LIMITA FUNKCE

- Fce f má v bodě a limitu L, jestliže k libovolně zvolenému okolí bodu L existuje ryzí (redukované) okolí bodu a tak, že pro všechna x z tohoto okolí náleží hodnoty $f(x)$ do zvoleného okolí L

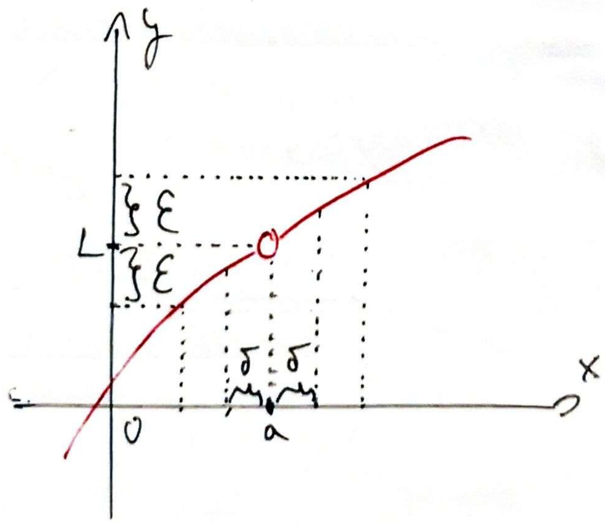
- $\lim\limits_{x \to a} f(x) = L \iff$ … vlastní lim. ve vlastním bodě

$$(\forall \varepsilon > 0 \ \exists \delta > 0); (\forall x \in D(f)) : (x \in (a-\delta; a+\delta) \setminus \{a\} \Rightarrow f(x) \in (L-\varepsilon; L+\varepsilon))$$

$$(\forall \varepsilon > 0 \ \exists \delta > 0); (\forall x \in D(f)) : (x \in R_\delta(a) \Rightarrow f(x) \in O_\varepsilon(L))$$

$$(\forall \varepsilon > 0 \ \exists \delta > 0); (\forall x \in D(f)) : (0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

- okolí bodu
  - $O_\varepsilon(a) = (a-\varepsilon; a+\varepsilon)$ = epsilonové okolí bodu a
- ryzí okolí bodu – nebo redukované
  - $R_\varepsilon(a) = (a-\varepsilon; a+\varepsilon) \setminus \{a\}$ = epsilonové ryzí okolí bodu a
    - někdy zapisujeme $^*O_\varepsilon(a)$



→ goniometrické fce

$$0 \leq \cos(x) \leq \frac{\sin(x)}{x} \leq 1 \qquad \text{pro } x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$$

$$\Rightarrow x=0: \; 1 \leq \frac{\sin(x)}{x} \leq 1 \quad \Rightarrow \quad \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\cot g(x)}{x} = \infty$$

$$\to \lim_{x\to 0} \frac{1}{\frac{\sin(x)}{x}} = \lim_{x\to 0} \frac{1}{1} \Rightarrow \lim_{x\to 0} \frac{x}{\sin(x)} = 1$$

$$\to \lim_{x\to 0} \frac{\frac{\sin(x)}{\cos(x)}}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{1} \Rightarrow \lim_{x\to 0} \frac{tg(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{x}{\sin(x)} = \lim_{x\to 0} \frac{tg(x)}{x} = \lim_{x\to 0} \frac{x}{tg(x)} = 1$$

→ příklady

- $$\lim_{x\to 0} \frac{\cot g(5x)}{\cot g(2x)} = \lim_{x\to 0} \frac{tg(2x)}{tg(5x)} = \lim_{x\to 0} \frac{5x \cdot 2x \cdot \overset{1}{tg(2x)}}{\underset{1}{tg(5x)} \cdot 5x \cdot 2x} = \lim_{x\to 0} \frac{2x}{5x} = \underline{\underline{\frac{2}{5}}}$$

- $$\lim_{x\to 0} \frac{2x}{\sin(x) + \sin(3x)} = \lim_{x\to 0} \frac{2x}{x \cdot \frac{\sin(x)}{x} + 3x \cdot \frac{\sin(3x)}{3x}} = \lim_{x\to 0} \frac{2x}{4x} = \underline{\underline{\frac{1}{2}}}$$

- $$\lim_{x\to 0} \frac{tg^2(x)}{\cos(x)-1} = \lim_{x\to 0} \frac{\frac{\sin^2(x)}{\cos^2(x)}}{\cos(x)-1} = \lim_{x\to 0} \frac{\sin^2(x)}{\cos^2(x)(\cos(x)-1)}$$

$$1 = \sin^2(x) + \cos^2(x)$$
$$\sin^2(x) = 1 - \cos^2(x)$$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{\cos^2(x)(\cos(x)-1)} = \lim_{x\to 0} \frac{(1-\cos(x))(1+\cos(x))}{\cos^2(x)(\cos(x)-1)} = \lim_{x\to 0} \frac{-1-\cos(x)}{\cos^2(x)} = \frac{-1-1}{1} = \underline{\underline{-2}}$$

- $$\lim_{x\to 0} \frac{x^2}{\cos(x)-1} = \lim_{x\to 0} \frac{x^2(\cos(x)+1)}{\cos^2(x)-1} = \lim_{x\to 0} \frac{x^2(\cos(x)+1)}{-\sin^2(x)} = \lim_{x\to 0} \frac{1+1}{-1} = \underline{\underline{-2}}$$

→ příklady

• 152/4

• $\lim_{x\to\pi} \frac{\cos^2(x)-3\cos(x)-4}{\cos^2(x)-4\cos(x)-5} = \lim_{x\to\pi} \frac{(\cos(x)+1)(\cos(x)-4)}{(\cos(x)+1)(\cos(x)-5)} = \lim_{x\to\pi} \frac{-5}{-6} = \underline{\underline{\frac{5}{6}}}$

• $\lim_{x\to -\frac{\pi}{4}} \frac{4+2\operatorname{cotg}(x)-2\operatorname{cotg}^2(x)}{\operatorname{cotg}^2(x)-1} = \lim_{x\to -\frac{\pi}{4}} \frac{-2(\operatorname{cotg}^2(x)-\operatorname{cotg}(x)-2)}{(\operatorname{cotg}(x)-1)(\operatorname{cotg}(x)+1)} =$

$= \lim_{x\to -\frac{\pi}{4}} \frac{-2(\operatorname{cotg}(x)+1)(\operatorname{cotg}(x)-2)}{(\operatorname{cotg}(x)-1)(\operatorname{cotg}(x)+1)} = \lim_{x\to -\frac{\pi}{4}} \frac{-2(-3)}{-2} = \underline{\underline{-3}}$

• 153

• $\lim_{x\to 1} \frac{2-\sqrt{x+3}}{\sqrt{x}-1} = \lim_{x\to 1} \frac{\frac{1}{-2\sqrt{x+3}}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 1} \frac{-\sqrt{x}}{\sqrt{x+3}} = \underline{\underline{-\frac{1}{2}}}$

• $\lim_{x\to\frac{\pi}{4}} \frac{\operatorname{tg}(x)-1}{\operatorname{cotg}(x)-1} = \lim_{x\to\frac{\pi}{4}} \frac{\frac{\sin(x)-\cos(x)}{\cos(x)}}{\frac{\cos(x)-\sin(x)}{\sin(x)}} = \lim_{x\to\frac{\pi}{4}} \frac{\sin(x)(\sin(x)-\cos(x))}{-\cos(x)(\sin(x)-\cos(x))} =$

$= \lim_{x\to\frac{\pi}{4}} (-\operatorname{tg}(x)) = \underline{\underline{-1}}$

• $\lim_{x\to\frac{\pi}{6}} \frac{2\sin^2(x)-\cos(2x)}{\sqrt{2\sin(x)}-1} = \lim_{x\to\frac{\pi}{6}} \frac{3\sin^2(x)-\cos^2(x)}{\sqrt{2\sin(x)}-1} = \lim_{x\to\frac{\pi}{6}} \frac{4\sin^2(x)-1}{\sqrt{2\sin(x)}-1} =$

$S: 2\sin(x) = a: \lim_{x\to\frac{\pi}{6}} \frac{(a-1)(a+1)(\sqrt{a}+1)}{a-1} = \lim_{x\to\frac{\pi}{6}} (2\sin(x)+1)(\sqrt{2\sin(x)}+1) = \underline{\underline{4}}$

• $\lim_{x\to 0} \frac{\cos^2(x)-1+\sin(2x)}{x} = \lim_{x\to 0} (-2\cos(x)\cdot\sin(x) + \cos(2x)\cdot 2) = \underline{\underline{2}}$

$= \lim_{x\to 0} \frac{-\sin^2(x)+2\sin(x)\cos(x)}{x} = \lim_{x\to 0} \frac{\sin(x)(2\cos(x)-\sin(x))}{x} = \underline{\underline{2}}$

pořád $\frac{0}{0}$ $\qquad$ 1

• $\lim_{x\to 9} \frac{\sqrt{x}-3}{\sqrt{2x}-3\sqrt{2}} = \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{2x}+3\sqrt{2})}{2x-18} = \lim_{x\to 9} \frac{(x-9)(\sqrt{2x}+3\sqrt{2})}{2(x-9)(\sqrt{x}+3)} =$

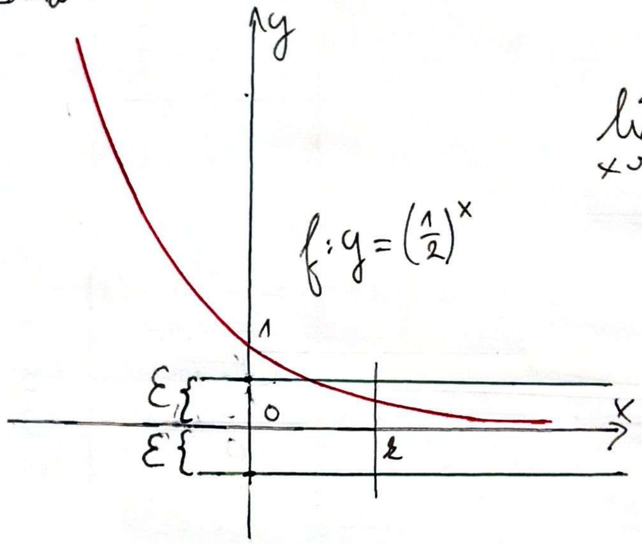
$= \lim_{x\to 9} \frac{\sqrt{2x}+3\sqrt{2}}{2\sqrt{x}+6} = \frac{3\sqrt{2}+3\sqrt{2}}{6+6} = \frac{6\sqrt{2}}{12} = \underline{\underline{\frac{\sqrt{2}}{2}}}$

→ vlastní limita v nevlastním bodě

$$\lim_{x\to\infty} f(x)=L \iff (\forall \varepsilon>0\ \exists k\in\mathbb{R});(\forall x\in D(f)):(x>k \Rightarrow f(x)\in O_\varepsilon(L))$$

$$\iff (\forall \varepsilon>0\ \exists k\in\mathbb{R});(\forall x\in D(f)):(x>k \Rightarrow |f(x)-L|<\varepsilon)$$

$$\lim_{x\to-\infty} f(x)=L \iff (\forall \varepsilon>0\ \exists k\in\mathbb{R});(\forall x\in D(f)):(x<k \Rightarrow f(x)\in O_\varepsilon(L))$$



$$\lim_{x\to\infty} f(x)=0$$

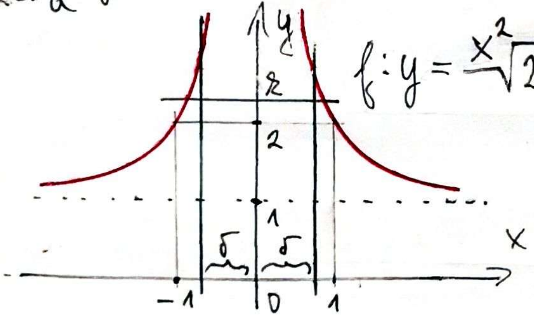
$$f: y=\left(\tfrac{1}{2}\right)^x$$

→ nevlastní limita ve vlastním bodě

$$\lim_{x\to a} f(x)=\infty \iff (\forall k\in\mathbb{R}\ \exists \delta>0);(\forall x\in D(f)):(x\in R_\delta(a) \Rightarrow f(x)>k)$$

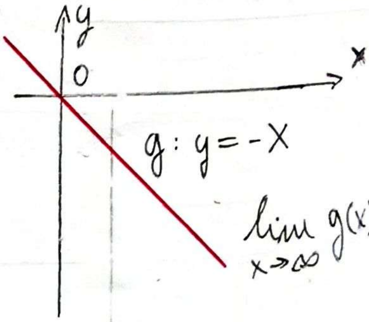
$$\iff (\forall k\in\mathbb{R}\ \exists \delta>0);(\forall x\in D(f)):(0<|x-a|<\delta \Rightarrow f(x)>k)$$

$$\lim_{x\to a} f(x)=-\infty \iff (\forall k\in\mathbb{R}\ \exists \delta>0);(\forall x\in D(f)):(x\in \hat{R}_\delta(a) \Rightarrow f(x)<k)$$



$$f: y=\sqrt[x^2]{2}=2^{\frac{1}{x^2}}$$

$$\lim_{x\to 0} f(x)=\infty$$



$$g: y=-x$$

$$\lim_{x\to\infty} g(x)=-\infty$$

→ nevlastní limita v nevlastním bodě

$$\lim_{x\to\infty} f(x)=\infty \iff (\forall k\in\mathbb{R}\ \exists q\in\mathbb{R});(\forall x\in D(f)):(x>q \Rightarrow f(x)>k)$$

$$\lim_{x\to\infty} f(x)=-\infty \iff (\forall k\in\mathbb{R}\ \exists q\in\mathbb{R});(\forall x\in D(f)):(x>q \Rightarrow f(x)<k)$$

$$\lim_{x\to-\infty} f(x)=\infty \iff (\forall k\in\mathbb{R}\ \exists q\in\mathbb{R});(\forall x\in D(f)):(x<q \Rightarrow f(x)>k)$$

$$\lim_{x\to-\infty} f(x)=-\infty \iff (\forall k\in\mathbb{R}\ \exists q\in\mathbb{R});(\forall x\in D(f)):(x<q \Rightarrow f(x)<k)$$

→ *příklady*

11/d) $\lim_{x\to-\infty}\frac{2x^4-x^3+4}{5x^4+x^3+2}=\lim_{x\to-\infty}\frac{2-0+0}{5+0+0}=\frac{2}{5}$

13/b) $\lim_{x\to\infty}\sqrt{\frac{4x^2-x}{3x^2}}=\lim_{x\to\infty}\sqrt{\frac{4-0}{3}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

13/e) $\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{x+1}=\lim_{x\to\infty}\frac{\sqrt{1+0}}{1+0}=1$

13/g) $\lim_{x\to\infty}\frac{\sqrt{x+2}+3\sqrt{x^2-6}}{2x+1}=\lim_{x\to\infty}\frac{0+3\sqrt{1-0}}{2+0}=\frac{3}{2}$

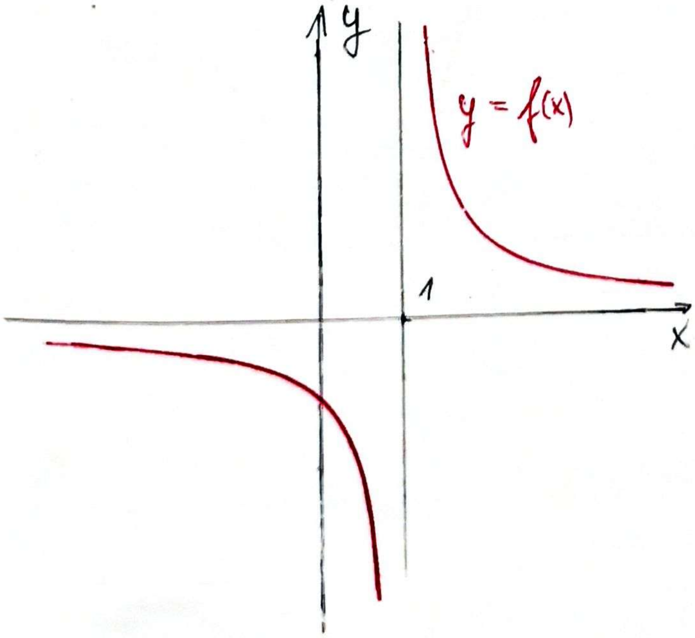
→ *jednostranné limity*

$$\lim_{x\to a^+}f(x)=L \iff (\forall\varepsilon>0\ \exists\delta>0):(x\in\overbrace{(a;a+\delta)}^{R_\delta^+(a)}\Rightarrow f(x)\in O_\varepsilon(L))$$

$$\lim_{x\to a^-}f(x)=\infty \iff (\forall k\in\mathbb{R}\ \exists\delta>0):(x\in\overbrace{(a-\delta;a)}^{R_\delta^-(a)}\Rightarrow f(x)>k)$$

$$\lim_{x\to a^+}f(x)\neq\lim_{x\to a^-}f(x)\Rightarrow\lim_{x\to a}f(x)\ \text{neexistuje}$$

• $f(x)=\frac{1}{x-1}$



$\lim_{x\to1^+}\frac{1}{x-1}=\frac{1}{0^+}=\infty$

↳ $x>1$

$\lim_{x\to1^-}\frac{1}{x-1}=\frac{1}{0^-}=-\infty$

↳ $x<1$

$\Rightarrow \lim_{x\to1}\frac{1}{x-1}$ neexistuje

→ *příklady*

• $f: y = \dfrac{2x+3}{2-x}$ → $(2x+3):(-x+2) = -2$

$$\begin{array}{l} -(2x-4) \\ \hline 7 \end{array}$$

$$f(x) = \frac{-7}{x-2} - 2$$

$\Rightarrow S[2; -2]$

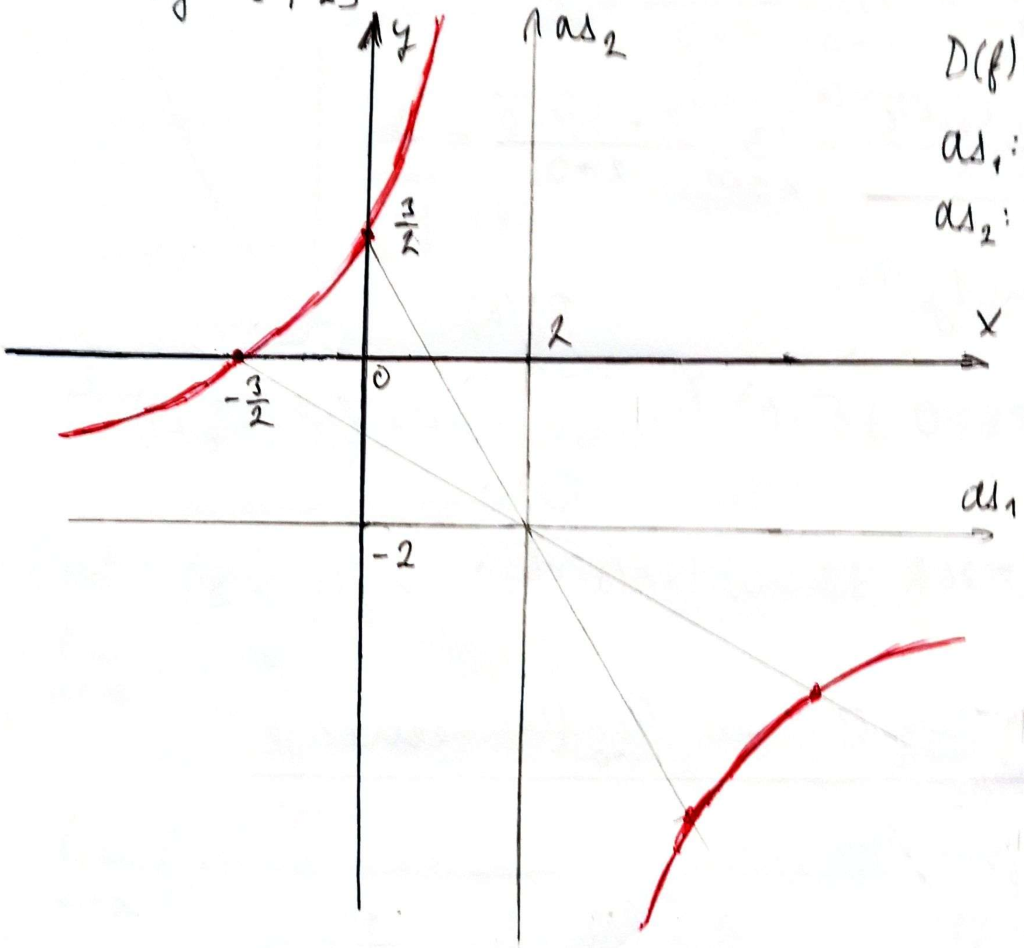
→ P_x: $2x = -3 \Rightarrow P_x[-\frac{3}{2}; 0]$

→ $P_y = [0; \frac{3}{2}]$

$D(f) = \mathbb{R} \setminus \{2\}$

$as_1: y = -2$

$as_2: x = 2$



• $\lim\limits_{x \to 2^+} \dfrac{2x+3}{2-x} = \dfrac{7}{0^-} = \underline{-\infty}$

• $\lim\limits_{x \to 2^-} \dfrac{2x+3}{2-x} = \dfrac{7}{0^+} = \underline{\infty}$

• $\lim\limits_{x \to \infty} \dfrac{2x+3}{2-x} = \dfrac{2+0}{0-1} = \underline{-2}$

• $\lim\limits_{x \to -\infty} \dfrac{2x+3}{2-x} = \dfrac{2+0}{0-1} = \underline{-2}$

• $f: y = \frac{x^2-1}{x^2+x-6} = \frac{(x-1)(x+1)}{(x+3)(x-2)} \Rightarrow D(f) = \mathbb{R} \setminus \{2; -3\}$

• $\lim_{x \to \infty} \frac{x^2-1}{x^2+x-6} = \frac{1-0}{1+0-0} = \underline{\underline{1}} \quad \wedge \quad \lim_{x \to -\infty} \frac{x^2-1}{x^2+x-6} = \underline{\underline{1}} \Rightarrow as$

• $\lim_{x \to 2^+} f(x) = \frac{1 \cdot 3}{5 \cdot 0^+} = \frac{3}{0^+} = \underline{\underline{\infty}}$

• $\lim_{x \to 2^-} f(x) = \frac{3}{5 \cdot 0^-} = \underline{\underline{-\infty}}$

• $\lim_{x \to -3^+} f(x) = \frac{8}{0^+(-5)} = \frac{8}{0^-} = \underline{\underline{-\infty}}$

• $\lim_{x \to -3^-} f(x) = \frac{8}{0^-(-5)} = \frac{8}{0^+} = \underline{\underline{\infty}}$

$P_x: 0 = x^2-1 \Rightarrow x = \pm 1$

$P_{x_1} = [1; 0] \quad P_{x_2} = [-1; 0]$

$P_y: y = \frac{-1}{-6} \Rightarrow P_y = [0; \frac{1}{6}]$