)$ is not wqo by $\sqsubseteq_m$. Note that a (nonempty) set $X \in \mathcal{P}(Q)$ can be encoded as a tree T_X of height 2 with $|X|$ leaves labeled by the elements of X. Then clearly $X \sqsubseteq_m Y \iff T_X \leq_h T_Y$. Thus if $\mathsf{T}^{\mathcal{L}}_{<\omega}(Q)$ were wqo by $\leq_h$, then $\mathcal{P}(Q)$ would be wqo by $\sqsubseteq_m$, but that is not the case. □

4.6 Results About Trees of Transfinite Height

In this section, we survey results about well-quasi-ordering order-theoretic trees that have arbitrary transfinite height. The only ordering relation we consider is the homeomorphic embedding relation; therefore, we will not state it explicitly.

Proposition 4.43 (Galvin (unpublished, quoted from [Tho88a]))**.** *The class of all trees of height $\omega + 1$ is not wqo.*

Proof. A proof of this statement is implicit in [Tho88a], where Thomas uses Galvin's idea to show that the class of all infinite graphs is not wqo by minors. See Theorem 6.2 of [Pit24] for a sketch of Thomas' proof. □

Galvin's counter-example might make our quest seem futile, but Laver [Lav78] discovered a large class of trees that includes trees of all ordinal heights and is bqo: the class of all *σ-scattered* trees. This directly builds upon Laver's earlier work on scattered linear orders [Lav71]. We briefly present his results on linear orders, as they demonstrate the strength and flexibility of Laver's generalization of Nash-Williams' infinite tree theorem.

Scattered linear orders Rather than studying specific linear orders, it is more natural to study their *order types*, which can be defined in a similar fashion as the isomorphism types of graphs (see the footnote in Section 1.4). If ϕ and ψ are (linear) order types, we say that ϕ *embeds* into ψ and write $\phi \leq \psi$ if ψ contains a subordering isomorphic to ϕ (hence if L and M are linearly ordered sets having type ϕ and ψ, respectively, then there is an order-preserving injection from L into M). An order type is *scattered* if the order type of the rationals $\mathbb{Q}$ does not embed into it, and it is *σ-scattered* if it is a countable union of scattered types. More precisely, ϕ is σ-scattered if every linear order L of type ϕ can be written as $L = \bigcup_{n\in\omega} L_n$ such that the type of each L_n is scattered. Given a quasi-order Q, we denote by $\mathcal{M}(Q)$ the class of all pairs (ϕ, ℓ) where ϕ is a σ-scattered (linear) order type and $\ell\colon \phi \to Q$ is a labeling function. Note that we are abusing notation for clarity. We order $\mathcal{M}(Q)$ by the rule that $(\phi, \ell_\phi) \leq (\psi, \ell_\psi)$ if there is an embedding $f\colon \phi \to \psi$ such that $\ell_\phi(x) \leq \ell_\psi(f(x))$ for all $x \in \phi$.

Theorem 4.44 (Laver [Lav71])**.** *If Q is bqo, then $\mathcal{M}(Q)$ is bqo.*

Proof sketch. Laver defines a hierarchy $\mathcal{H}_\alpha(Q) \subseteq \mathcal{M}(Q)$ for $\alpha \in \mathrm{On}$, and shows that the members of $\mathcal{H}(Q) := \bigcup_{\alpha\in\mathrm{On}} \mathcal{H}_\alpha(Q)$ have some nice properties with respect to embeddings. In particular, each labeled order type $(\phi, \ell_\phi) \in \mathcal{H}(Q)$ can be represented as a Q^+-labeled tree $T(\phi)$ for some bqo Q^+ in such a way that if $T(\phi) \preceq T(\psi)$, then $\phi \leq \psi$. Since earlier in the same paper he generalized

Nash-Williams' theorem for bqo-labeled trees, it follows by the order-reflecting property (see Observation 3.11) of this construction that $\mathcal{H}(Q)$ is bqo. Finally, he shows that each order type from $\mathcal{M}(Q)$ can be obtained by combining a finite number of orders types from $\mathcal{H}(Q)$ in a bqo-preserving manner. □

Laver then obtained a positive answer for Fraïssé's conjecture:

Corollary 4.45. *The class of all countable linear orders is wqo.*

Proof. This is an easy consequence of the previous theorem because every countable order type is σ-scattered. Indeed, if L is a countable linear order, then we can partition it into countably many singletons. The type of a singleton is clearly scattered, so the order type of L is σ-scattered. □

Scattered trees A tree T is *scattered* if the full binary tree of height ω does not homeomorphically embed into it (hence $2^{<\omega}$ plays an analogous role to that of $\mathbb{Q}$ in the definition of scattered order types). A tree T is *σ-scattered* if it can be written as $T = \bigcup_{n\in\omega} T_n$, where each T_n is a scattered tree such that whenever $x \in T$ and $x <_T y$ for some $y \in T_n$, then $x \in T_n$. Notice that every countable tree T is σ-scattered since it can be written as $T = \bigcup\{(\leftarrow, x] \mid x \in T\}$, where

$$(\leftarrow, x] \coloneqq \{y \in T \mid y \leq x\}.$$

In particular, $2^{<\omega}$ is σ-scattered. Given a quasi-order Q, we denote by $\mathcal{T}(Q)$ the class of all Q-labeled trees that are σ-scattered.

Theorem 4.46 (Laver [Lav78])**.** *If Q is bqo, then $\mathcal{T}(Q)$ is bqo.*

Note that we defined trees as having a unique minimal element (the root) to highlight the connection with graph theory, but order-theoretic trees are often defined without this restriction. Both Nash-Williams' and Laver's theorems extend naturally to trees with multiple minimal elements by Theorem 3.13 (iv).[5]

Pseudo-trees A *pseudo-tree* is a partially ordered set $(T, <)$ such that $(\leftarrow, x]$ is a chain for each $x \in T$ (so a tree is a well-founded pseudo-tree). A pseudo-tree is *well-branched* if all of its meets $x \wedge y$ exist. If S and T are well-branched pseudo-trees, then a *homeomorphic embedding* of S into T is an injection $\varphi\colon S \to T$ such that $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ for all $x, y \in S$. For every pseudo-tree T, we define a well-branched pseudo-tree $\hat{T}$ whose elements are the sets $(\leftarrow, x] \cap (\leftarrow, y]$ for $x, y \in T$, ordered by inclusion, and we let $i_T\colon T \to \hat{T}$ be defined by $i_T(x) \coloneqq (\leftarrow, x]$. Using this, we say that $\varphi\colon S \to T$ is an *embedding* of pseudo-trees S and T if there exists a homeomorphic embedding $\hat{\varphi}\colon \hat{S} \to \hat{T}$ such that $\varphi \circ i_T = i_S \circ \hat{\varphi}$. We can thus order the class of all pseudo-trees by the rule that $S \leq T$ if there exists an embedding of S into T. Corominas [Cor85] proved the following:

Theorem 4.47 (Corominas)**.** *The class of all countable pseudo-trees is bqo.*

Generalization to classes of partial orders On closer inspection, the theorems above are variations on a common theme. McKay [McK14] generalized the notion of being σ-scattered to partial orders and proved a characterization theorem that captures all of the results mentioned in this section and shows that a much wider class of partial orders is bqo. However, the precise statement of McKay's theorem is beyond the scope of this thesis.

[5]Alternatively, the argument for the forest version of Theorem 4.16 applies here as well.

5 Well-Quasi-Ordering Graphs

In this chapter, we study well-quasi-orderings of graphs with respect to the four basic graph containment relations: the minor relation $\preceq$, the topological minor relation $\preceq_t$, the subgraph relation $\subseteq$, and the induced subgraph relation $\subseteq_i$. For a broader overview, including additional relations not considered here, see [Liu20]. We prove Theorem 2 in Section 5.3 and Theorem 3 in Section 5.4.

It is straightforward to verify that, in the order listed above, these containment relations become strictly stronger. More precisely, we have

$$H \subseteq_i G \implies H \subseteq G \implies H \preceq_t G \implies H \preceq G,$$

while the converse implications fail in general. These relations are partial orders on the class of all finite graphs (if we identify isomorphic graphs), but only quasi-orders on the class of all graphs. Indeed, if we let $K_{\geq k}$ denote the disjoint union of the cliques K_n for $n \geq k$, then $K_{\geq k} \subseteq_i K_{\geq m} \subseteq_i K_{\geq k}$ for all $k, m \in \omega$.

We say that a graph parameter π is *bounded* for a class of graphs $\mathcal{G}$ if there exists $k \in \omega$ such that $\pi(G) < k$ for all $G \in \mathcal{G}$. If we let Q be a quasi-order, then a *Q-labeled graph* is a graph G endowed with a function $\ell_G \colon V_G \to Q$.

5.1 WQO by the Minor Relation

Wagner conjectured[1] that finite graphs are wqo by the minor relation. This was famously proved by Robertson and Seymour [RS04] in a series of around 20 papers and is widely regarded as one of the deepest results in 20th-century discrete mathematics. To highlight this, we quote Diestel [Die17]:

> Our goal [...] is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: *in every infinite set of graphs there are two such that one is a minor of the other.* This *graph minor theorem*, inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes well over 500 pages.

Note that in order to prove the graph minor theorem, it suffices to show that there are no infinite antichains of finite graphs, as there are clearly no infinite decreasing sequences: taking a proper minor always reduces either the number of vertices or the number of edges.

5.1.1 Tree Decompositions

A core concept in the proof of the graph minor theorem is *tree-width*, a parameter that measures how similar a given graph is to a tree. In particular, a connected graph on at least two vertices has tree-width 1 if and only if it is a tree.

Definition 5.1. A *tree decomposition* of a graph G is a pair (T, X) where T is a tree and $X = \langle X_t \mid t \in V_T \rangle$ is a family of finite subsets of V_G called *bags* of T with the following properties:

[1] Throughout the Graph Minors series, Robertson and Seymour refer to the statement as "Wagner's conjecture" and cite [Wag70], but according to Diestel [Die17], Wagner always insisted that he never conjectured it, even after the theorem was proved.

(i) $\bigcup\{X_t \mid t \in V_T\} = V_G$,

(ii) for every edge $e \in E_G$ there exists $t \in V_T$ such that $e \subseteq X_t$, and

(iii) for each $v \in V_G$, the set $\{t \mid v \in X_t\}$ induces a connected subgraph in T.

The *width* of a tree decomposition (T, X) is $\max\{|X_t| - 1 \mid t \in V_T\}$, provided this maximum exists; otherwise the width is undefined.

Remark. Tree decompositions of infinite width have also been studied; see for example the foundational paper [RST92] by Robertson, Seymour, and Thomas.

Definition 5.2. The *tree-width* of a graph G is the minimum width of a tree decomposition of G. If the tree-width of G is defined, we denote it by $\mathrm{tw}(G)$.

Every finite graph clearly admits a tree decomposition (take a single bag containing all vertices), so finite graphs have defined tree-width. As for infinite graphs, an analog of the de Bruijn–Erdős theorem exists that characterizes the tree-width of infinite graphs in terms of their finite subgraphs:

Theorem 5.3 (Thomas [Tho88b])**.** *Let G be an arbitrary graph, and let $k \in \omega$. Then $\mathrm{tw}(G) \leq k \iff \mathrm{tw}(H) \leq k$ for every finite subgraph H of G.*

Lemma 5.4. *If G is a graph with defined tree-width, then the following holds.*

(a) $H \preceq G \implies \mathrm{tw}(H) \leq \mathrm{tw}(G)$. *…tree-width is minor-monotone*

(b) $\delta(G) \leq \mathrm{tw}(G)$. *…$\delta(G)$ denotes the minimal degree of G*

(c) $\chi(G) \leq \mathrm{tw}(G) + 1$. *…$\chi(G)$ denotes the chromatic number of G*

(d) $\omega(G) \leq \mathrm{tw}(G) + 1$. *…$\omega(G)$ denotes the clique number of G*

Proof. Note that due to Theorem 5.3, we can, without loss of generality, assume that G is finite. Let (T, X) be a tree decomposition of G with minimal width.

(a) Since we assume that G is finite, a graph isomorphic to H can be obtained from G by deleting edges and vertices and by contracting edges. Observe that all these operations allow us to define a tree decomposition for the smaller graph: after deleting an edge e, there is no need to change the decomposition at all; and after deleting a vertex v, remove it from all bags that contain it. Finally, after contracting an edge uv to form a new vertex w, replace u and v in the bags containing them with w. Either way, the size of bags did not increase.

(b) If T contains only a single bag, then the claim trivially holds since the width is $|V_G| - 1$. Otherwise, let $t \in V_T$ be a leaf, p be the unique neighbour of t, and note that without loss of generality $X_t \setminus X_p \neq \varnothing$; if not, we remove t from T and try again. We can thus let $v \in X_t \setminus X_p$. Observe that v appears only in X_t from (iii). Therefore, all neighbours of v must appear in X_t from (ii), and we have

$$\delta(G) \leq \deg(v) \leq |X_t| - 1 \leq \mathrm{tw}(G).$$

(c) Notice that from (a) and (b), every subgraph of G contains a vertex of degree at most $\mathrm{tw}(G)$. This allows us to easily define a proper coloring of G using $\mathrm{tw}(G) + 1$ colors by recursively coloring low-degree vertices.

(d) We claim that whenever $K \subseteq V_G$ induces a clique in G, then there exists $t \in V_T$ such that $K \subseteq X_t$. Note that letting $|K| = \omega(G)$ yields (d). Define for $v \in K$ the set $T_v \coloneqq \{t \in V_T \mid v \in X_t\}$. We know from (iii) that each T_v induces a subtree in T, and from (ii) that whenever $u, v \in K$, then $T_u \cap T_v \neq \varnothing$. This implies that $\bigcap\{T_v \mid v \in K\} \neq \varnothing$, proving the claim: indeed, we can choose a rooting of T and let T_v be such that its root r_v is the furthest away from the root of T. Since each T_u intersects T_v, clearly $r_v \in T_u$ for all $u \in K$. □

It should now be an easy exercise to show that forests have tree-width ≤ 1, cycles have tree-width 2, the complete graph K_n has tree-width $n-1$, and the complete bipartite graph $K_{n,m}$ has tree-width $\min(n, m)$. It can also be shown that series-parallel graphs have tree-width ≤ 2, k-trees have tree-width k, and the $n \times n$ planar grid has tree-width n.

Algorithmic aspects of tree-width Tree-width has many important algorithmic applications. Although we will not explore this direction in detail, we briefly note its computational relevance. Deciding whether $\mathrm{tw}(G) \leq k$ is NP-complete when both G and k are part of the input. However, once a tree decomposition of width bounded by some constant k is obtained for a graph, it allows us to efficiently solve many problems that would otherwise be NP-hard, such as finding a maximum independent set or a minimal vertex cover of G using dynamic programming across the bags of the decomposition. See [Fia26] for more details.

5.1.2 The Graph Minor Theorem

One of the first steps towards the graph minor theorem was showing that graph classes with bounded tree-width are wqo by the minor relation [RS90] (this is known as the *bounded graph minor theorem*). The basic idea of the proof is that graphs of bounded tree-width are "tree-shaped," so one can adapt the ideas contained in Nash-Williams' proof [Nas63] of Kruskal's tree theorem and apply them to "tree-shaped" graphs instead. Proving the full graph minor theorem is much more complicated and involves analyzing tree decompositions of graphs without a K_n minor and studying how they interact with different surfaces. Diestel provides a sketch of the proof in Section 12.7 of [Die17].

Theorem 5.5 (Robertson–Seymour Graph Minor Theorem)**.** *The class of all finite graphs is wqo by the minor relation.*

This theorem has profound implications, probably the most important one being the following application of Theorem 2.7. We say that a property φ is *minor-closed* if whenever G has φ and H is a minor of G, then H has φ.

Corollary 5.6. *For every minor-closed property φ there exists a finite set of forbidden minors $H_1, \ldots, H_k$ such that G has $\varphi \iff$ no H_i is a minor of G.*

The set of minimal obstructions of φ (that is, non-isomorphic graphs that are $\preceq$-minimal among those that do not have φ) is called its *Kuratowski set*,[2] and results describing these sets are often referred to as *excluded minor theorems*. Probably the most well-known such theorem is due to Wagner: that a graph is planar $\iff$ it contains neither K_5 nor $K_{3,3}$ as a minor. Archdeacon [Arc81] proved a similar theorem for the projective plane, with the number of forbidden minors being 35. The above corollary implies that such a set exists for every surface, but the complete Kuratowski set is currently not known for any surface other than the two mentioned above.

[2]Named after Kuratowski's theorem, which states that a graph is planar $\iff$ it contains neither K_5 nor $K_{3,3}$ as a topological minor.

Algorithmic implications In their quest to prove the graph minor theorem, Robertson and Seymour [RS95] have found an algorithm that tests for any fixed graph H whether H is a minor of the input graph G in time $O(|V_G|^3)$. An algorithm with quadratic running time was later discovered by Kawarabayashi, Kobayashi, and Reed [KKR12].

Corollary 5.7. *Every minor-closed property φ can be decided in polynomial time.*

Proof. If $H_1, \ldots, H_k$ is the Kuratowski set of φ, then testing whether G has φ reduces to testing the k assertions of whether H_i is a minor of G. □

This result has deep implications for complexity theory of graph algorithms, including the resolution of some long-standing open problems [Die17]. However, its practical applicability is limited since one has to know a specific obstruction set for φ, and there is an enormous hidden constant depending on H. As Robertson and Seymour put it, "the algorithm is not practically feasible."

Proof-theoretic strength Friedman, Robertson, and Seymour [FRS87] have shown that the bounded graph minor theorem stated above is logically equivalent, over a system called RCA_0 (which is even weaker than Peano arithmetic), to Friedman's extension of Kruskal's theorem, which we studied in Section 4.3. Recall that the extended Kruskal's theorem cannot be proved in the (very strong) logical system Π^1_1-CA_0, and as a consequence, the bounded graph minor theorem (and thus also the full graph minor theorem) cannot be proved in Π^1_1-CA_0 either. According to Freund [Fre22], this is "one of the most spectacular manifestations of Gödel's theorems in mathematics." The considerable complexity of the proof of the graph minor theorem has made it difficult to determine an upper bound on its logical strength, and the first known upper bound was found only recently by Krombholz and Rathjen [KR19].

Friedman, Robertson, and Seymour [FRS87] also considered several finite forms of the graph minor theorem in the spirit of Friedman's finite versions of Kruskal's theorem, which we mentioned in Section 4.2. For example:

Theorem 5.8. *For all k and p, there exists a natural number n such that for any sequence $G_1, \ldots, G_n$ of graphs satisfying $\operatorname{tw}(G_m) \le p$ and $|G_m| \le k + m$, there are indices $i < j \le n$ such that $G_i \preceq G_j$.*

The truth of this statement follows from a simple compactness argument similar to the proof of Theorem 4.9, and it can be shown [FRS87] to be unprovable in fairly strong logical systems, including ATR_0.

Extension to labeled graphs For quasi-orders P and Q, denote by $\mathsf{G}(P, Q)$ the class of all triplets (G, ℓ, ϕ), where G is a finite graph, and $\ell\colon V_G \to P$ and $\phi\colon E_G \to Q$ are functions. We extend the minor relation to $\mathsf{G}(P, Q)$ by letting

$$(H, \ell_H, \phi_H) \preceq (G, \ell_G, \phi_G)$$

if there exists a family $\langle V_v \mid v \in V(H) \rangle$ of disjoint subsets of $V(G)$ that satisfies:

(i) each V_v induces a nonempty connected subgraph of G,

(ii) for each $v \in V(H)$, there is $x \in V_v$ such that $\ell_H(v) \le_P \ell_G(x)$,

(iii) for each $uv \in E_H$, there are $x \in V_u$ and $y \in V_v$ such that $xy \in E_G$ and $\phi_H(uv) \le_Q \phi_G(xy)$.

In the final paper of their Graph Minor series [RS10], Robertson and Seymour proved the following:

Theorem 5.9 (Robertson, Seymour). *If P and Q are wqos, then $\mathsf{G}(P, Q)$ is wqo.*

Remark. The graph minor theorem (in both its standard and labeled variants) applies not only to graphs but also to directed graphs, multigraphs with loops, hypergraphs, and even to hypergraphs where the vertices of an edge are ordered.

5.1.3 Minors and Infinite Graphs

Thomas proved that the graph minor theorem does not generalize to infinite graphs in its full strength, but the bounded graph minor theorem does.

Theorem 5.10 (Thomas [Tho88a]). *The class of all (finite or infinite) graphs is not wqo by $\preceq$. In fact, already graphs of size $2^\omega = |\mathbb{R}|$ are not wqo by $\preceq$.*

Proof. See Theorem 6.2 of [Pit24] for a sketch of the proof. □

One might wonder whether the assumption that the graphs have size continuum is necessary, but in fact a much stronger statement holds:

Theorem 5.11 (Komjáth [Kom95]). *For every uncountable cardinal κ, there is a family of 2^κ graphs of size κ such that none is a minor of another.*

Remark. Pitz [Pit20] recently found a simpler proof, which also implies that for every uncountable cardinal κ, there is a family of 2^κ order-theoretic trees of height $\omega+1$ and size κ such that there is no homeomorphic embedding of one into another. See Theorem 6.4 of [Pit24] for a sketch of the proof.

Thomas [Tho88a] also asks whether countable graphs are wqo, and this question remains unanswered to this day as the main open problem in the field.

Conjecture 5.12 (Thomas). *The class of all countable graphs is wqo by minors.*

However, Thomas gave a positive answer for many classes of graphs:

Theorem 5.13 (Thomas [Tho89]). *Any class of (finite or infinite) graphs with bounded tree-width is wqo by the minor relation.*

Corollary 5.14. *If H is a finite planar graph and $\mathcal{G}$ is the class of all (finite or infinite) graphs that do not have H as a minor, then $\mathcal{G}$ is wqo by minors.*

Proof. Robertson and Seymour proved [RS86] that for every finite planar graph H, there is a constant k such that if G is a finite graph that does not have H as a minor, then $\mathrm{tw}(G) \leq k$ (this is called the *grid minor theorem* since it is enough to show that it holds for all planar $n \times n$ grids). If we now denote by $\mathcal{H}$ the class of all finite subgraphs of the graphs from $\mathcal{G}$, then we have $\mathrm{tw}(G) \leq k$ for all $G \in \mathcal{H}$ since they avoid the planar minor H. Theorem 5.3 implies that in fact $\mathrm{tw}(G) \leq k$ for all $G \in \mathcal{G}$, so $\mathcal{G}$ is wqo by the previous theorem. □

5.2 WQO by the Topological Minor Relation

In the 1940s, Vázsonyi conjectured that two classes of finite graphs are wqo by the topological minor relation: trees and *subcubic graphs* (graphs with maximum degree at most 3). Kruskal [Kru60] and Tarkowski [Tar60] independently gave a positive answer to the first conjecture, and Nash-Williams [Nas65a] extended it to infinite trees. Mader [Mad72] later generalized Kruskal's theorem to graphs with a bounded number of disjoint cycles, and Fellows, Hermelin, and Rosamond [FHR12] generalized it to graphs that have a *feedback vertex set* (a set of vertices intersecting each cycle) of bounded size.

The first generalization can be derived from the forest version of Kruskal's theorem by choosing an appropriate labeling [Liu20], and the second generalization is equivalent to the first due to a classical theorem of Erdős and Pósa [EP65] that a graph has a bounded number of disjoint cycles $\iff$ it has a feedback vertex set of bounded size.

5.2.1 Vázsonyi's Subcubic Graph Conjecture

The second conjecture of Vázsonyi is thus of greater interest than these generalizations of Kruskal's theorem, as it concerns a fundamentally different class of graphs. After remaining open for many years, it was finally confirmed to be true by Robertson and Seymour as a consequence of their graph minor theorem.

Lemma 5.15. *If G is subcubic and $H \preceq G$, then $H \preceq_t G$*

Proof. Let $\langle V_v \,|\, v \in V(H) \rangle$ be a family of disjoint vertex sets in G that witness H being a minor of G, and let G' be a subgraph of G obtained by keeping at most 1 edge between each pair of sets V_u and V_v so that $uv \in E_H \iff$ there is an edge between V_u and V_v in G'. Next, for each $v \in V(H)$, mark those vertices of V_v from which there is an edge in G' to some V_u where $u \neq v$, and find in each V_v a $\subseteq$-minimal set V'_v such that V'_v induces in G' a tree containing all marked vertices of V_v. It is easy to verify that V'_v contains a central vertex c_v of degree at most 3 such that the unique paths (possibly of length 0) from c_v via V'_v to the marked vertices of V_v are internally vertex-disjoint. By combining these paths with the edges of G' between distinct V_u and V_v, we obtain a homeomorphic embedding (see Definition 1.36) of H in G that maps each $v \in V(H)$ to c_v. □

Remark. The requirement that G is subcubic cannot be weakened: one can easily find graphs H and G with maximum degree 4 such that $H \preceq G$ but $H \not\preceq_t G$.

Hence, if a class of subcubic graphs is wqo by $\preceq$, it is also wqo by $\preceq_t$.

Corollary 5.16 (Robertson, Seymour)**.** *The class of all finite subcubic graphs is wqo by the topological minor relation.*

Corollary 5.17 (Thomas)**.** *For every k, the class of all (finite or infinite) subcubic graphs of tree-width at most k is wqo by the topological minor relation.*

5.2.2 Robertson's Conjecture

The class of all finite graphs is not wqo by the topological minor relation. To see this, consider for $n \geq 1$ the graph R_n obtained from a path on $n+1$ vertices by doubling each edge. If one prefers to work with simple graphs, one can subdivide one of the parallel edges. We call R_n the *Robertson chain* of length n. The *ends* of R_n are the ends of the original path. Let R_n^* be the graph obtained from R_n by identifying its ends (gluing them together). It is easy to see that $\{R_n^* \mid n \geq 3\}$ is an infinite $\preceq_t$-antichain. Note that this also shows that degrees being at most 3 in Theorem 5.16 is optimal.

Another infinite antichain can be constructed by letting R'_n be the graph obtained from R_n by attaching two leaves to each end of R_n. In fact, every known infinite antichain of finite graphs contains arbitrarily long Robertson chains as topological minors. Motivated by this, Robertson conjectured[3] in the 1980s that for every n, the class $\mathcal{R}_n$ of all finite graphs that do not contain R_n as a topological minor is wqo by the topological minor relation. Note that this generalizes all of the results mentioned above: the case $n = 1$ is the unlabeled forest version of Kruskal's tree theorem, the case $n = 2$ implies Vázsonyi's subcubic graph conjecture since no subcubic graph contains R_2 as a topological minor, and similarly, the case $n = 2k - 1$ implies Mader's theorem for graphs with no k disjoint cycles (and thus also the theorem for graphs having a feedback vertex set of bounded size).

Ding [Din96] proved a weakening of Robertson's conjecture: that the class of all finite graphs that do not contain R_n as a minor is wqo by the topological minor relation. The full conjecture has been settled only recently by Liu in his thesis [Liu14] under the supervision of Thomas, with the proof now appearing in a series of papers starting with [LT24].

Theorem 5.18 (Liu, Thomas)**.** *For every $n \geq 1$, the class $\mathcal{R}_n$ is wqo by $\preceq_t$.*

One might notice that Robertson's conjecture is not optimal in the sense that there are classes of graphs closed under taking topological minors that are wqo by $\preceq_t$ and yet contain all Robertson chains. Indeed, $\{R_n \mid n \geq 1\}$ itself is not an $\preceq_t$-antichain; but this is resolved if the vertices are labeled. For a quasi-order Q, we extend the topological minor relation to Q-labeled graphs by letting

$$(H, \ell_H) \preceq_t (G, \ell_G)$$

if there exists a homeomorphic embedding (see Definition 1.36) $f\colon V_H \to V_G$ of H in G such that $\ell_H(v) \leq \ell_G(f(v))$ for all $v \in V_H$. For $n \geq 1$ denote by $\mathcal{R}_n(Q)$ the class of all Q-labeled graphs (G, ℓ) such that $G \in \mathcal{R}_n$.

Theorem 5.19 (Liu, Thomas)**.** *Let $n \geq 1$. If Q is wqo, then $\mathcal{R}_n(Q)$ is wqo.*

Observe that when Q is not trivial (it contains at least two non-equivalent elements), then this is optimal: let $x, y \in Q$ be such that $x \not\leq y$, and label the ends of R_n with x, while labeling the other vertices with y. Then $\{R_n \mid n \geq 1\}$ is an antichain with respect to the topological minor relation that respects labels.

Remark. Liu and Thomas also provide a complete characterization of classes of unlabeled graphs that are wqo by $\preceq_t$, but the proof is yet to be published [Liu20].

5.2.3 Nash-Williams' Immersion Conjecture

Immersions are a weakening of the topological minor relation in which we allow the paths of homeomorphic embeddings to intersect, but we still require them to be edge-disjoint. More formally, if H and G are graphs, then we say that H has a *weak immersion* in G if there exists an injection $f\colon V_H \to V_G$ and a family $\langle P_{uv} \mid uv \in E_H \rangle$ of edge-disjoint paths in G such that each path P_{uv} connects $f(u)$ to $f(v)$. This is a *strong immersion* if no internal vertex of any P_{uv} belongs to

[3] The conjecture is mentioned in [Din96], but Liu [LT24] states that "the conjecture has been circulated in the community since the late 1980s or earlier."

$f[V_H]$. Note that further requiring the paths P_{uv} to be internally vertex-disjoint would make this a homeomorphic embedding. One can again extend these notions to Q-labeled graphs by demanding that $\ell_H(v) \leq \ell_G(f(v))$ for each $v \in V_H$.

Nash-Williams conjectured [Nas64] that the class of all finite graphs is wqo by the weak immersion relation, and he later [Nas65a] proposed the same for the strong immersion relation. It is worth noting that Lemma 5.15 holds for immersions as well, so either of these conjectures would imply Vázsonyi's conjecture for subcubic graphs. In fact, the minor, topological minor, weak immersion, and strong immersion are equivalent for subcubic graphs [Liu20], even though the immersion relations and the minor relation are incomparable. Indeed, it is easy to find graphs H and G with maximum degree 4 such that H is a minor of G, but H has no weak immersion in G, and graphs H' and G' with maximum degree 4 such that H' has a strong immersion in G', but H' is not a minor of G'.

A labeled version of the weak immersion conjecture was proved by Robertson and Seymour [RS10] as a consequence of the hypergraph version of Theorem 5.9. They, in fact, proved it for loopless multigraphs, obtaining an unlabeled version for multigraphs that may have loops by labeling each vertex with the number of loops incident with it and applying the loopless labeled version.

Theorem 5.20 (Robertson, Seymour)**.** *If Q is wqo, then the class of all finite Q-labeled graphs is wqo by the weak immersion relation that respects labels.*

The strong version of the immersion conjecture remains open, but Robertson and Seymour [RS10] remark: "It seemed to us at one time that we had a proof of the stronger [conjecture], but even if it was correct it was very much more complicated, and it is unlikely that we will write it down."

5.3 WQO by the Induced Subgraph Relation

In this section, we study classes of graphs that are well-quasi-ordered by the induced subgraph relation. We extend a theorem of Ding, which states that classes of finite graphs with bounded tree-depth (or equivalently, not containing paths of arbitrary length as subgraphs) are wqo by the induced subgraph relation. We show that it holds for infinite graphs as well, proving Theorem 2. Finally, we mention some results about hereditary graph classes defined by a finite number of forbidden induced subgraphs.

First, observe that there are two obvious infinite antichains with respect to the subgraph and induced subgraph relations: the set $\{C_n \mid n \geq 3\}$ consisting of cycles, and the set $\{F_n \mid n \geq 1\}$ consisting of *forks*. A *fork* of length n is the graph obtained from a path on $n+1$ vertices by attaching two leaves to each end of the original path, and we denote it by F_n. Ding proved that these two antichains are the only obstructions for subgraph-closed classes to be wqo.

Theorem 5.21 (Ding [Din92])**.** *Let $\mathcal{G}$ be a class of finite graphs closed under taking subgraphs. Then the following statements are equivalent:*

(a) *$\mathcal{G}$ is wqo by the subgraph relation,*

(b) *$\mathcal{G}$ is wqo by the induced subgraph relation,*

(c) *$\mathcal{G}$ contains only finitely many cycles and forks.*

5.3.1 Tree-Depth

Tree-depth is a graph parameter that measures how close a given graph is to a star (a complete bipartite graph $K_{1,n}$). The term "tree-depth" was introduced by Nešetřil and Ossona de Mendez in [NO06] and studied extensively in [NO12], although equivalent notions had been considered earlier, notably by Ding [Din92] in the context of well-quasi-orderings.

In this section, we review basic properties of tree-depth following [NO12], with the distinction that we extend the definition to possibly infinite graphs, which makes it necessary to modify some of the proofs. Note that we no longer require order-theoretic trees to have a unique minimal element (the root). The *closure* of an order-theoretic tree T is the graph

$$\operatorname{clos}(T) \coloneqq (T, \{\{x, y\} \in [T]^2 \mid x <_T y\}).$$

Definition 5.22. The *tree-depth* $\operatorname{td}(G)$ of a graph G is the minimum height of an order theoretic tree T such that $G \subseteq \operatorname{clos}(T)$.

Observation 5.23. *If κ is a cardinal number, then $\operatorname{td}(K_\kappa) = \kappa$.*

Proof. Clearly $K_\kappa = \operatorname{clos}(\kappa)$. Here, we treat $(\kappa, <)$ as a tree with a single branch of length κ. And if T is a tree of height less than κ, then no vertex of $\operatorname{clos}(T)$ can have degree $|\kappa \setminus \{0\}|$ because all branches of T have length less than κ. □

Since the subgraph relation is transitive and every graph is a subgraph of a large enough complete graph, the above observation implies that every graph has its tree-depth defined. Notice that if $f: V_G \to T$ is a mapping witnessing that $G \subseteq \operatorname{clos}(T)$ (see Definition 1.32), then $G \subseteq \operatorname{clos}(f[V_G])$, where we interpret $f[V_G] \subseteq T$ as a subtree of T. Since the height of a subtree is bounded by the height of the original tree, we get the following observation:

Observation 5.24. *For every graph G there exists an order-theoretic tree T of height $\operatorname{td}(G)$ such that $G \subseteq \operatorname{clos}(T)$ and $|T| = |G|$.*

It is easy to see that if $H \subseteq G$, then $\operatorname{td}(H) \leq \operatorname{td}(G)$. We show that the same is true for all minors of G, meaning that for every ordinal α, the class of all graphs with tree-depth at most α is closed under taking minors.

Lemma 5.25. $H \preceq G \implies \operatorname{td}(H) \leq \operatorname{td}(G)$.

Proof. Let $\langle V_v \mid v \in V(H) \rangle$ be a family of disjoint vertex sets in G that witness H being a minor of G, and let $f: V(G) \to T$ be an injection witnessing that $G \subseteq \operatorname{clos}(T)$. We will show that $H \subseteq \operatorname{clos}(T)$. Because each set V_v induces a connected subgraph in G, and the downsets

$$(\leftarrow, t] \coloneqq \{s \in t \mid s \leq_T t\}$$

for $t \in T$ are chains, the subtrees $f[V_v] \subseteq T$ have unique minimal elements t_v. Because the sets V_v for $v \in V(H)$ are disjoint and f is injective, $t_u \neq t_v$ for $u \neq v$. Define a map $g: V(H) \to T$ by $g(v) \coloneqq t_v$. We claim that g witnesses H being a subgraph of $\operatorname{clos}(T)$. Indeed, let $uv \in E_H$ and let $x \in V_u$ and $y \in V_v$ be such that $xy \in E_G$. Then either $f(x) <_T f(y)$ or $f(y) <_T f(x)$; assume without loss of generality that the first case occurred. Then also $t_u \leq_T f(x) <_T f(y)$. Since $t_v \leq_T f(y)$ and the downset of $f(y)$ is a chain, t_u and t_v are comparable; that is, $t_u <_T t_v$ or $t_v <_T t_u$. In either case, $g(u)g(v) \in E(\operatorname{clos}(T))$. □

Calculating tree-depth Next, observe that if G is the disjoint union of a family of graphs $\langle G_\alpha \mid \alpha < \kappa \rangle$, where κ is a cardinal number, then

$$\operatorname{td}(G) = \sup\{\operatorname{td}(G_\alpha) \mid \alpha < \kappa\}.$$

We can use this to recursively characterize graphs with finite tree-depth, since if $\operatorname{td}(G)$ is finite, then this supremum is a maximum.

Proposition 5.26. *If G has finite tree-depth, then*

$$\operatorname{td}(G) = \begin{cases} 1, & \textit{if } |G| = 1, \\ 1 + \min\{\operatorname{td}(G - v) \mid v \in V_G\}, & \textit{if } G \textit{ is connected and } |G| > 1, \\ \max\{\operatorname{td}(G_\alpha) \mid \alpha < \kappa\}, & \textit{otherwise}, \end{cases}$$

where $\langle G_\alpha \mid \alpha < \kappa \rangle$ are the components of G.

Proof. Due to our previous discussion, it suffices to verify the second condition. Let G be connected, let T be a tree of height $\operatorname{td}(G) < \omega$, and let $f \colon V_G \to T$ witness that $G \subseteq \operatorname{clos}(T)$. Since G is connected, we can assume that T has a unique root r. Note that there exists $v \in V_G$ such that $f(v) = r$, since otherwise $\operatorname{td}(G) \leq \operatorname{H}(T) - 1$, which is false. Observe that $G - v \subseteq \operatorname{clos}(T \setminus \{r\})$, so $\operatorname{td}(G - v) \leq \operatorname{td}(G) - 1$. This proves that

$$\operatorname{td}(G) \geq 1 + \min\{\operatorname{td}(G - v) \mid v \in V_G\}.$$

To show the other inequality, suppose that $v \in V_G$ and let $f \colon V(G - v) \to S$ witness that $G - v \subseteq \operatorname{clos}(S)$ for some tree S of height $\operatorname{td}(G - v)$. Pick $r \notin S$ and let $T \coloneqq S \cup \{r\}$ inherit the order of S, with the addition that $r <_T x$ for all $x \in S$. In other words, T is a tree with root r whose branch-trees are the "components" of S. Observe that $G \subseteq \operatorname{clos}(T)$, as witnessed by $f \cup \{(v, r)\}$. Hence $\operatorname{td}(G) \leq \operatorname{td}(G - v) + 1$ for all $v \in V_G$, proving the second inequality. □

Tree-depth and paths It should be easy to see that the tree-depth of a path on n vertices is logarithmic in n. Indeed, $\operatorname{td}(P_1) = 1$ and for $n \geq 2$ we have

$$\operatorname{td}(P_n) = \min_{1 \leq i \leq n-2} \Big(1 + \max\{\operatorname{td}(P_i),\, \operatorname{td}(P_{n-1-i})\}\Big) = 1 + \operatorname{td}(P_{\lceil n/2 \rceil}).$$

Clearly $\operatorname{td}(P_n) = \log_2 n + o(1)$, and a more careful examination reveals that

$$\operatorname{td}(P_n) = \lceil \log_2(n + 1) \rceil.$$

Lemma 5.27. *A class of graphs $\mathcal{G}$ has (finitely) bounded tree-depth $\iff$ there is a natural number $n \in \omega$ such that $P_n \not\subseteq G$ for all $G \in \mathcal{G}$.*

Proof. ($\Rightarrow$) Let $\operatorname{td}(G) < k$ for each $G \in \mathcal{G}$ and let $n = 2^k - 1$. Then $P_n \not\subseteq G$ for any $G \in \mathcal{G}$ since $\operatorname{td}(P_n) = \lceil k \rceil \geq k$ and tree-depth is minor-monotone.

($\Leftarrow$) We claim that if $P_n \not\subseteq G$, then $\operatorname{td}(G) < n$. Suppose that $\operatorname{td}(G) \geq n$. We can without loss of generality assume (i) that G has finite tree-depth; otherwise, we take a suitable subgraph, and (ii) that G is connected; otherwise, we pick a component with maximal tree-depth. Pick $v_1 \in V_G$ and let G_1 be a component of $G - v_1$ with maximal tree-depth. For $i \geq 1$, choose $v_{i+1} \in V(G_i)$ such that v_{i+1} is a neighbour of v_i, and let G_{i+1} be a component of $G_i - v_{i+1}$ with maximal tree-depth. By inductively applying Proposition 5.26, we get $\operatorname{td}(G_i) \geq \operatorname{td}(G) - i$. Since $\operatorname{td}(G) \geq n$, we have $\operatorname{td}(G_{n-1}) \geq 1$. In particular, G_{n-1} is nonempty, so v_n is defined. But then $v_1 v_2 \ldots v_n$ is a path on n vertices in G, a contradiction. □

Tree-depth and colorings Tree-depth can also be characterized using certain coloring notions. A *centered coloring* of a graph G by k colors is a mapping $f\colon V_G \to k$ such that in every connected subgraph $H \subseteq G$, some color appears exactly once. A *rank coloring* of G by k colors is a mapping $f\colon V_G \to k$ such that every path between two vertices with the same color contains a vertex with a greater color. Note that every rank coloring is a centered coloring, and that all centered colorings are proper (no two adjacent vertices can have the same color).

Theorem 5.28 (Nešetřil, Ossona de Mendez [NO06])**.** *If G has finite tree-depth, then* $\operatorname{td}(G)$*, the minimum number of colors in a centered coloring of G, and the minimum number of colors in a rank coloring of G are all equal.*

We omit the proof, as the argument given in [NO06] extends to infinite graphs with essentially no changes.

The tree-depth compactness theorem We use the coloring characterization of tree-depth described above to prove a compactness theorem.

Theorem 5.29. *Let G be an arbitrary graph, and let k be a positive integer. Then* $\operatorname{td}(G) \le k \iff \operatorname{td}(H) \le k$ *for every finite subgraph H of G.*

We prove this using an important theorem of Rado [Rad49], which we formulate in graph-theoretic terms for clarity. Let V be an arbitrary set of vertices, and let $\mathcal{A} = \langle A_v \,|\, v \in V \rangle$ be a family of nonempty finite sets. We interpret A_v as the set of admissible colors for the vertex v. Let $\mathcal{A}(U) := \langle A_u \,|\, u \in U \rangle$ for $U \subseteq V$. A *selector* on $\mathcal{A}(U)$ is a mapping f with domain U such that $f(u) \in A_u$ for all $u \in U$. We interpret f as a coloring of the subgraph induced by U.

Theorem 5.30 (Rado's Selection Principle)**.** *Let $\mathcal{A} = \langle A_v \,|\, v \in V \rangle$ be a family of nonempty finite sets. For every finite $U \subseteq V$, let f_U be a selector on $\mathcal{A}(U)$. Then, there exists a selector g on $\mathcal{A}(V)$ with the property that for each finite $W \subseteq V$, there is a finite $U \subseteq V$ such that $W \subseteq U$ and $g \restriction W = f_U \restriction W$.*

Proof. See Section 7.1 of [Smo26]. □

Proof of Theorem 5.29. Suppose that every finite subgraph H of G has tree-depth at most k, and let f_H be a rank coloring of H using at most k colors. We invoke Rado's selection principle and find a coloring $g\colon V_G \to k$ with the property described above. Assume for contradiction that g is not a rank coloring. Then there is a path $v_0 v_1 \dots v_n$ in G such that $g(v_i) \le g(v_0) = g(v_n)$ for all $i \le n$. Put $W := \{v_i \,|\, i \le n\}$. By the choice of g, there exists $U \subseteq V_G$ such that $W \subseteq U$ and g agrees with $f_{G[U]}$ on W. But then $f_{G[U]}$ is not a rank coloring of $G[U]$, a contradiction. Hence g is a rank coloring of G by k colors and $\operatorname{td}(G) \le k$. □

Tree-depth and tree-width We will show that the tree-width of a graph is at most its tree-depth. Together with the fact that tree-depth is minor-monotone, Theorem 5.13, and Theorem 2.7, this implies that for every $n \in \omega$, the class $\mathcal{D}_n$ of all (finite or infinite) graphs with tree-depth at most n can be characterized by a finite set of minimal forbidden minors (graphs with tree-depth greater than n, minimal in the minor order). By compactness, these obstructions coincide with those for the class of finite graphs of tree-depth at most n. They were studied by Giannopoulou and Thilikos [GT09], and a complete list is known for $n \le 3$.

On the other hand, $\mathcal{D}_n$ also admits a characterization by a finite set of minimal forbidden subgraphs. We establish this in the next section by extending Ding's theorem. Once again, the obstructions are the same in both the finite and infinite settings. These were investigated by Dvořák [Dvo07], who determined the set of subgraph obstructions for $n \leq 3$.

Lemma 5.31. *If* $\mathrm{td}(G)$ *is finite, then* $\mathrm{tw}(G) \leq \mathrm{td}(G) - 1$.

Proof. Let F be an order-theoretic tree of height $\mathrm{td}(G)$, and let $f \colon V_G \to F$ be a map witnessing that $G \subseteq \mathrm{clos}(F)$. Choose $r \notin F$ and let $T \coloneqq F \cup \{r\}$ be a tree with the order inherited from F, with the addition that $r <_T x$ for all $x \in F$. In other words, T is a tree with root r whose branch-trees are the "components" of F. Since T has finite height, it can be interpreted as a graph-tree. Abusing notation, we define a tree decomposition (T, X) of G by letting

$$X_t \coloneqq \{v \in V_G \mid f(v) \leq_T t\}$$

for $t \in T$. This is indeed a tree decomposition: clearly $\bigcup\{X_t \mid t \in T\} = V_G$, and if u and v form an edge with $f(u) <_T f(v)$, then $\{u, v\} \subseteq X_{f(v)}$. Finally, for each $v \in V_G$, the set $\{t \mid v \in X_t\}$ corresponds to the subtree of T sprouting from $f(v)$. Therefore, (T, X) is a tree decomposition of G. The size of its bags is bounded from above by the length of the branches of F, which is at most $\mathrm{H}(F) = \mathrm{td}(G)$. The width of (T, X) is thus at most $\mathrm{td}(G) - 1$. □

5.3.2 Ding's Theorem

Every graph class with bounded tree-depth is wqo by the minor relation due to Theorem 5.13, because it has bounded tree-width from Lemma 5.31. Ding [Din92] further proved that, for finite graphs, bounded tree-depth classes are wqo by the induced subgraph relation. In this section, we show that Ding's result extends to infinite graphs, proving Theorem 2.

We extend the induced subgraph relation to labeled graphs by letting

$$(H, \ell_H) \subseteq_i (G, \ell_G)$$

if there exists an injection $f \colon V_H \to V_G$ such that

$$uv \in E_H \iff f(u)f(v) \in E_G$$

and $\ell_H(v) \leq \ell_G(f(v))$ for all $v \in V_H$.

Theorem 5.32 (Ding [Din92])**.** *The following statements hold for every* $n \geq 1$.

(a) *If Q is wqo, then the class of all finite Q-labeled graphs with tree-depth at most n is wqo by the induced subgraph relation.*

(b) *If Q is wqo, then the class of all finite Q-labeled graphs that do not contain P_n as a subgraph is wqo by the induced subgraph relation.*

These two statements are clearly equivalent due to Lemma 5.27. Notice that Ding's theorem is not optimal in the sense that there are classes of finite graphs closed under taking subgraphs that are wqo by $\subseteq_i$, yet have unbounded tree-depth (for instance, the class of all disjoint unions of paths). Theorem 5.21 provides a more complete characterization. However, Theorem 5.32 becomes optimal if Q contains at least two non-equivalent elements $x \not\leq y$. We can then consider paths P_n for $n \geq 3$ with their end-vertices labeled by x and internal vertices labeled by y. Observe that this is an infinite antichain.

Proof of Ding's theorem using Theorem 5.19 (cf. [Liu20]). Let $G_0, G_1, \dots$ be an infinite sequence of finite Q-labeled graphs without a P_n subgraph. Define a new wqo Q' by adding an extra element x into Q, incomparable to all original $y \in Q$. For each $i \in \omega$, let G_i' be the graph obtained from G_i by replacing each edge by a Robertson chain of length 2 (we identify the ends of the Robertson chain with the vertices of the original edge) and labeling the middle vertices of these chains with x. Then $G_i' \preceq_t G_j'$ with respect to $Q' \iff G_i \subseteq G_j$ with respect to Q. It is also easy to see that if G_i does not contain P_n as a subgraph, then G_i' does not contain R_{2n} as a topological minor. By Theorem 5.19 there are $i < j$ such that $G_i' \preceq_t G_j'$ and thus also $G_i \subseteq G_j$. Hence, the class of all finite Q-labeled graphs without a P_n subgraph is wqo by the subgraph relation, and by Theorem 5.21 also by the induced subgraph relation. □

We show that Theorem 5.32 extends to classes of possibly infinite graphs. Our proof follows the ideas already present in Ding's original argument, replacing certain uses of wqo theory with corresponding results from bqo theory. For this reason, we omit Ding's original proof of Theorem 5.32, as doing so would largely duplicate our presentation. We denote by $\mathcal{D}_n(Q)$ the class of all (finite or infinite) Q-labeled graphs with tree-depth at most n.

Theorem 5.33. *Let $n \in \omega$. If Q is bqo, then $\mathcal{D}_n(Q)$ is bqo by $\subseteq_i$.*

Proof. We proceed by induction on n. The case $n = 1$ holds by Theorem 3.13 (iv). Assume that $n > 1$, and that the claim holds for $n - 1$. Let $f : B \to \mathcal{D}_n(Q)$ be a $\mathcal{D}_n(Q)$-array. We need to find $s \triangleleft t$ in B such that $f(s) \subseteq_i f(t)$. For $s \in B$, denote by (G_s, ℓ_s) the Q-labeled graph $f(s)$. Color an element $s \in B$ red if G_s has tree-depth 1, and blue otherwise. Lemma 3.9 implies that there exists a monochromatic sub-barrier $B' \subseteq B$. If B' is red, then $f \restriction B'$ is a good array from the base case $n = 1$, and thus f is also good. Assume that B' is blue; then every graph in the sub-array $f \restriction B'$ has tree-depth at least 2.

Define an enriched bqo $Q^+ \coloneqq Q \times \{0, 1\}$ ordered by the rule that

$$(x, e) \leq (x', e')$$

if $x \leq x'$ and $e = e'$. Note that Q^+ indeed is bqo from Proposition 3.12 (a) and Theorem 3.13 (ii). For each $s \in B'$, invoke Proposition 5.26 and pick a vertex $v_s \in V(G_s)$ such that $\mathrm{td}(G_s - v_s) \leq n - 1$. Consider the Q-array $g : B' \to Q$ defined by $g(s) \coloneqq \ell_s(v_s)$. Lemma 3.10 implies that there is a barrier $B'' \subseteq B'$ such that whenever $s, t \in B''$ and $s \triangleleft t$, then $g(s) \leq g(t)$.

For $s \in B''$ put $H_s \coloneqq G_s - v_s$ and define a Q^+-labeling of H_s by letting $\ell_s^+(u) \coloneqq (\ell_s(u), e_s(u))$ for $u \in V(H_s)$, where $e_s(u)$ is 1 if u and v_s are adjacent in G_s, and 0 otherwise. Define a $\mathcal{D}_{n-1}(Q^+)$-array $f'' : B'' \to \mathcal{D}_{n-1}(Q^+)$ by letting $f''(s) \coloneqq (H_s, \ell_s^+)$. From the induction hypothesis, $\mathcal{D}_{n-1}(Q^+)$ is bqo by $\subseteq_i$, and so there exists a pair $s \triangleleft t$ in B'' such that $(H_s, \ell_s^+) \subseteq_i (H_t, \ell_t^+)$. Let a mapping $\varphi : V(H_s) \to V(H_t)$ witness this. Since ℓ_s^+ and ℓ_t^+ exactly encode which vertices of H_s and H_t are adjacent to v_s and v_t, respectively, and since $\ell_s(v_s) \leq \ell_t(v_t)$ by our choice of B'', we deduce that $\varphi \cup \{(v_s, v_t)\}$ is a witness of $(G_s, \ell_s) \subseteq_i (G_t, \ell_t)$. □

5.3.3 Hereditary Graph Classes

Ding's theorem states that for every n, the class $\mathcal{P}_n$, consisting of all finite graphs that do not contain P_n as a subgraph, is wqo by the induced subgraph relation. It is natural to ask what happens when we forbid P_n as an *induced* subgraph.

More generally, we say that a class of finite graphs is *monogenic* or *H-free* if it is characterized by a single forbidden induced subgraph H. It is not difficult to see that a class of (finite) graphs $\mathcal{G}$ can be characterized in terms of forbidden induced subgraphs $\iff$ $\mathcal{G}$ is closed under taking induced subgraphs. Indeed, one only needs to take the $\subseteq_i$-minimal elements of the class of all (finite) graphs not present in $\mathcal{G}$. Such graph classes are said to be *hereditary*. Note that we only consider finite graphs in this section.

Theorem 5.34 (Damaschke [Dam90])**.** *The class of all H-free graphs is wqo by the induced subgraph relation $\iff$ H is an induced subgraph of P_4.*

Before we prove this, note that P_4-free graphs are called *cographs*. The *join* $G \otimes H$ of graphs G and H is the graph $(G^c + H^c)^c$, where G^c denotes the complement of G. Notice that the subgraph of $G \otimes H$ induced by V_G is exactly G. Cographs can be characterized in multiple ways [CLB81]:

Lemma 5.35. *The following conditions are equivalent for a graph G.*

(i) *G is P_4-free (a cograph).*

(ii) *G can be obtained from start graphs K_1 by the operations of disjoint union and complementation.*

(iii) *G can be obtained from start graphs K_1 by the operations of disjoint union and join.*

It is easy to see that the third characterization allows us to represent a cograph G as a binary order-theoretic tree T_G labeled by $Q := \{0, 1, 2\}$ that shows how the start graphs have been combined to form G. The leaves of T_G are the vertices of G and are labeled with 2. The labels 0 and 1 correspond to the operations $+$ and $\otimes$, respectively. We leave the formal definition of T_G to the reader. Observe that T_G has the property that $uv \in E_G \iff$ the meet $u \wedge v$ has label 1. We denote the labeling function of T_G by ℓ_G and let Q be ordered by equality.

Lemma 5.36. *Let H and G be cographs. If $T_H \preceq T_G$, then $H \subseteq_i G$.*

Proof. Let $\varphi\colon T_H \to T_G$ be a homeomorphic embedding, and note that each leaf of T_H is mapped to a leaf of T_G since $\ell_H(v) = \ell_G(\varphi(v))$ for all $v \in T_H$. Moreover, we have $uv \in E_H \iff \ell_H(u \wedge v) = 1$ and $\varphi(u \wedge v) = \varphi(u) \wedge \varphi(v)$. Combining this with

$$\ell_H(u \wedge v) = \ell_G\big(\varphi(u) \wedge \varphi(v)\big) = 1 \iff \varphi(u)\varphi(v) \in E_G$$

yields that $uv \in E_H \iff \varphi(u)\varphi(v) \in E_G$. Therefore φ witnesses H being an induced subgraph of G. □

Proof of Theorem 5.34. Let $\mathcal{H}$ be the class of all H-free graphs for some fixed graph H, and assume that $\mathcal{H}$ is wqo. Since cycles form an infinite $\subseteq_i$-antichain, $\mathcal{H}$ contains only finitely many cycles. Hence there exists some n such that $H \subseteq_i C_k$ for every $k \geq n$. It is easy to see that H must be *path-like*; that is, an induced subgraph of a path. Recall that

$$H \subseteq_i G \iff H^c \subseteq_i G^c.$$

Consequently, $\mathcal{H}$ is wqo $\iff$ $\mathcal{H}^c := \{G^c \mid G \in \mathcal{H}\}$ is wqo, and $\mathcal{H}^c$ is the class of all H^c-free graphs. Thus H^c is also path-like. Finally, observe that H and H^c are path-like $\iff$ $H \subseteq_i P_4$. Hence $H \subseteq_i P_4$ is a necessary condition for $\mathcal{H}$ to be wqo. Lemma 5.36 implies that P_4-free graphs are wqo due to Kruskal's theorem, so the above condition is also sufficient. □

Remark. The above proof also shows that if Q is wqo, then the class of all P_4-free Q-labeled graphs is wqo by $\subseteq_i$. We simply need to encode the labels of a given graph G into the labels of T_G, which is easily done via the product quasi-ordering.

A corollary of Theorem 5.34 is that P_5-free graphs are no longer wqo. However, Damaschke also showed that the class of all (P_5, K_3)-free graphs is again wqo. We say that a class of graphs is *bigenic* or (H_1, H_2)*-free* if it is characterized by two $\subseteq_i$-incomparable forbidden induced subgraphs, H_1 and H_2.

Hereditary classes, and bigenic classes in particular, are an active area of research [DLP18; Bon+21; Lop24] due to their strong connection to a graph parameter known as *clique-width.* Clique-width is closely related to tree-width; it generalizes tree-width in the sense that graph classes with bounded tree-width have bounded clique-width, but not necessarily vice versa. Because clique-width is $\subseteq_i$-monotone (though not minor-monotone: a graph's clique-width can be significantly smaller than that of its minor), the property of having bounded clique-width is itself a hereditary property.

It is important to note that not all graph classes with bounded clique-width are wqo; for example, every cycle has a clique-width of at most 4, yet the set of all cycles forms an infinite $\subseteq_i$-antichain. Daligault, Rao, and Thomassé [DRT10] raised the inverse question: does every hereditary class that is wqo by $\subseteq_i$ have bounded clique-width? This was soon after disproved by Lozin, Razgon, and Zamaraev [LRZ15]. However, their counterexample relied on a hereditary class of graphs whose set of minimal forbidden induced subgraphs is infinite. The question remains open for *finitely defined* hereditary classes — that is, classes whose set of minimal forbidden induced subgraphs is finite.

Conjecture 5.37. *If a finitely defined hereditary graph class $\mathcal{G}$ is wqo by the induced subgraph relation, then $\mathcal{G}$ has bounded clique-width.*

Note that Theorem 5.34 confirms this conjecture for monogenic classes since a graph has clique-width $\leq 2 \iff$ it is P_4-free. All results for bigenic classes continue to support the conjecture as well. Dabrowski, Lozin, and Paulusma [DLP18] state that currently, there are only two bigenic graph classes left for which the conjecture remains to be verified:

- $(K_3, P_2 + P_4)$-free graphs, and
- $(\overline{P_1 + P_4}, P_2 + P_3)$-free graphs.

5.4 WQO by the Subgraph Relation

The question of which graph classes closed under taking subgraphs are wqo by the subgraph relation is answered by Theorem 5.21. In this section, we consider classes of graphs that are *not* closed under taking subgraphs, and while doing so prove Theorem 3. The monogenic classes introduced in the previous section are a natural candidate for non-closed classes. For a finite graph H, denote by $\mathcal{G}(H)$ the class of all (finite or infinite) H-free graphs.

Lemma 5.38. *$\mathcal{G}(H)$ is closed under taking subgraphs $\iff$ H is a clique.*

Proof. ($\Leftarrow$) This is true since $K_n \subseteq_i G \iff K_n \subseteq G$ for every graph G.

($\Rightarrow$) Suppose that there are vertices u and v in H that are not adjacent. Then the graph $H + uv$ is H-free, but the subgraph $H \subseteq H + uv$ is not. □

Lemma 5.39. *If $\mathcal{G}(H)$ is wqo by $\subseteq$, then H is an induced subgraph of a path.*

Proof. If a class of graphs is wqo by $\subseteq$, then it may contain only finitely many cycles since cycles form an infinite $\subseteq$-antichain. Hence $H \subseteq_i C_k$ for all sufficiently large k, and it is easy to see that H must be as described. □

We will now show that if H is an independent set, then $\mathcal{G}(H)$ is wqo, proving Theorem 3. Recall that E_n denotes the independent set on n vertices.

Theorem 5.40. *For every $n \in \omega$, the class $\mathcal{G}(E_n)$ is wqo by the subgraph relation.*

This is equivalent to the statement that graph classes with bounded independence number are wqo because

$$E_n \not\subseteq_i G \iff \alpha(G) < n.$$

I would like to express my thanks to professor Nešetřil for pointing out the finite-graphs case of this theorem.

Proof. Let $G_0, G_1, G_2, \dots$ be a sequence of graphs with independence number at most n. Suppose first that all of these graphs are finite. If their size is bounded, then one of them has to repeat; that is, there are indices $i < j$ such that $G_i = G_j$. Hence, assume that $|G_i| \to \infty$ as $i \to \infty$. Since these graphs contain no large independent sets, Ramsey's theorem implies that $\omega(G_i) \to \infty$ as $i \to \infty$. In particular, there exists $i > 1$ such that $\omega(G_i) \geq |V(G_1)|$, and thus $G_1 \subseteq G_i$.

We can therefore assume that all of the graphs in the sequence are infinite. We invoke the famous Erdős–Dushnik–Miller theorem ([DM41], see also [ER56] for extensions), which states that if κ is an infinite cardinal, then every graph of size κ contains either a clique of size κ or an independent set of size $\aleph_0$. Since $\alpha(G_i) \leq n$, we have $\omega(G_i) = |G_i|$ for all $i \in \omega$. Because ordinals are well-ordered, there are indices $i < j$ such that $|G_i| \leq |G_j|$, and thus $G_i \subseteq G_j$. □

Notice that this implies that the restriction to subgraph-closed graph classes in Theorem 5.21 is necessary, since from Theorem 5.34 we get that:

Observation 5.41. *For $n \geq 3$, the subclass of $\mathcal{G}(E_n)$ consisting of finite graphs is not wqo by the induced subgraph relation. Thus $\mathcal{G}(E_n)$ is not wqo by $\subseteq_i$ either.*

Remark. One can also obtain this directly by realizing that E_n-free graphs are complements of K_n-free graphs. Hence $\mathcal{G}(E_n)$ is wqo by $\subseteq_i \iff \mathcal{G}(K_n)$ is wqo by $\subseteq_i$, since $H \subseteq_i G \iff \overline{H} \subseteq_i \overline{G}$. But $\mathcal{G}(K_3)$ is not wqo by $\subseteq_i$ because all forks (see the beginning of Section 5.3) are K_3-free.

Next, we show that the labeled version of Theorem 5.40 fails in general, but that it holds for "almost complete" graphs. We extend the subgraph relation to labeled graphs by letting

$$(H, \ell_H) \subseteq (G, \ell_G)$$

if there exists an injection $f\colon V_H \to V_G$ such that

$$uv \in E_H \implies f(u)f(v) \in E_G$$

and $\ell_H(v) \leq \ell_G(f(v))$ for all $v \in V_H$.

Proposition 5.42. *There exists a bqo set Q such that the class of all finite Q-labeled graphs G with $\alpha(G) \leq 2$ is not wqo by $\subseteq$.*

Proof. Let $Q = (\{0,1\}, =)$. For $n \geq 2$, define a graph G_n by taking two copies of K_n and adding an even cycle between them. More formally, G_n has $2n$ vertices, partitioned into two cliques: $X_n = \{x_1, \ldots, x_n\}$, where every vertex is labeled 0, and $Y_n = \{y_1, \ldots, y_n\}$, where every vertex is labeled 1. We also add *cross-edges* to form a cycle of length $2n$. Specifically, add edges $x_i y_i$ for all $1 \leq i \leq n$, and $x_{i+1} y_i$ for all $1 \leq i < n$, plus the closing edge $x_1 y_n$. Clearly, $\alpha(G_n) \leq 2$.

We claim that this is an infinite $\subseteq$-antichain. Assume for contradiction that there is a mapping $f\colon V(G_n) \to V(G_m)$ witnessing G_n being a label-respecting subgraph of G_m for some $n \neq m$. Then $f[X_n] \subseteq X_m$ and $f[Y_n] \subseteq Y_m$, and every cross-edge of G_n must be mapped to a cross-edge of G_m. Therefore f witnesses that $C_{2n} \subseteq C_{2m}$, which is false. □

This stands in contrast to all other graph-related wqo results presented in this thesis so far. While in all other cases, the wqo property was true for wqo-labeled graphs as well, this counter-example shows that no analogous extension of Theorem 5.40 to labeled graphs is possible.

Notice that although the graphs used in the counter-example had no large independent sets, the difference $|V_G| - \omega(G)$ was unbounded. We will show that this is the only obstruction to a labeled version of Theorem 5.40. We say that a (possibly infinite) graph G is an *almost k-clique* for some $k \in \omega$, if there exists a clique $C \subseteq V_G$ such that $|V_G \setminus C| \leq k$. Note that we abuse notation here, identifying a clique with the set of its vertices. For a quasi-ordered class Q, denote by $\mathcal{K}_k(Q)$ the class of all Q-labeled almost k-cliques.

Theorem 5.43. *Let $k \in \omega$. If Q is bqo, then $\mathcal{K}_k(Q)$ is bqo by $\subseteq_i$, and thus by $\subseteq$.*

Remark. If we restrict ourselves to finite graphs, then the same implication holds for wqos as well. One can easily modify the proof below to show this.

Proof. The case $k = 0$ is the $\sqsubseteq_1$ version of Theorem 3.13 (iv). For $k > 0$, we proceed by induction. Let $f\colon B \to \mathcal{K}_k(Q)$ be a $\mathcal{K}_k(Q)$-array. For $s \in B$ denote by (G_s, ℓ_s) the Q-labeled graph $f(s)$. As in the proof of Theorem 5.33, we define an enriched bqo $Q^+ \coloneqq Q \times \{0,1\}$ ordered by the rule that

$$(x, e) \leq (x', e')$$

if $x \leq x'$ and $e = e'$. For $s \in B$, let $C_s \subseteq V(G_s)$ be a clique in G_s such that $|V(G_s) \setminus C_s| \leq k$. If B contains a sub-barrier B' such that $|V(G_s) \setminus C_s| = 0$ for all $s \in B'$, then the sub-array $f \restriction B'$ is good from the base case $k = 0$, so f is also good. We can therefore (by Lemma 3.9) assume that there exist vertices $v_s \in V(G_s) \setminus C_s$ for all $s \in B$. We can also assume (by Lemma 3.10) that whenever we have $s \triangleleft t$, then $\ell_s(v_s) \leq \ell_t(v_t)$.

Note that $H_s \coloneqq G_s - v_s$ is an almost $(k-1)$-clique. Define a Q^+-labeling ℓ_s^+ of H_s by letting $\ell_s^+(u) \coloneqq (\ell_s(u), e_s(u))$ for $u \in V(H_s)$, where $e_s(u)$ is 1 if u and v_s are adjacent in G_s, and 0 otherwise. From the induction hypothesis, $\mathcal{K}_{k-1}(Q^+)$ is bqo, so there are $s \triangleleft t$ in B such that $(H_s, \ell_s^+) \subseteq_i (H_t, \ell_t^+)$. By the same argument as in the proof of Theorem 5.33 we have that also $(G_s, \ell_s) \subseteq_i (G_t, \ell_t)$. □

6 Generalized WQOs

In this chapter, we study certain natural generalizations of well-quasi-orderings. We begin by reviewing basic results on graph homomorphisms and rigid relations following [HN04]. We then formulate Vopěnka's principle and establish several equivalent characterizations, which lead to the concept of *class-wqos*. Lastly, we consider *κ-wqos*, a related notion introduced by Shelah [She82].

Digraphs A *digraph* G is a pair (V, E) where V is an arbitrary nonempty set of *vertices* and $E \subseteq V \times V$ is a set of directed *arcs*. We say that G is *irreflexive* or *symmetric* when the relation E is irreflexive or symmetric, respectively. We adopt the same notations as for graphs, since graphs can be viewed as irreflexive symmetric digraphs. A *subgraph* of a digraph G is any digraph H such that $V_H \subseteq V_G$ and $E_H \subseteq E_G \cap V_H^2$. The *corresponding symmetric digraph* of a graph G is obtained by replacing each edge $\{u, v\} \in E_G$ with the two arcs (u, v) and (v, u). An *orientation* of a graph G is a digraph obtained by replacing each edge $\{u, v\} \in E_G$ with exactly one of the arcs (u, v) or (v, u). Finally, the *underlying graph* of a digraph G is the graph with the same vertices as G, in which $\{u, v\}$ is an edge $\iff$ at least one of (u, v) or (v, u) is an arc of G. We say that a digraph G is *connected* if its underlying graph is connected. If clear from the context, we shall write $uv \in E$ instead of $(u, v) \in E$.

6.1 The Homomorphism Order

Let H and G be any digraphs. A *homomorphism* of H to G, written $f\colon H \to G$, is a mapping $f\colon V_H \to V_G$ such that

$$uv \in E_H \implies f(u)f(v) \in E_G.$$

We write $H \to G$ and we say that H is *homomorphic* to G if there exists a homomorphism of H to G, and we write $H \not\to G$ otherwise. It is easy to see that the homomorphism relation $\to$ is a quasi-order on the class of all graphs. Note that it suffices to consider homomorphisms for digraphs, as the definition can be applied to graphs via their corresponding symmetric digraphs.

We state a few simple observations about homomorphisms to build intuition. A *walk* of length k in a graph (digraph) G is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that each consecutive pair $v_i v_{i+1}$ is an edge (arc) of G. A walk is *closed* if $v_0 = v_k$. Walks and closed walks with distinct vertices (apart from $v_0 = v_k$ for closed walks) in digraphs are called *directed paths* and *directed cycles*, respectively. We denote the directed path on n vertices by $\vec{P}_n$, and the directed cycle on n vertices by $\vec{C}_n$. Note that $\vec{C}_2$ is the corresponding symmetric digraph of K_2. It is easy to see that homomorphisms map walks to walks and closed walks to closed walks. In particular, a homomorphism of a graph H to a graph G maps paths and cycles of H to walks and closed walks of G, respectively.

Furthermore, notice that if G is bipartite, then $G \to K_2$. If G has at least one edge then obviously also $K_2 \to G$. From transitivity, we get that $H \to G$ for any two bipartite graphs H and G with at least one edge. In particular, $C_{2k} \to C_{2l}$ for all $k, l \geq 2$. On the other hand:

Observation 6.1. $C_{2k+1} \to C_{2l+1} \iff 1 \leq l \leq k$.

Proof. An odd cycle of length n has no closed odd walks shorter than n, but it has a closed walk of any odd length greater than or equal to n. $\square$

The *odd girth* $g_o(G)$ of a graph G is the length of a shortest odd cycle in G.

Proposition 6.2. *If $H \to G$, then $g_o(G) \le g_o(H)$.*

Proof. Let $f\colon H \to G$ be a homomorphism. Every odd cycle in H of length k is mapped by f to a closed walk in G of the same length. If this walk is not a cycle, it can be divided into two shorter closed walks, one of them having odd length. By iterating this, we find an odd cycle in G of length at most k. $\square$

Corollary 6.3. *If $g_o(H) < g_o(G)$, then $H \not\to G$.*

Homomorphisms and colorings Let κ be a cardinal number. A *κ-coloring* of a graph G is a mapping $f\colon V_G \to \kappa$ such that $uv \in E_G \implies f(u) \neq f(v)$. The *chromatic number* $\chi(G)$ of G is the least κ such that a κ-coloring of G exists. Recall that the complete graph K_κ has vertex set $\kappa = \{\alpha \mid \alpha < \kappa\}$, and observe that the homomorphisms $f\colon G \to K_\kappa$ are precisely the κ-colorings of G. We can thus view homomorphisms to a graph H as generalized colorings, where V_H represents the set of available colors, and $xy \in E_H$ indicates that the colors x and y are compatible. A homomorphism $f\colon G \to H$ is then an assignment of colors to vertices of G such that each pair of adjacent vertices receives compatible colors.

Proposition 6.4. *If $H \to G$, then $\chi(H) \le \chi(G)$.*

Proof. Let $h\colon H \to G$ be a homomorphism. Whenever $f\colon G \to K_\kappa$ is a κ-coloring of G, then $h \circ f$ is a κ-coloring of H. $\square$

Corollary 6.5. *If $\chi(H) < \chi(G)$, then $G \not\to H$.*

Isomorphisms Let H and G be any digraphs. An *isomorphism* of H to G is a bijective mapping $f\colon V_H \to V_G$ such that

$$uv \in E_H \iff f(u)f(v) \in E_G.$$

Equivalently, a map f is an isomorphism $\iff$ f is bijective and f, f^{-1} are both homomorphisms. An isomorphism $f\colon G \to G$ is called an *automorphism* of G. Similar to graphs, we identify isomorphic digraphs.

Cores and retracts If $H \to G$ and $G \to H$, then we say that H and G are *homomorphically equivalent*. Note that this is indeed an equivalence relation. Furthermore, due to Propositions 6.2 and 6.4 we get the following observation:

Observation 6.6. *Homomorphically equivalent graphs have the same odd girth and the same chromatic number.*

Let H be a subgraph of a digraph G. A *retraction* of G to H is a homomorphism $r\colon G \to H$ such that $r(v) = v$ for all $v \in V_H$. If a retraction of G to H exists, we say that G *retracts to* H and that H is a *retract* of G. Clearly, any graph is homomorphically equivalent to its retracts.

Observe that if $\chi(G) = n$ and $K_n \subseteq G$, then K_n is a retract of G. However, K_n does not have a retract other than itself, and the same holds for odd cycles. Such graphs are called *cores*. More formally, a finite digraph G is a *core* if it does not retract to any proper subgraph (a subgraph of G not isomorphic to G).

Lemma 6.7. *A finite digraph G is a core $\iff$ it is not homomorphic to any proper subgraph of itself.*

Proof. ($\Leftarrow$) If G retracts to a proper subgraph, then it is homomorphic to it.

($\Rightarrow$) If G is homomorphic to a proper subgraph of itself, then let H be a proper subgraph of G minimal in $|V_H|$ such that there is a homomorphism $f\colon G \to H$. Then $H \not\to H'$ for any proper subgraph $H' \subseteq H$, and hence any homomorphism of H to H must be an automorphism of H. Thus $h \coloneqq f \restriction V_H$ is an automorphism, and clearly $f \circ h^{-1}$ is a retraction of G to H. □

We can thus find a subgraph of G that is a core by repeatedly taking homomorphisms to proper subgraphs until it is no longer possible.

Theorem 6.8. *Every finite digraph G is homomorphically equivalent to a unique (up to isomorphism) core H. We say that H is* the core *of G.*

For example, the core of any bipartite graph with at least one edge is K_2.

Proof. Let H and H' be retracts of G that are cores. Since G and its retracts are homomorphically equivalent, there exist homomorphisms $f\colon H \to H'$ and $g\colon H' \to H$. The previous lemma implies that $f \circ g$ and $g \circ f$ must be automorphisms, so H and H' are isomorphic. □

Corollary 6.9. *A finite digraph G is a core $\iff$ every homomorphism of G to itself is an automorphism.*

The homomorphism order Write $H \leq G$ if $H \to G$. Since $\leq$ is a quasi-order, we can write $H < G$ and consider $\equiv$-equivalent elements. When restricted to finite digraphs, this has the nice property that every equivalence class contains a unique core, which we may choose as the representative of that class. Denote by $\mathcal{C}$ the class of all such representatives that are graphs (recall that we view graphs as a subclass of digraphs), and by $\mathcal{D}$ the class of all digraphs. While it might not seem that way, the orders $(\mathcal{C}, \leq)$ and $(\mathcal{D}, \leq)$ are extremely rich.

Theorem 6.10 (Hedrlín [Hed69])**.** *The order $(\mathcal{C}, \leq)$ is countably universal, meaning that every countable partial order $(P, \preceq)$ embeds into $(\mathcal{C}, \leq)$.*

Here, an *embedding* of $(P, \preceq)$ into $(\mathcal{C}, \leq)$ is an injection $\varphi\colon P \to \mathcal{C}$ such that for all $x, y \in P$, we have $x \preceq y \iff \varphi(x) \leq \varphi(y)$.

Proof. See [Fia+17] for a simpler proof. □

To see that something like this might hold, notice that complete graphs form an infinite chain isomorphic to ω, and that odd cycles form an infinite chain isomorphic to the reverse order ω^*. To construct an infinite antichain, we use a classical result of Erdős [Erd59], by which for all integers $g \geq 3$ and $k \geq 1$, there exists a graph $S(g, k)$ with odd girth at least g and chromatic number at least k. By our previous discussion regarding these two notions, it is easy to see that we can use the graphs $S(g, k)$ to construct an infinite antichain in $\mathcal{C}$ by letting G_i be the core of $S(i, i)$ and keeping only a subset of these graphs so that $i < j$ implies $\chi(G_i) < \chi(G_j)$ and $\mathrm{g}_o(G_i) < \mathrm{g}_o(G_j)$. This also means that $(\mathcal{C}, \leq)$ is not wqo.

Theorem 6.11 (Hedrlín, Pultr; see Chapter II of [PT80])**.** *The order $(\mathcal{D}, \leq)$ is universal, meaning that every partially ordered set $(P, \preceq)$ embeds into $(\mathcal{D}, \leq)$. In fact, even the order of connected graphs is universal [PT80, Ch. IV, Thm. 3.3].*

Remark. Something can also be said about partially ordered proper classes, but set-theoretic complications arise. We will come back to this in the more suitable setting of Section 6.4.

Theorem 6.12 (Welzl [Wel82])**.** *The order* $(\mathcal{C}, \leq)$ *is dense, meaning that for every pair of graphs* $G_1 < G_2$ *there exists* H *such that* $G_1 < H < G_2$*, with the single exception of* $K_1 < K_2$*.*

Proof. See [Neš99] for a simpler proof of the more general statement that the class of *all* graphs is dense, with the single exception of $K_1 < K_2$. ☐

For example: $K_2 < \cdots < C_7 < C_5 < C_3 = K_3$.

Theorem 6.13 (Fiala, Hubička, Long, Nešetřil [Fia+17])**.** *The order* $(\mathcal{C}, \leq)$ *has the fractal property, meaning that for every pair of graphs* $G_1 < G_2$*, distinct from* $K_1 < K_2$*, the order* $(\mathcal{C}, \leq)$ *embeds into the interval* $\{H \mid G_1 < H < G_2\}$*.*

6.2 Rigid Relations

In this section, we study rigid graphs and digraphs, following Chapter 4 of [HN04].

We say that a digraph G is *asymmetric* if it has no automorphism other than the identity, and *rigid* if it has no endomorphism (a homomorphism $f : G \to G$) other than the identity. Note that a finite digraph G is rigid $\iff$ it is asymmetric and a core. We call a class $\mathcal{F}$ of digraphs an *incomparable family* if every pair of digraphs $H \neq G$ from $\mathcal{F}$ satisfies that $H \not\to G$ and $G \not\to H$. To avoid trivialities, we shall not consider single-vertex digraphs to be rigid.

Finding rigid digraphs is not difficult: for instance, every directed path $\vec{P}_n$ on at least two vertices is rigid. This can be easily generalized:

Lemma 6.14. *Any acyclic digraph that has a directed Hamiltonian path is rigid.*

By an *acyclic digraph*, we mean a digraph that does not contain any directed cycle $\vec{C}_n$ as a subgraph. A *directed Hamiltonian path* in a digraph G is a directed path in G that visits every vertex.

Proof. No endomorphism can identify two vertices, as this would create a directed cycle due to the Hamiltonian path, so the graph is a core. Note that the Hamiltonian path is unique (otherwise there would be a directed cycle), and so there is no automorphism other than the identity. ☐

This allows us to construct many rigid digraphs. Consider the digraph $\vec{P}_n(i, j)$ obtained from $\vec{P}_n$ by adding an arc from the i-th vertex of the path to the j-th vertex of the path for some $j > i + 1$ (see Figure 6.1). These digraphs are rigid, and the above proof also shows that $\vec{P}_n(i, j) \to \vec{P}_n(i', j')$ only if $i = i'$ and $j = j'$.

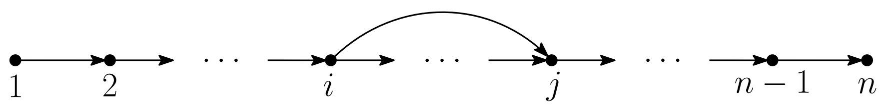

Figure 6.1: The rigid digraph $\vec{P}_n(i, j)$.

Proposition 6.15. *Let* $n \geq 3$*. Then the digraphs* $\vec{P}_n(i, j)$ *for* $1 \leq i \leq n - 2$ *and* $i + 2 \leq j \leq n$ *form an incomparable family of rigid digraphs.*

Finding rigid *graphs* is substantially harder and has a long tradition [Neš09]. There exists a unique minimal rigid graph, and it has 8 vertices and 14 edges. It was discovered by Pavol Hell and Jaroslav Nešetřil, and is depicted in Figure 6.2.

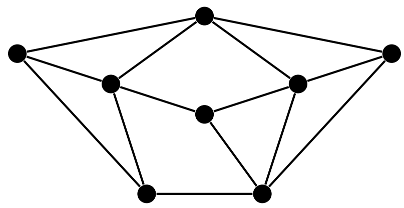

Figure 6.2: The smallest rigid graph.

A basic technique for constructing rigid graphs is finding asymmetric *critical* graphs. Here, a graph G is *critical* if $\chi(G') < \chi(G)$ for all subgraphs $G' \subseteq G$ with fewer vertices. Note that every critical graph is a core. Using this observation, we can argue that the graphs H_k introduced in Figure 6.3 are rigid.

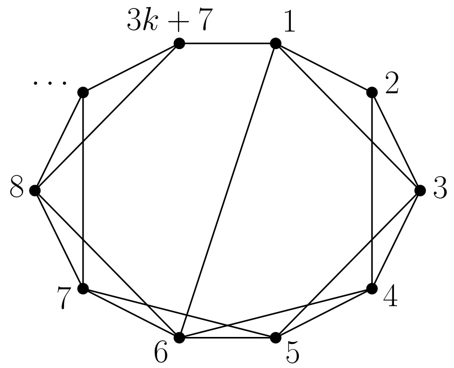

Figure 6.3: The graphs H_k for $k \geq 1$, discovered by Pavol Hell.

Proposition 6.16. *Each graph H_k is rigid.*

Proof. Observe that $\chi(H_k) = 4$. Indeed, suppose we attempt to 3-color H_k, without loss of generality assigning colors 1, 2, and 0 to vertices 1, 2, and 3, respectively. If we now try to extend the coloring to vertices $4, 5, \ldots$, the color of a vertex n must be n modulo 3, resulting in two adjacent vertices 1 and $3k + 7$ having the same color. Moreover, deleting any vertex allows us to 3-color the graph in the obvious circular fashion, so H_k is a core.

To see that H_k is asymmetric, notice that any automorphism must preserve degrees and hence take vertex 6 to itself (it is the only vertex with degree five). Then 1 must be mapped to itself since the edge $\{1, 6\}$ is the only edge incident with 6 and not in a triangle. Then 2 must map to itself as the only vertex of degree three not adjacent to another vertex of degree three, and thus 3 must also map to itself since it is the only vertex adjacent to both 1 and 2. It is now easy to continue like this, showing that the automorphism must be the identity. □

Thus there exist arbitrarily large rigid graphs. But something much stronger is true: asymptotically, *almost all* finite graphs are rigid (see Theorem 4.7 of [HN04]). Next, we turn to infinite graphs and digraphs.

Theorem 6.17 (Vopěnka, Pultr, Hedrlín [VPH65]). *There is a rigid digraph on every infinite vertex set.*

We present a simpler construction due to Nešetřil [Neš02].

Proof. We will construct a rigid digraph of size κ for every infinite cardinal κ, obtaining the claim since we assume the axiom of choice, so every set is bijective with a cardinal number. Let $X = \kappa$ and $X' = \{0\} \times \kappa$. We will treat these sets as disjoint copies of κ, denoting by β' the element $(0, \beta) \in X'$. Further let $\{a, b, c, a', b', c'\}$ be six additional vertices disjoint from X and X'.

We first make the arcs $\beta\gamma$ and $\beta'\gamma'$ for all $\beta < \gamma < \kappa$, the arcs $\beta\beta'$ for all $\beta < \kappa$, and the arcs $0a, ab, bc, c0, b0$ and $0'a', a'b', b'c', c'0', a'c'$. Finally, choose for every limit ordinal $\beta < \kappa$ with countable cofinality a strictly increasing sequence $\langle \beta_n \mid n \in \omega \rangle$ such that $\sup_n \beta_n = \beta$, and add a disjoint path of length $2n+3$ from β to β'_n for each n (see Figure 6.4). Note that the vertex set of the resulting digraph G has size at most $\kappa + \kappa + \kappa \cdot \omega = \kappa$.

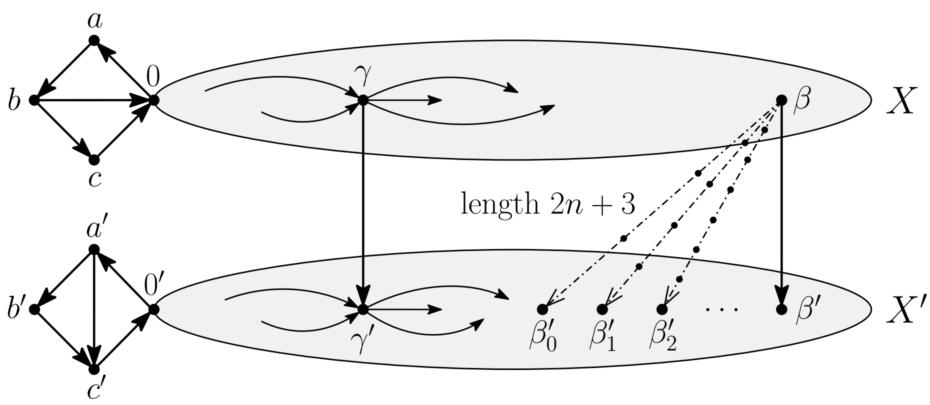

Figure 6.4: The infinite rigid digraph G.

We claim that G is rigid. Let $f : G \to G$ be a homomorphism. Since G is acyclic with the exception of $A \coloneqq \{a, b, c, 0\}$ and $A' \coloneqq \{a', b', c', 0'\}$, these sets cannot be mapped anywhere else. Moreover, $f(0) = 0$ or $f(0) = 0'$ as the vertices 0 and $0'$ are distinguished by the large cliques which include them. However, while the graphs induced by A and A' are isomorphic, there is no homomorphism between them that would map 0 to $0'$. Therefore $f(0) = 0$ and $f(0') = 0'$. It follows easily that f restricted to $A \cup A'$ is the identity, and that f maps X to X and X' to X'. Since the arcs $\beta\beta'$ are the only arcs between X and X', we have $f(\beta)' = f(\beta')$ for every $\beta < \kappa$, allowing us to focus only on the top half of the digraph. The forward arcs $\beta\gamma$ ensure that f is order preserving: whenever $\beta < \gamma$, then $f(\beta) < f(\gamma)$. We will refer to this property as *monotonicity*.

Assume that f is not the identity mapping and let α be the first ordinal such that $f(\alpha) \neq \alpha$; necessarily $f(\alpha) > \alpha$, since $f(\alpha) < \alpha$ would imply $f(f(\alpha)) < f(\alpha)$ by monotonicity, contradicting the minimality of α. Put $\gamma_0 \coloneqq \alpha$ and $\gamma_{n+1} \coloneqq f(\gamma_n)$, constructing a sequence $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$ with limit $\beta \coloneqq \sup_n \gamma_n$. Recall that we chose a cofinal sequence $\langle \beta_n \mid n \in \omega \rangle$ for β. It is easy to see that limit ordinals must be mapped to limit ordinals, and that the sequence chosen for $f(\beta)$ must be $\langle f(\beta_n) \mid n \in \omega \rangle$ due to the paths joining β to the vertices β'_n. Note that the sequence $\langle f(\gamma_n) \mid n \in \omega \rangle$ is the sequence $\langle \gamma_n \mid 1 \leq n \in \omega \rangle$. From now on, we will write $[\beta_n]$ instead of $\langle \beta_n \mid n \in \omega \rangle$ for simplicity. Since the sequences $[\beta_n]$ and $[\gamma_n]$ are both cofinal in β, they are interlacing in the sense that after each γ_n there is some β_m and conversely. Due to the monotonicity of f, the sequences $[f(\gamma_n)]$ and $[f(\beta_n)]$ are interlacing as well, and hence also $[\gamma_n]$ and $[f(\beta_n)]$. Thus $\sup_n f(\beta_n) = \sup_n \gamma_n = \beta$ and we have $f(\beta) = \beta$. Since we previously concluded that $\langle f(\beta_n) \,|\, n \in \omega \rangle$ is the sequence chosen for $f(\beta)$, we see that $f(\beta_n) = \beta_n$. But this is impossible because there are some n and m such that $\gamma_n < \beta_m < \gamma_{n+1}$, from which $f(\gamma_n) = \gamma_{n+1} > \beta_m = f(\beta_m)$, contradicting the monotonicity of f. $\square$

The indicator technique It is not hard to imagine that it should be possible to turn our infinite rigid digraph into an infinite rigid *graph* of equal cardinality by replacing each arc with a suitable finite rigid graph. Formally, a *replacement graph* is a finite graph J together with two of its vertices a and b called *connector vertices*. Given a digraph G, we denote by $G * J$ the graph obtained from G by replacing each arc $xy \in E_G$ with an isomorphic copy J_{xy} of J, identifying x with a and y with b.

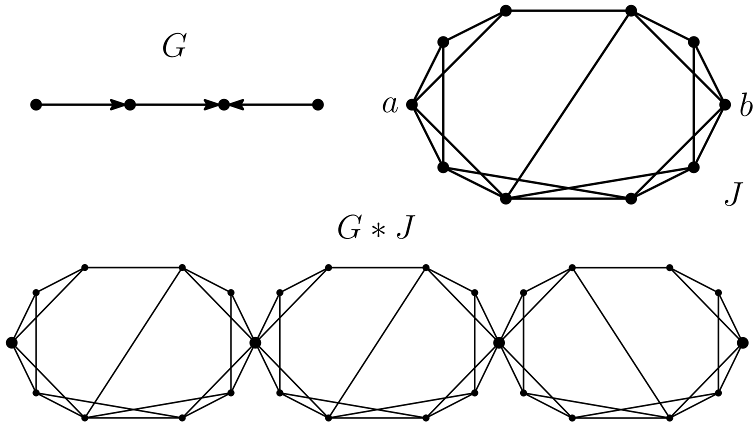

Figure 6.5: The replacement operation.

We say that J is a *strong* replacement graph if for any irreflexive digraph G, and any homomorphism $f\colon J \to G * J$, the homomorphic image $f[J]$ is contained in some copy J_{xy}. Rigid, strong replacement graphs are called *indicators*, and they are very useful:

Theorem 6.18. *Let J be an indicator graph.*

(i) *If G is a rigid digraph, then $G * J$ is a rigid graph.*

(ii) *If $\langle G_i \,|\, i \in I \rangle$ is an incomparable family of digraphs, then $\langle G_i * J \,|\, i \in I \rangle$ is an incomparable family of graphs.*

Before proceeding with the proof, we first introduce some notation. We call the vertices of $G * J$ that correspond to the original vertices of G the *branch vertices*, and the other vertices inside the individual copies of the replacement graph the *inner vertices*. Given a homomorphism $f\colon G \to H$, we define a homomorphism

$$(f * J)\colon G * J \to H * J$$

as follows. If v is a branch vertex of $G * J$, then $(f * J)(v) := f(v)$. And if v is an inner vertex in some copy J_{xy} of J in $G * J$, then $(f * J)(v)$ is the corresponding inner vertex in the copy $J_{f(x)f(y)}$ of J in $H * J$.

Lemma 6.19. *Let J be an indicator. If G and H are irreflexive digraphs without isolated vertices, then each homomorphism $h\colon G * J \to H * J$ is equal to $f * J$ for some homomorphism $f\colon G \to H$.*

Proof. Since G has no isolated vertices, each vertex of $G * J$ belongs to some copy J_{xy}. Since J is strong, each copy J_{xy} of G maps to a copy J_{uv} of H. Since J is rigid, this mapping takes corresponding vertices identically to each other. Hence restricting h to the branch vertices of $G * J$ defines a mapping f of the vertices of G to the vertices of H. Clearly f is a homomorphism satisfying $h = f * \mathcal{J}$. □

Proof of Theorem 6.18. Rigid digraphs must be irreflexive and without isolated vertices. Moreover, each member of an incomparable family must be irreflexive, and if it has some isolated vertices, we can simply forget them since they do not influence whether a homomorphism is possible. □

We will now show that the previously introduced graphs H_k can serve as indicators. Note that each H_k is *triangle-connected*; that any two of its vertices $u \neq v$ are joined by a *triangle path*, i.e., a path $u = v_1, v_2, \ldots, v_p = v$ of length at least 2 in which each v_i is adjacent to v_{i+2}, as long as $i + 2 \leq p$.

Lemma 6.20. *If J is rigid and triangle-connected, then it is a strong replacement graph with respect to any pair of nonadjacent connector vertices.*

Proof. Let $f\colon J \to G * J$ be a homomorphism to any irreflexive digraph G. Notice that the homomorphic image $f[J]$ is also triangle-connected. Suppose that $f[J]$ were not contained in a single copy J_{xy}; that some vertex of J was mapped to $u \in J_{xy}$, but another vertex of J was mapped to $v \notin J_{xy}$. Let $u = v_1, v_2, \ldots, v_p = v$ be a triangle path joining u and v in $f[J]$, and suppose that v_r is the last vertex on the path that is in J_{xy}. But then v_{r-1} and v_{r+1} are not adjacent (by our assumption on the connector vertices of J), a contradiction. □

Corollary 6.21. *There is a rigid graph on every infinite vertex set.*

Proof. Take the rigid digraph from Theorem 6.17 and replace its arcs with the indicator H_1. Note that the resulting graph is connected. □

Corollary 6.22. *For any infinite cardinal κ, there exists an incomparable family of 2^κ many rigid graphs of size κ.*

Remark. There are exactly 2^κ non-isomorphic graphs of size κ in total.

Proof. Let G be the rigid graph of size κ from the previous corollary. Since G is connected, its edge set also has cardinality κ. Since G is rigid, no two of the 2^κ possible orientations of G are isomorphic, and they form an incomparable family of rigid digraphs. We obtain an incomparable family of rigid graphs by replacing all arcs with the indicator H_1 once more. □

The indicator technique, sometimes also referred to as the "arrow construction," was pioneered by Hedrlín and Pultr in [HP65]. Other applications of this method can be found in Chapter IV of [PT80].

The edge-based replacement operation We can also consider *edge-based replacement graphs J*, where we choose two *connector edges* a_1a_2 and b_1b_2. Before performing the replacement operation on a digraph G, we first split each vertex v of G into two vertices v_1 and v_2. The replacement operation yields a graph $G \mathbin{\bar{*}} J$ obtained from G by replacing each arc $xy \in E_G$ with an isomorphic copy J_{xy} of J, identifying x_i with a_i and y_i with b_i for $i = 1, 2$.

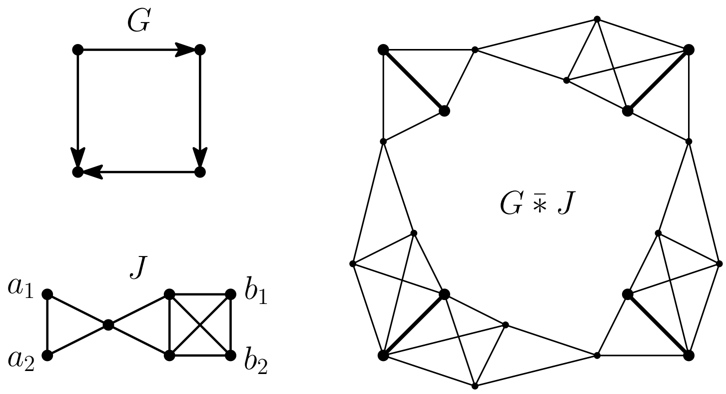

Figure 6.6: The edge-based replacement operation.

Consider the edge-based replacement graphs H_k^+, introduced in Figure 6.7. By essentially the same argument as in the proof of Proposition 6.16, these graphs are rigid. Moreover, that they have chromatic number 5. It is not difficult to see that this construction can be generalized to produce rigid graphs with arbitrarily large chromatic number.

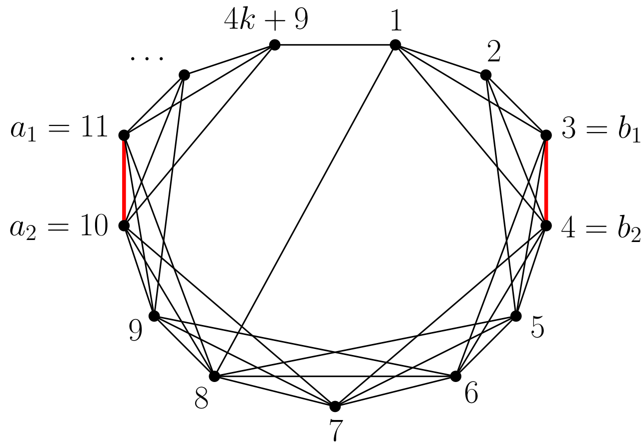

Figure 6.7: The graphs H_k^+ for $k \geq 1$.

Notice that the graphs H_k^+ are *square-connected*, meaning that any two vertices $u \neq v$ are joined by a *square path*, i.e., a path $u = v_1, v_2, \ldots, v_p = v$ of length at least 3 in which each v_i is adjacent to v_{i+2} as long as $i + 2 \leq p$, and to v_{i+3} as long as $i + 3 \leq p$. This is clearly a strengthening of triangle-connectivity.

Theorem 6.23. *Let J be one of the graphs H_k^+ from Figure 6.7.*

(i) *If G is a connected digraph, then $G \bar{*} J$ is a triangle-connected graph.*

(ii) *If G is a rigid digraph, then $G \bar{*} J$ is a rigid graph.*

(iii) *If $\langle G_i \mid i \in I \rangle$ is an incomparable family of digraphs, then $\langle G_i \bar{*} J \mid i \in I \rangle$ is an incomparable family of graphs.*

Remark. The graphs H_k would also work, but the choice of H_k^+ allows for generalization: any square-connected graph J with non-adjacent connector edges (the subgraph of J induced by a_1, a_2, b_1, b_2 is disconnected) would work for (ii) and (iii), and it is easy to describe a property that yields (i).

Proof. The edge-based replacement graphs H_k^+ are strong in the sense that for any irreflexive digraph G, and any homomorphism $f\colon J \to G \bar{*} J$, the homomorphic image $f[J]$ is contained in some copy J_{xy}. Indeed, one can verify this by going over the proof of Lemma 6.20. The claim then follows by the same argument as the one we used to prove Theorem 6.18. □

Corollary 6.24. *For any positive integer n, there exists an incomparable family consisting of n triangle-connected indicator graphs.*

Proof. The graphs $\vec{P}_n(i,j)$ introduced in the previous section are connected, and they form an incomparable family of rigid graphs. □

6.3 Relational Structures

Relational structures may be viewed as generalizations of digraphs. In particular, binary relational systems correspond to digraphs with arcs of multiple colors. This perspective will be useful in understanding Vopěnka's principle.

Let A and X be sets. An *A-ary relation* on X is a subset $R \subseteq \{\alpha \mid \alpha\colon A \to X\}$ of the set of all mappings α from A into X. If A is finite, say $|A| = n$, then we usually regard these mappings as n-tuples $(x_1, \ldots, x_n) \in X^n$. If R is an A-ary relation on X and S is an A-ary relation on Y, a mapping $f\colon X \to Y$ is a *homomorphism* of R to S if $\alpha \in R \implies \alpha \circ f \in S$ holds for all $\alpha \in R$. For $(x_1, \ldots, x_n) \in R$, this means $\big(f(x_1), \ldots, f(x_n)\big) \in S$.

A *signature* Σ is a family $\langle A_i \mid i \in I\rangle$, where I is a set. A *relational structure* of type $\Sigma = \langle A_i \mid i \in I\rangle$ on a set X is a family $\langle R_i \mid i \in I\rangle$, where R_i is an A_i-ary relation on X. If $\mathcal{R} = \langle R_i \mid i \in I\rangle$ and $\mathcal{S} = \langle S_i \mid i \in I\rangle$ are relational structures of the same type on sets X and Y, respectively, then a *homomorphism* of $\mathcal{R}$ to $\mathcal{S}$ is a mapping $f\colon X \to Y$ that is a homomorphism of R_i to S_i for each $i \in I$. We shall adopt the notation and terminology we use for homomorphisms of digraphs. We will, for instance, say that a relational structure $\mathcal{R}$ is *rigid* if the only homomorphism of $\mathcal{R}$ to $\mathcal{R}$ is the identity.

A *binary I-system* is any relational structure of type $\Sigma = \langle A_i \mid i \in I\rangle$ where $A_i = 2$ for all $i \in I$. Alternatively, a binary I-system $G = \langle E_i \mid i \in I\rangle$ on a set V is a digraph with vertex set V and $|I|$ types of arcs. We will therefore adopt the terminology for digraphs.

Given a binary I-system G and a family of indicators $\mathcal{J} = \langle J_i \mid i \in I\rangle$, we denote by $G * \mathcal{J}$ the graph obtained from G by replacing each arc xy of type $i \in I$ with an isomorphic copy $J_{xy}^{(i)}$ of J_i, as in the definition of the replacement operation for digraphs. We call the vertices of $G * \mathcal{J}$ that correspond to the original vertices of G the *branch vertices*, and the other vertices inside the individual copies of the indicators the *inner vertices*.

Given a homomorphism $f\colon G \to H$, we define a homomorphism

$$(f * \mathcal{J})\colon G * \mathcal{J} \to H * \mathcal{J}$$

as follows. If v is a branch vertex of $G * \mathcal{J}$, then $(f * \mathcal{J})(v) := f(v)$. And if v is an inner vertex in a copy $J_{xy}^{(i)}$ of J_i in $G * \mathcal{J}$, then $(f * \mathcal{J})(v)$ is the corresponding vertex in the copy $J_{f(x)f(y)}^{(i)}$ of J_i in $H * \mathcal{J}$.

Theorem 6.25. *Let $\mathcal{J} = \langle J_i \mid i \in I\rangle$ be an incomparable family of triangle-connected indicators. If G and H are binary I-systems without loops (the relations are irreflexive) and without isolated vertices (each vertex is contained in some* *arc), then each homomorphism $h\colon G * \mathcal{J} \to H * \mathcal{J}$ is equal to $f * \mathcal{J}$ for some homomorphism $f\colon G \to H$.*

Proof. Because there are no isolated vertices, each vertex of $G*\mathcal{J}$ belongs to a copy of some J_i. Because there are no loops and the indicators are triangle-connected, the same argument as in the proof of Lemma 6.20 shows that the homomorphic image of each copy of J_i must be contained in a copy of J_j for some $j \in I$. Since the indicators are incomparable, $j = i$. Since J_i is rigid, this mapping takes corresponding vertices identically to each other. Hence, restricting h to the branch vertices of $G * \mathcal{J}$ defines a mapping f of the vertices of G to the vertices of H. It is easy to see that f is the desired homomorphism $G \to H$. □

6.4 Vopěnka's Principle

In Section 6.2, we proved that there is a proper class of non-isomorphic rigid graphs, but it is not clear whether these graphs can be made *mutually* rigid; that is, an incomparable family. Petr Vopěnka conjectured in the 1960s that it cannot be done, and this statement is now known as *Vopěnka's principle*:

VP : There is no proper class of rigid graphs that is an incomparable family.

Vopěnka's principle can be seen as a (very strong) large cardinal axiom; weaker than a huge cardinal but much stronger than a measurable cardinal.[1] It in fact implies that there is a proper class of measurable cardinals [AR94, Theorem A.6]. In particular, Vopěnka's principle cannot be proved from the axioms of ZFC.

Set theory point of view We seem to be quantifying over proper classes, which the language of ZFC does not allow. When formulated in ZFC, Vopěnka's principle makes the assertion for each first-order formula defining a proper class of graphs; let us call this *Vopěnka's scheme* (VS). The definition of VP given above is a single assertion in the language of second-order set theory, and thus a proper study of this principle must be carried out in a theory whose language allows one to quantify over classes, such as the von Neumann–Bernays–Gödel set theory NBG[2] (see, for example, [FBL73, pp. 119–135]).[3]

The question of whether the full Vopěnka's principle is stronger than Vopěnka's scheme, and whether they have the same first-order consequences, was settled only quite recently by Hamkins [Ham16]. He proved that while VP and VS are not equivalent over NBG, it is true that NBG + VP is a conservative extension of ZFC + VS, meaning that they have exactly the same consequences regarding sets.

From now on, we will work in NBG and assume the *axiom of global choice*, the statement that every class has a choice function.[4]

[1]An uncountable cardinal number κ is *measurable* if there exists a non-principal ultrafilter $\mathcal{F}$ on κ such that any intersection of less than κ sets from $\mathcal{F}$ is still in $\mathcal{F}$.

[2]NBG is a conservative extension of ZF; hence if ZF is consistent, then so is NBG. It is important that even though statements of NBG can quantify over classes, its comprehension axiom is *predicative*: we cannot define new classes using formulas with class quantifiers.

[3]Alternatively, one can study Vopěnka's principle in terms of an inaccessible cardinal κ so that $(V_\kappa, \in) \models$ Vopěnka's principle in which classes are interpreted as arbitrary subsets $X \subseteq V_\kappa$ of the universe, as is done in Kanamori's famous monograph [Kan94].

[4]The axiom of global choice has the same consequences regarding sets as the usual axiom of choice. Furthermore, NBG with global choice is a conservative extension of ZFC. Since the consistency of ZF implies the consistency of ZFC, it thus also implies the consistency of NBG with global choice.

An equivalent formulation of Vopěnka's principle due to Adámek, Rosický, and Trnková [ART88], particularly useful in the study of large cardinals, is the following: For every first-order language, any proper class of structures in this language admits an elementary embedding[5] between two of its members.[6] We do not pursue this direction further and instead refer the reader to [Kan94] and [Bag+15].

Category theory point of view The above characterization of VP in terms of graphs is auxiliary: Theorem 6.25 together with Corollary 6.24 imply that VP is equivalent to the same assertion (there is no proper class of mutually rigid "objects," instead of graphs) for arbitrary binary I-systems with finitely many relations. However, much more is true: VP is equivalent to the same assertion for arbitrary relational structures! One can take this even further using *category theory*; this was, in fact, also the original motivation behind Vopěnka's principle.

The standard category theory formulation of Vopěnka's principle is:

CatVP : No locally presentable category has a large discrete full subcategory.

Briefly, a *category* consists of *objects* $A \in \mathcal{O}$ and *morphisms* $m \in \mathcal{M}$. Each morphism m has two objects associated with it: its *domain* and its *codomain.* One can think of a morphism m with domain A and codomain B as an arrow $A \xrightarrow{m} B$. Additionally, whenever $A \xrightarrow{n} B$ and $B \xrightarrow{m} C$, then there exists a morphism $n \circ m$ such that $A \xrightarrow{n \circ m} C$.[7] Every category has to satisfy two axioms. First, for every object A, there exists an *identity morphism* $A \xrightarrow{1_A} A$ such that $1_A \circ m = m$ and $n \circ 1_A = n$ whenever the compositions are defined. Second, whenever $A \xrightarrow{p} B$, $B \xrightarrow{n} C$, and $C \xrightarrow{m} D$, then $p \circ (n \circ m) = (p \circ n) \circ m$.

A category is *small* if both $\mathcal{O}$ and $\mathcal{M}$ are sets, and is *large* otherwise (at least one of them is a proper class). It is *discrete* if the only morphisms that exist are the identity morphisms. A category $\mathcal{D}$ is a *subcategory* of a category $\mathcal{C}$ if every object of $\mathcal{D}$ is an object of $\mathcal{C}$, and every morphism of $\mathcal{D}$ is a morphism of $\mathcal{C}$. It is a *full* subcategory if for every pair of objects A and B included in $\mathcal{D}$, every morphism m from $\mathcal{C}$ such that $A \xrightarrow{m} B$ is also in $\mathcal{D}$. We say that a category $\mathcal{D}$ can be *fully embedded* into a category $\mathcal{C}$ if there exists a full subcategory of $\mathcal{C}$ that is isomorphic to $\mathcal{D}$ (in the natural sense).

The *category of digraphs* **Gra** consists of all digraphs, with the morphisms being the homomorphisms between them. The *category of ordinals* **Ord** consists of ordinals, and the morphisms are induced by the standard ordinal comparison relation (whenever $\alpha \leq \beta$, we add a morphism $\alpha \xrightarrow{m} \beta$). The *category of sets* **Set** consists of all sets, and the morphisms are mappings between them. More precisely, the morphisms with domain A and codomain B are pairs (f, B), where $f \colon A \to B$ is a mapping. Finally, the *category of relational structures of type* Σ, denoted by $\mathbf{Rel}(\Sigma)$, has objects $(X, \mathcal{R})$, where X is a set and $\mathcal{R}$ is a relational structure of type Σ on X, with morphisms being homomorphisms.

[5]For two structures $\mathcal{M} = \langle M, \dots \rangle$ and $\mathcal{M}' = \langle M', \dots \rangle$ of a shared language $\mathcal{L}$, an injection $j \colon M \to M'$ is an *elementary embedding* of $\mathcal{M}$ into $\mathcal{M}'$ if for any formula $\varphi(v_1, \dots, v_n)$ of $\mathcal{L}$ and any $x_1 \dots, x_n \in M$, we have $\mathcal{M} \models \varphi(x_1, \dots, x_n) \iff \mathcal{M}' \models \varphi\big(j(x_1), \dots, j(x_n)\big)$. If also $M \subseteq M'$ and j is the identity map on M, then $\mathcal{M}$ is an *elementary substructure* of $\mathcal{M}'$.

[6]If we replaced "elementary embedding" with "elementary substructure" or "substructure," this would be provably false. Consider the structures $\mathcal{M}_\alpha = \langle \alpha + 1, \{\alpha\} \rangle$ in a language with equality and a single unary predicate (but we do not include $\in$). None of them is a substructure of another, but there is an elementary embedding taking α to β for $\omega \leq \alpha < \beta$.

[7]It is standard to write the composition of $A \xrightarrow{n} B$ and $B \xrightarrow{m} C$ as $m \circ n$ rather than $n \circ m$, but I decided to be consistent with the diagrammatic order for the composition of functions used throughout this thesis.

Theorem 6.26. *Every relational category* $\mathbf{Rel}(\Sigma)$ *is fully embeddable into* $\mathbf{Gra}$.

Proof. See Theorem 5.3 in Chapter II of [PT80]. ☐

Corollary 6.27. $\mathsf{VP} \iff$ *There is no proper class of rigid structures of the same type that forms an incomparable family.*

Proof. If such structures existed, they would induce an incomparable family of rigid digraphs. Using the indicator technique, we would obtain an incomparable family of rigid graphs, contradicting VP. ☐

We omit the formal definition of a *locally presentable* category. In essence, they are categories that can be built from a set of small, "well-behaved" building blocks. The categories $\mathbf{Gra}$ and $\mathbf{Rel}(\Sigma)$ are locally presentable, while $\mathbf{Ord}$ is not.

Let us now return to the principle CatVP via the following theorem:

Theorem 6.28 (Adámek, Rosický, Trnková [ART88])**.** *Every locally presentable category can be fully embedded into* $\mathbf{Gra}$.

Proof. See Theorem 2.65 of [AR94]. ☐

Corollary 6.29. $\mathsf{VP} \iff \mathsf{CatVP}$

Proof. If a locally presentable category contained a full discrete subcategory, then the category of digraphs $\mathbf{Gra}$ would also contain a full discrete subcategory. Accordingly, there would be a proper class of mutually rigid digraphs and thus also a proper class of mutually rigid graphs (again by the indicator technique). ☐

In particular, if we use the fact that every relational category $\mathbf{Rel}(\Sigma)$ is locally presentable, we obtain Corollary 6.27.

Another equivalent formulation of Vopěnka's principle in terms of categories, first observed by Adámek, Rosický, and Trnková [ART88], is that the category of ordinals $\mathbf{Ord}$ cannot be fully embedded into $\mathbf{Gra}$. In other words:

Theorem 6.30. $\mathsf{VP} \iff$ *There is no sequence of digraphs* $\langle G_\alpha \mid \alpha \in \mathrm{On} \rangle$ *such that whenever* $\alpha \leq \beta$*, then there is a unique homomorphism* $G_\alpha \to G_\beta$*, and when* $\alpha > \beta$*, then* $G_\alpha \not\to G_\beta$.

Remark. We can replace "digraphs" with "graphs." Indeed, suppose that there is no such "bad" sequence of graphs, and we want to show that no such sequence $\langle G_\alpha \mid \alpha \in \mathrm{On} \rangle$ of digraphs can exist either. Suppose it did. Choose an indicator J and construct a corresponding sequence $\langle G_\alpha * J \mid \alpha \in \mathrm{On} \rangle$ of graphs. Note that all G_α have to contain at least two vertices and be rigid. Since rigid digraphs with at least two vertices contain neither loops nor isolated vertices, Lemma 6.19 implies that $\langle G_\alpha * J \mid \alpha \in \mathrm{On} \rangle$ is a "bad" sequence of graphs.

We now turn to representability. A category is *thin* if it contains at most one morphism $A \xrightarrow{m} B$ for any pair of its objects (A, B). It is easy to see that thin categories are exactly the categories induced by a quasi-order (in the sense we defined $\mathbf{Ord}$). If a category is isomorphic to a subcategory (not necessarily full) of $\mathbf{Set}$, it is said to be *concrete*. Intuitively, such categories can be naturally "realized" using sets. It is an easy exercise to show that all of the categories mentioned above are concrete.

One can prove that every small category is concrete and, furthermore, fully embeddable into $\mathbf{Gra}$; this yields Theorem 6.11 since every partially ordered set corresponds to a small thin category. A category is *universal* if every concrete category can be fully embedded into it. Hedrlín and Kučera proved, under a fairly reasonable set-theoretic assumption,[8] that **Gra** is universal and thus fully embeds every thin category.

Theorem 6.31 (Hedrlín, Kučera; see Chapter III of [PT80])**.** *It is consistent with the axioms of* NBG *that every partially ordered class* $(P, \preceq)$ *embeds into the homomorphism order of digraphs* $(\mathcal{D}, \leq)$*.*

The study of rigid objects and representability of categories arose in Prague in the 1960s with the works of Zdeněk Hedrlín, Aleš Pultr, Luděk Kučera, Věra Trnková, and others. This came to be known as the Prague School of Category Theory, and its developments up to the late 1970s are summarized in [PT80]. A short and very readable account of the finite case of some of these results appears in Chapter 4 of [HN04]. Chapter 6 of [AR94] contains a plethora of category theory results regarding Vopěnka's principle. Finally, see [Awo06] for an accessible introduction to category theory.

WQO theory point of view We say that a quasi-ordered class Q is *class-wqo* if for every sequence

$$\langle q_\alpha \,|\, \alpha \in \mathrm{On} \rangle$$

of elements of Q, there are $\alpha < \beta$ such that $q_\alpha \leq q_\beta$. We will now show that Vopěnka's principle is equivalent to the statement that the class of all graphs $\mathcal{G}$ is class-wqo by the homomorphism and induced subgraph relations.

Theorem 6.32. *The following statements are equivalent.*

(i) *Vopěnka's principle.*

(ii) *There is no sequence* $\langle G_\alpha \,|\, \alpha \in \mathrm{On} \rangle$ *of rigid graphs such that* $G_\alpha \not\to G_\beta$ *and* $G_\beta \not\to G_\alpha$ *for every pair* $\alpha \neq \beta$*.*

(iii) *For every sequence* $\langle G_\alpha \,|\, \alpha \in \mathrm{On} \rangle$ *of graphs, there are* $\alpha < \beta$ *such that* $G_\alpha \to G_\beta$*. In other words,* $\mathcal{G}$ *is class-wqo by the homomorphism relation.*

(iv) *For every sequence* $\langle G_\alpha \,|\, \alpha \in \mathrm{On} \rangle$ *of graphs, there are* $\alpha < \beta$ *such that* $G_\alpha \subseteq_i G_\beta$*. In other words,* $\mathcal{G}$ *is class-wqo by the induced subgraph relation.*

The equivalence of statements (iii) and (iv) should feel at least a bit surprising, since the induced subgraph relation is much "stronger" (more restrictive) than the homomorphism relation.

Proof. Clearly (i)⇒(ii). We will show (ii)⇒(i) by constructing a well-ordering of any proper class of graphs $\mathcal{H}$, isomorphic to On. For each $\alpha \in \mathrm{On}$, let $\mathcal{H}_\alpha$ be the *set* of all graphs $G \in \mathcal{H}$ with rank $\varrho(G) = \alpha$ (see the footnote in Section 1.4). Since we assume the axiom of choice, each $\mathcal{H}_\alpha$ can be well-ordered. Invoke the axiom of global choice, *choose* a well-ordering $\prec_\alpha$ for each $\mathcal{H}_\alpha$, and well-order $\mathcal{H}$ by letting $H \prec G$ if $\varrho(H) < \varrho(G)$, or if $\varrho(H) = \varrho(G) = \alpha$ and $H \prec_\alpha G$. This is

[8] Recall that an ultrafilter $\mathcal{F}$ is non-principal $\iff \bigcap \mathcal{F} = \varnothing$. We say that an ultrafilter $\mathcal{F}$ is κ-complete, for a cardinal number κ, if any intersection of less than κ sets from $\mathcal{F}$ is still in $\mathcal{F}$. The assumption of Hedrlín and Kučera is denoted by M and it reads: "there exists a cardinal κ such that every κ-complete ultrafilter is principal." Notice that this implies that there cannot be a proper class of measurable cardinals (see footnote 1). Thus, if we accept M, then Vopěnka's principle does not hold, and vice versa. However, M is provably consistent with the axioms of set theory, while VP might not be (depends if you believe in huge cardinals).

clearly a well-ordering in which every initial segment $\{G \in \mathcal{H} \mid G \prec H\}$ for $H \in \mathcal{H}$ is a set, and it is well known that any such ordering is isomorphic to On.

Further note that (iv)⇒(iii)⇒(ii). It thus suffices to show (ii)⇒(iv).

Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be a sequence of graphs, $G_\alpha = (V_\alpha, E_\alpha)$, and without loss of generality assume that no two of them are isomorphic (otherwise the claim trivially holds). Since there are only set-many non-isomorphic graphs of any given cardinality, there exists a proper class $A \subseteq \mathrm{On}$ such that $\aleph_0 \leq |V_\alpha| < |V_\beta|$ for all pairs $\alpha < \beta$ in A. Recall that we view graphs as irreflexive symmetric digraphs, and for each $\alpha \in A$, consider the binary relational system

$$G_\alpha^+ = (V_\alpha, E_\alpha, R_\alpha, K_\alpha, C_\alpha),$$

where R_α is a rigid graph on V_α given by Corollary 6.21, K_α is the complete graph on V_α, and C_α is the complement of E_α. Let $\mathcal{J}$ be a family of 4 incomparable triangle-connected indicators given by Corollary 6.24, and put $H_\alpha \coloneqq G_\alpha^+ * \mathcal{J}$.

The graphs H_α are rigid from Theorem 6.25 due to R_α and our choice of $\mathcal{J}$. By (ii), there are ordinals $\alpha \neq \beta$ in A such that there exists a homomorphism $h\colon H_\alpha \to H_\beta$. Let $f\colon G_\alpha^+ \to G_\beta^+$ be a homomorphism satisfying $h = f * \mathcal{J}$, given by Theorem 6.25. Notice that f must be injective due to K_α. Therefore $\alpha < \beta$; otherwise, we would have $|V_\alpha| > |V_\beta|$, making an injection from V_α into V_β impossible. We claim that f witnesses that G_α is an induced subgraph of G_β. We already know that it is an injective map that preserves the edges E_α of G_α, and it also preserves the non-edges of G_α due to C_α. □

I would like to thank professor Nešetřil for showing me this elegant argument. A similar argument, using K_α, is already present in [ART88, Lemma 2]. Bagaria and Wilson [BW23, Proposition 2.1] observed that one can modify it by adding R_α to obtain (ii)⇒(iii). As we have seen, further including C_α yields (ii)⇒(iv).

6.5 Results of Shelah

A notion similar to that of class-wqo was considered by Shelah [She82]. Given a quasi-order Q that is not wqo, it is natural to ask whether there exists a cardinal number κ such that every sequence $\langle q_\alpha \mid \alpha < \kappa\rangle$ of elements of Q contains indices $\alpha < \beta < \kappa$ such that $q_\alpha \leq q_\beta$. If this is true, we say that Q is *κ-wqo*.[9] For instance, if Q is a set of size $|Q| = \kappa$, then Q is κ^+-wqo. We call the least cardinal κ such that Q is κ-wqo the *well-ordering number* of Q (provided such a cardinal exists). Thus wqo theory concerns itself with the case $\kappa = \omega$. It is easy to see that if Q is κ-wqo, then Q admits no antichains of size κ, and no sequences $\langle q_\alpha \mid \alpha < \kappa\rangle$ such that if $\alpha < \beta$, then $q_\alpha > q_\beta$. In contrast to the case $\kappa = \omega$, the converse implication does not hold in general without additional assumptions, such as κ being weakly compact (one can then reuse our proof of Theorem 2.4).

The result most relevant for us is that the class of all trees is κ-wqo for a mildly large cardinal κ. More precisely, let the class T_ω of all order-theoretic trees of height at most ω be ordered by the rule that $S \leq T$ if there exists an injective level-embedding (see Definition 4.17) of S into T. Then:

Theorem 6.33. *The well-ordering number of T_ω is the first beautiful cardinal.*

[9] Pouzet [Pou72] suggested a completely different notion of α-wqo for countable ordinals $\alpha < \omega_1$ of the form $\alpha = \omega^\beta$ for some β. This was subsequently thoroughly studied by Marcone [Mar94]. Under Pouzet's definition, α-wqos serve as approximations of the bqo property: ω-wqo coincides with wqo, and Q is bqo $\iff$ Q is α-wqo for all $\alpha < \omega_1$. Moreover, it is possible to show that Q is α-wqo $\iff$ $Q^{<\alpha}$ is wqo, and it follows that Q is bqo $\iff$ $Q^{<\omega_1}$ is wqo.

Corollary 6.34. *If κ is a beautiful cardinal, then the class of all graph-trees is κ-wqo by the subgraph and induced subgraph relations.*

We will define these beautiful cardinals shortly, but we first mention one more result from [She82]. In order to prove Theorem 6.33, Shelah had to generalize the notion of being bqo. A *κ-I-barrier B* ("I" for increasing) consists of finite increasing sequences from κ (instead of ω), and Q is κ-I-bqo if every mapping f from a κ-I-barrier to Q satisfies the usual requirement of bqo theory. However, this most natural generalization cannot prove an analogue of Nash-Williams' partition theorem (see Lemma 3.9), which is crucial in establishing the basics of bqo theory. Shelah is therefore led to propose a stronger notion. He defines Q to be $[\kappa;\lambda]$-I-bqo if for every κ-I-barrier B, coloring $\chi\colon B \to \lambda$, and mapping $f\colon B \to Q$, there exist two elements $s \lhd t$ of the same color $\chi(s) = \chi(t)$ such that $f(s) \leq f(t)$. Finally, Q is defined to be $[\kappa]$-bqo if it is $[\kappa;\lambda]$-I-bqo for every $\lambda < \kappa$. Hence Q is bqo $\iff$ it is $[\omega]$-bqo, but in general, being $[\kappa]$-bqo is a stronger property than being κ-I-bqo.

Recall that a linear order is *scattered* if the order of the rationals does not embed into it. We say that it is *λ-scattered* if it is a union of at most λ scattered orders, and we denote by $\mathcal{M}_\lambda$ the class of all such orders. Laver [Lav71] showed that $\mathcal{M}_\omega$ is bqo (see Section 4.6). Shelah investigated the case of uncountable λ.

Theorem 6.35. *If κ is the first beautiful cardinal greater than $\lambda > \aleph_0$, then $\mathcal{M}_\lambda$ is $[\kappa]$-bqo, and its well-ordering number is κ.*

Beautiful cardinals An uncountable cardinal number κ is *weakly compact* if for all cardinals $\mu < \kappa$ and $k < \omega$, in every partition of $[\kappa]^k$ into μ color classes, there exists a subset $A \subseteq \kappa$ of size κ such that $[A]^k$ is monochromatic. In the partition calculus of Erdős and Rado [ER56], this can be written as

$$\kappa \longrightarrow (\kappa)^k_\mu.$$

Interestingly, if an uncountable cardinal number κ satisfies $\kappa \longrightarrow (\kappa)^2_2$, then this already implies that κ is weakly compact (see [Jec03] or [BŠ01]). Since weakly compact cardinals are inaccessible, ZFC cannot prove their existence.

Recall that $[A]^{<\omega}$ denotes the set of all finite subsets of A; this can also be viewed as the set of all finite, strictly increasing sequences of elements of A.

Definition 6.36. For cardinals κ and μ, the expression

$$\kappa \longrightarrow (\omega)^{<\omega}_\mu$$

means that for every mapping $f\colon [\kappa]^{<\omega} \to \mu$, there exists a subset $A \subseteq \kappa$ ordered according to ω such that $f \restriction [A]^k$ is constant (every k-set of A has the same color) for all $k \in \omega$. Note that the value of f on $[A]^n$ might differ for different n.

A cardinal κ is *ω-Erdős* if $\kappa \longrightarrow (\omega)^{<\omega}_2$. The existence of an ω-Erdős cardinal is significantly stronger than that of a weakly compact cardinal (as demonstrated by Friedman [Fri01]), yet it is still weak enough to be consistent with the axiom of constructibility[10] $\mathbf{V} = L$, as proved by Silver [Sil70]. Since measurable cardinals

[10]We denote by $\mathbf{V}$ the class of all sets (the universe of set theory). The *constructible universe* L was introduced by Gödel [Göd40] in order to prove the consistency of the continuum hypothesis (the assertion that $2^{\aleph_0} = \aleph_1$). The hierarchy of *constructible sets* is defined recursively as follows:

$$L_0 \coloneqq \varnothing, \quad L_{\alpha+1} \coloneqq \mathrm{Def}(L_\alpha), \quad L_\lambda \coloneqq \bigcup_{\alpha<\lambda} L_\alpha, \quad L \coloneqq \bigcup_{\alpha\in\mathrm{On}} L_\alpha,$$

imply that $\mathbf{V} \neq L$, the existence of an ω-Erdős cardinal is far weaker than the existence of a measurable cardinal, which itself is much, much weaker than Vopěnka's principle.

Moreover (see [Jec03]), if κ is ω-Erdős, then

$$\kappa \longrightarrow (\omega)^{<\omega}_{\mu}$$

holds for all $\mu < \kappa$.

Beautiful cardinals are a weakening of ω-Erdős cardinals. Their motivation is natural in the context of the $[\kappa]$-bqos introduced above.

Definition 6.37. For cardinals κ and μ, the expression

$$\kappa \xrightarrow{w} (\omega)^{<\omega}_{\mu}$$

means that for every mapping $f\colon [\kappa]^{<\omega} \to \mu$, there is a subset $A = \{\alpha_n \mid n \in \omega\}$ of κ such that $\alpha_i < \alpha_j$ for $i < j$, with the following property: For every $n \in \omega$, we have $f(\{\alpha_0, \dots, \alpha_n\}) = f(\{\alpha_1, \dots, \alpha_{n+1}\})$.

We say that an uncountable cardinal number κ is *beautiful* if $\kappa \xrightarrow{w} (\omega)^{<\omega}_{\mu}$ for all $\mu < \kappa$. Clearly, every ω-Erdős cardinal is beautiful. We mention that beautiful cardinals are limits of weakly compact cardinals, and that if κ is beautiful, then it is also beautiful in the constructible universe L.[11]

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where λ is a limit ordinal, and $\mathrm{Def}(X)$ denotes the set of subsets of X definable by first-order formulas over $(X, \in)$ with parameters from X. While the power set $\mathcal{P}(X)$ contains all subsets of X, the set $\mathrm{Def}(X)$ contains only those subsets that can be explicitly described using the sets that we have already constructed earlier. Formally, a subset $Y \subseteq X$ belongs to $\mathrm{Def}(X)$ if there is a first-order formula $\phi(u, v_1, \dots, v_n)$ and parameters $p_1, \dots, p_n \in X$ such that $Y = \{x \in X \mid (X, \in) \models \phi(x, p_1, \dots, p_n)\}$. When we say the formula is evaluated over $(X, \in)$, we mean that all quantifiers $\forall$ and $\exists$ in ϕ are restricted to range strictly over the elements of X, rather than the entire universe $\mathbf{V}$ (so a quantifier $\forall a$ is interpreted as $\forall a \in X$).

The *axiom of constructibility* $\mathbf{V} = L$ is the statement that every set is constructible.

Gödel showed that L is what we call a *transitive inner model* of ZF: Given any model $(M, \in^M)$ of ZF, the class L^M within this model is transitive, and $(L^M, \in^L)$, where $\in^L$ is the restriction of $\in^M$ to L^M, is also a model of ZF. Gödel further showed that in this inner model, the axiom of choice AC and the generalized continuum hypothesis GCH (the assertion that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α) are satisfied regardless of M. Hence, if ZF is consistent (has a model M), then ZFC and ZFC + GCH are also consistent (they have a model, namely L^M).

[11] A basic property of L is that $L_\alpha \cap \mathrm{On} = \alpha$ holds for all ordinals α, and that if M is a model of set theory and $\alpha \in M$ is an ordinal in M, then $\alpha \in L^M$ is an ordinal in L^M as well. Even though α is still the same set, we have deleted all non-constructible sets, which might affect the properties of the sets we retained (perhaps we deleted some important bijections). The reason why α remains an ordinal is that "x is an ordinal" is a Δ_0 property (it can be described using a formula containing only bounded quantifiers $\forall y \in z$ and $\exists y \in z$), and Δ_0 formulas are absolute, meaning that their truth value is preserved in transitive subclasses $A \subseteq M$.

Conclusion and Open Problems

In this thesis, we studied well-quasi-orderings of graphs and order-theoretic trees by various containment relations.

Trees In Section 4.5, we proved, while avoiding bqo theory, that out-trees are wqo by the homomorphism relation, and we extended this result to trees with bqo-labeled leaves. We also constructed a counterexample for trees with wqo-labeled leaves. Notably, the trees in our counter-example were not finitely branching. Since we restricted ourselves to trees without infinite branches, a counter-example with finitely branching trees cannot exist in that setting by Kőnig's lemma because finite trees are wqo from Kruskal's tree theorem.

This restriction arises from the fact that in trees without infinite branches, every branch contains a leaf, enabling our proof of Theorem 4.41. One may nevertheless consider tree homomorphisms of leaf-labeled trees with infinite branches. Extending Theorem 4.41 to such trees would likely require techniques from the proof of Laver's generalization of Nash-Williams' theorem for infinite trees.

Conjecture 1. *If Q is bqo, then the class of all leaf-labeled trees over Q of height at most ω is bqo by the tree-homomorphism relation.*

Tree-depth In Section 5.3, we extended the notion of tree-depth to arbitrary (possibly infinite) graphs and showed that its basic properties carry over to this more general setting. Using the coloring characterization of tree-depth, we then established a compactness theorem. We also extended Ding's theorem to infinite graphs, showing that if Q is bqo and n is a positive integer, then the class of all Q-labeled graphs with tree-depth at most n is bqo by the induced subgraph relation.

Two natural (and likely easy) problems concerning tree-depth are:

Problem 1. *Is it true that if $\mathrm{td}(G) = \alpha$, then for every $\beta < \alpha$, there exists a subgraph $H \subseteq G$ with $\mathrm{td}(H) = \beta$?*

Problem 2. *Construct for every ordinal α a graph with tree-depth α.*

Regarding the second problem, there is a simple construction for graphs of tree-depth $\alpha \leq \omega \cdot 2$, which can probably be generalized. Recall that if κ is a cardinal, then $\mathrm{td}(K_\kappa) = \kappa$. In particular, $\mathrm{td}(K_\omega) = \omega$. Moreover, the join of two graphs H and G is the graph $(H^c + G^c)^c$. Taking the join of K_ω with the disjoint union of ω_1 copies of K_λ for $\lambda \leq \omega$ yields a graph of tree-depth $\omega + \lambda$.

Subgraphs In Section 5.4, we asked for which finite graphs H, the class $\mathcal{G}(H)$ of all finite H-free graphs is wqo by the subgraph relation. We showed that a necessary condition is that H is an induced subgraph of a path, and proved that if H is an independent set, then the class of all (finite or infinite) H-free graphs is wqo by the subgraph relation. Equivalently, graph classes with bounded independence number are wqo by the subgraph relation. Surprisingly, we found that this result does not generalize to wqo- or bqo-labeled graphs, and we described a smaller class of graphs for which a labeled version can be recovered. However, the bqo in our counter-example contained incomparable elements, and I have not been able to produce a counter-example with a linearly ordered wqo (or equivalently, a well-ordered label set).

Conjecture 2. *If Q is well-ordered, then for every $n \in \omega$, the class of all (finite or infinite) Q-labeled graphs G with $\alpha(G) \leq n$ is wqo by the subgraph relation.*

A natural next direction is to consider graphs that are close to being an independent set:

Conjecture 3. *If H is a finite graph containing exactly one edge, then $\mathcal{G}(H)$ is wqo by the subgraph relation.*

We briefly describe the structure of such graph classes. Denote by H_n the graph with n vertices and one edge. The conjecture is known to hold for $n \leq 3$ by Theorem 5.34. Recall that a graph is H-free $\iff$ its complement is H^c-free. Thus $\mathcal{G}(H_3)$ is the class of all complete multipartite graphs, since H_3^c-free graphs are disjoint unions of cliques. To see this, notice that the edge relation in H_3^c-free graphs is transitive in the sense that whenever uv and vw are edges, then uw is also an edge. More generally, a graph is H_n^c-free $\iff$ any two of its inclusion-maximal cliques intersect in at most $n-3$ vertices (if two distinct maximal cliques shared $n-2$ vertices, taking those shared vertices plus one unique vertex from each clique would induce an H_n^c subgraph). Hence a graph G is H_n-free $\iff$ any two of its inclusion-maximal independent sets intersect in at most $n-3$ vertices.

Class-wqos In Section 6.4, we introduced the notion of class-wqo, and we proved that Vopěnka's principle is equivalent to the statement that the class of all graphs $\mathcal{G}$ is class-wqo by either of the following relations: homomorphisms, subgraphs, or induced subgraphs.

I believe that Vopěnka's principle warrants deeper investigation from this class-wqo perspective. For instance, Vopěnka's principle implies that $\mathcal{G}$ is class-wqo by both the minor and topological minor relations. However, Theorem 5.11 implies that $\mathcal{G}$ is not κ-wqo by either of these relations for any cardinal κ.

Problem 3. *Is Vopěnka's principle equivalent to the statement that $\mathcal{G}$ is class-wqo by the minor (or topological minor) relation?*

Problem 4. *Is there a natural class of graphs that is not κ-wqo for any cardinal number κ, yet can be shown to be class-wqo without any large cardinals?*

Later, in Section 6.5, we discussed Shelah's result that the well-ordering number of the class T_ω of all order-theoretic trees of height at most ω under injective level-embeddings is the first beautiful cardinal number. Consequently, the class of all graph-trees $\mathcal{T}$ is κ-wqo by the induced subgraph relation when κ is a beautiful cardinal number.

Remark. $\mathcal{T}$ is trivially wqo by the homomorphism relation since every tree is bipartite and is thus homomorphic to K_2. Investigating stronger forms of this observation was, in fact, the original motivation behind our theorem for out-trees.

Problem 5. *Are there other natural classes of graphs, besides trees, that are κ-wqo for beautiful cardinals κ?*

Note that ZFC cannot prove the existence of a cardinal number κ such that T_ω is κ-wqo because beautiful cardinals are large cardinals. While $\mathcal{T}$ could technically be κ-wqo for some cardinal κ smaller than the first beautiful cardinal, the above observation suggests that ZFC likely cannot prove that a cardinal number κ such that $\mathcal{T}$ is κ-wqo exists.

Problem 6. *Is it possible to prove that $\mathcal{T}$ is class-wqo without relying on any large cardinals?*

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