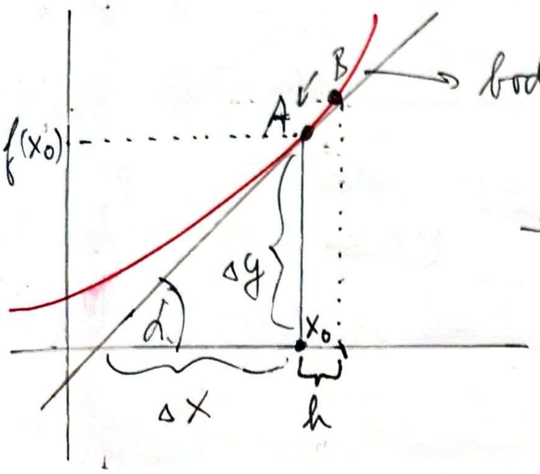
DERIVACE = směrnice tečny na graf

$$f'(x_0) = \operatorname{tg}\alpha = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



→ bod B přibližuju k bodu A

→ směrnice $= \frac{\Delta y}{\Delta x} = \operatorname{tg}(\alpha)$

- $(x^n)' = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + \binom{n}{1} x^{n-1} \cdot h + \ldots + \binom{n}{n} h^n - x^n}{h} =$

$$= \lim_{h \to 0} \left(n \cdot x^{n-1} + \binom{n}{2} x^{n-2} \underbrace{h + \ldots + h^{n-1}}_{\to 0} \right) = \underline{n \cdot x^{n-1}}$$

- $(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cdot \overset{\to 1}{\cos h} + \cos x \cdot \sin h - \sin x}{h} =$

$$= \lim_{h \to 0} \frac{\sin(x) \cdot 1 - \sin x + \cos x \sin h}{h} = \underline{\cos x}$$

- $(\cos x)' = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cdot \overset{\to 1}{\cos h} - \sin x \cdot \sin h - \cos x}{h} =$

$$= \lim_{h \to 0} \frac{-\sin x \cdot \sin h}{h} = \underline{-\sin x}$$

- $(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{\cos x \cdot \cos x - \sin x (-\sin x)}{\cos^2 x} = \underline{\frac{1}{\cos^2 x}}$

- $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \underline{\frac{1}{2\sqrt{x}}}$

- $(\operatorname{cotg} x)' = \lim_{h \to 0} \frac{\operatorname{cotg}(x+h) - \operatorname{cotg}(x)}{h} = \lim_{h \to 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos(x)}{\sin(x)}}{h} =$

$$= \lim_{h \to 0} \frac{\frac{\cos(x+h)\sin(x) - \cos(x) \cdot \sin(x+h)}{\sin(x+h)\sin(x)}}{h} = \lim_{h \to 0} \frac{\overbrace{\sin(x - x - h)}^{-\sin(h)}}{h \cdot \sin(x+h)\sin(x)} =$$

$$= \lim_{h \to 0} \frac{-1}{\sin(x+h)\sin(x)} = \underline{-\frac{1}{\sin^2(x)}}$$

Vzorce pro derivace

Sunday, April 11, 2021 14:19

- $n' = 0$
- $[x^n]' = n \cdot x^{n-1}$
- $[\sqrt{x}]' = \frac{1}{2\sqrt{x}}$
- $[n^x]' = n^x \cdot \ln(n)$
- $[e^x]' = e^x$
- $[\log_n x]' = \frac{1}{x \cdot \ln(n)}$
- $[\ln(x)]' = \frac{1}{x}$
- $[\sin(x)]' = \cos(x)$
- $[\cos(x)]' = -\sin(x)$
- $[\tan(x)]' = \frac{1}{\cos^2(x)}$
- $[\mathrm{cotg}(x)]' = -\frac{1}{\sin^2(x)}$
- $[\arcsin(x)]' = \frac{1}{\sqrt{1-x^2}}$
- $[\arccos(x)]' = -\frac{1}{\sqrt{1-x^2}}$
- $[\arctan(x)]' = \frac{1}{1+x^2}$
- $[\mathrm{arccotg}(x)]' = -\frac{1}{1+x^2}$
- $[n \cdot f]' = n \cdot f'$
- $[f \pm g]' = f' \pm g'$
- $[f \cdot g]' = f' \cdot g + f \cdot g'$
- $[f \cdot g \cdot h]' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$
- $\left[\frac{f}{g}\right]' = \frac{f' \cdot g - f \cdot g'}{g^2}$
- $[f(g)]' = f'(g) \cdot g'$
- $[f(g(h))]' = f'(g(h)) \cdot g'(h) \cdot h'$
- $[f^g]' = \left[e^{\ln(f^g)}\right]' = \left[e^{g \cdot \ln(f)}\right]' = e^{g \cdot \ln(f)} \cdot [g \cdot \ln(f)]'$
    - Z funkce na funkci udělám složenou funkci $e^x \wedge x = g \cdot \ln(f)$

→ příklady

- $f_5: y = \frac{1}{x^2} + \frac{4}{x^3} \Rightarrow f_5': y = \frac{-2}{x^3} + \frac{-12}{x^4}$ $D(f_5) = R \setminus \{0\}$, $D(f_5') = R \setminus \{0\}$

- $f_6: y = \frac{2}{x} - \frac{1}{7} \cdot \sqrt[5]{x^2} \Rightarrow f_6': y = \frac{-2}{x^2} - \frac{2}{35} \cdot x^{-\frac{3}{5}}$ $D(f_6) = R \setminus \{0\}$, $D(f_6') = R \setminus \{0\}$

- $f_7: y = 2\sin(x) + 3\cos(x) \Rightarrow f_7': y = 2\cos(x) - 3\sin(x)$ $D(f_7) = D(f_7') = R$

- $f_8: y = x^7 - 7\cos(x) \Rightarrow f_8': y = 7x^6 + 7\sin(x)$ $D(f_8) = D(f_8') = R$

- $f_9: y = 6\ln(x) - 9\log(x) \Rightarrow f_9': y = \frac{6}{x} - \frac{9}{x \cdot \ln(10)}$ $D(f_9) = R^+$, $D(f_9') = R \setminus \{0\}$

- $f_{10}: y = 3^x + 2 \cdot e^x \Rightarrow f_{10}': y = 3^x \cdot \ln(3) + 2 \cdot e^x$ $D(f_{10}) = D(f_{10}') = R$

- $h_1: y = x \cdot \sin(x) \Rightarrow h_1': y = \sin(x) + x\cos(x)$

- $h_2: y = (x^2-1) \cdot \sin(x) \Rightarrow h_2': y = 2x \cdot \sin(x) + (x^2-1) \cdot \cos(x)$

- $h_3: y = \sin(x) \cdot \operatorname{tg}(x) \Rightarrow h_3': y = \cos(x) \cdot \operatorname{tg}(x) + \sin(x) \cdot \frac{1}{\cos^2(x)}$

$$y = \cos(x) \frac{\sin(x)}{\cos(x)} + \frac{\sin(x)}{\cos^2(x)} = \sin(x) + \frac{\sin(x)}{\cos^2(x)}$$

- $h_4: y = \frac{2x-1}{x+3} \Rightarrow h_4': y = \frac{2(x+3) - (2x-1)}{(x+3)^2} = \frac{2x+6-2x+1}{(x+3)^2} = \frac{7}{(x+3)^2}$

- $h_5: y = \frac{x^2+2x}{1-x^2} \Rightarrow h_5': y = \frac{(2x+2)(1-x^2) - (x^2+2x)(-2x)}{(1-x^2)^2}$

$$y = \frac{2x - 2x^3 + 2 - 2x^2 + 2x^3 + 4x^2}{(1-x^2)^2} = \frac{2x^2+2x+2}{(1-x^2)^2}$$

- $h_6: y = \frac{\sin(x) + \cos(x)}{\sin(x) - \cos(x)}$

$$h_6': y = \frac{(\cos(x) - \sin(x))(\sin(x) - \cos(x)) - (\sin(x) + \cos(x))(\cos(x) + \sin(x))}{(\sin(x) - \cos(x))^2} =$$

$$= \frac{-(\sin(x) - \cos(x))^2 - (\sin(x) + \cos(x))^2}{1 - 2\sin(x)\cos(x)} =$$

$$= \frac{-1 + 2\sin(x)\cos(x) - 1 - 2\sin(x)\cos(x)}{1 - \sin(2x)} = \frac{-2}{1 - \sin(2x)}$$

příklady

1) $y = (x^3-2)^5 \Rightarrow \frac{dy}{dx} = 5(x^3-2)^4 \cdot 3x^2 = 15x^2(x^3-2)^4$

2) $y = (5-2x)^{-2} \Rightarrow \frac{dy}{dx} = -2(5-2x)^{-3} \cdot (-2) = 4(5-2x)^{-3}$

5) $y = \sin(2x) \Rightarrow \frac{dy}{dx} = \cos(2x) \cdot 2 = 2\cos(2x)$

6) $y = \sin\left(\frac{x}{3} + \frac{\pi}{2}\right) \Rightarrow \frac{dy}{dx} = \cos\left(\frac{x}{3} + \frac{\pi}{2}\right) \cdot \frac{1}{3} = \frac{1}{3}\cos\left(\frac{x}{3} + \frac{\pi}{2}\right)$

10) $y = e^{\frac{x}{2}} \Rightarrow \frac{dy}{dx} = e^{\frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{2}e^{\frac{x}{2}}$

11) $y = e^{1+\cos(x)} \Rightarrow \frac{dy}{dx} = e^{1+\cos(x)} \cdot (-\sin(x)) = -\sin(x) \cdot e^{1+\cos(x)}$

14) $y = \sqrt{\frac{1+e^x}{1-e^x}} \Rightarrow \frac{dy}{dx} = \frac{1}{2 \cdot \frac{\sqrt{1+e^x}}{\sqrt{1-e^x}}} \cdot \left(\frac{1+e^x}{1-e^x}\right)' = \frac{\sqrt{1-e^x}}{2\sqrt{1+e^x}} \cdot \left(\frac{1+e^x}{1-e^x}\right)' =$

$= \frac{\sqrt{1-e^x}}{2\sqrt{1+e^x}} \cdot \frac{e^x(1-e^x) - (1+e^x)(-e^x)}{(1-e^x)^2} = \frac{\sqrt{1-e^x}}{2\sqrt{1+e^x}} \cdot \frac{e^x - e^{2x} + e^x + e^{2x}}{(1-e^x)^2} =$

$= \frac{2e^x\sqrt{1-e^x}}{2(1-e^x)^2\sqrt{1+e^x}} = \frac{e^x}{(1-e^x)\sqrt{1-e^x}\sqrt{1+e^x}} = \frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$

15) $y = \sqrt{\frac{e^x - e^{-x}}{e^x + e^{-x}}} = \sqrt{\frac{e^{2x}-1}{e^{2x}+1}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{e^{2x}+1}}{2\sqrt{e^{2x}-1}} \cdot \left(\frac{e^{2x}-1}{e^{2x}+1}\right)' =$

$= \frac{\sqrt{e^{2x}+1}}{2\sqrt{e^{2x}-1}} \cdot \frac{e^{2x} \cdot 2(e^{2x}+1) - (e^{2x}-1)e^{2x} \cdot 2}{(e^{2x}+1)^2} =$

$= \frac{\sqrt{e^{2x}+1}}{2\sqrt{e^{2x}-1}} \cdot \frac{2 \cdot e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x}+1)^2} =$

$= \frac{4e^{2x}\sqrt{e^{2x}+1}}{2(e^{2x}+1)^2\sqrt{e^{2x}-1}} = \frac{2e^{2x}}{(e^{2x}+1)\sqrt{e^{2x}+1}\sqrt{e^{2x}-1}} = \frac{2e^{2x}}{(e^{2x}+1)\sqrt{e^{4x}-1}}$

17) $y = \ln(x^2+x+7) \Rightarrow \frac{dy}{dx} = \frac{1}{x^2+x+7} \cdot (2x+1) = \frac{2x+1}{x^2+x+7}$

18) $y = \ln(\sin(x)) - \ln(\cos(x)) \Rightarrow \frac{dy}{dx} = \frac{1}{\sin(x)}\cos(x) - \frac{1}{\cos(x)} \cdot (-\sin(x)) =$

$= \mathrm{tg}(x) + \mathrm{cotg}(x)$

$$20)\ y = \ln\frac{1+e^x}{1-e^x} \Rightarrow \frac{dy}{dx} = \frac{1-e^x}{1+e^x}\cdot\frac{e^x(1-e^x)-(1+e^x)(-e^x)}{(1-e^x)(1-e^x)} =$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{1-e^{2x}} = \frac{2e^x}{1-e^{2x}}$$

$$21)\ y = \ln\frac{1+\sin(x)}{1-\sin(x)} \Rightarrow \frac{dy}{dx} = \frac{1-\sin(x)}{1+\sin(x)}\cdot\frac{\cos(x)(1-\sin(x))-(1+\sin(x))(-\cos(x))}{(1-\sin(x))(1-\sin(x))} =$$

$$= \frac{\cos(x)-\sin(x)\cos(x)+\cos(x)+\sin(x)\cos(x)}{1-\sin^2(x)} = \frac{2\cos(x)}{\cos^2(x)} = \frac{2}{\cos(x)}$$

$$23)\ y = \ln(5e^x + x^5) \Rightarrow \frac{dy}{dx} = \frac{1}{5e^x+x^5}\cdot(5e^x+5x^4) = \frac{5e^x+5x^4}{5e^x+x^5}$$

$$24)\ y = \ln\left(tg\left(\frac{x}{2}\right)\right) \Rightarrow \frac{dy}{dx} = \frac{1}{tg\left(\frac{x}{2}\right)}\cdot\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot\frac{1}{2} = \frac{1}{2\cdot\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\cdot\cos^2\left(\frac{x}{2}\right)} = \frac{1}{\sin(x)}$$

$$26)\ y = \arccos\left(\frac{x}{5}\right) \Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-\frac{x^2}{25}}}\cdot\frac{1}{5} = -\frac{1}{\sqrt{25-x^2}}$$

$$27)\ y = \arcsin\sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}}$$

- $y = (e^{x\cdot\ln 2})^x = e^{2x\cdot\ln 2} \Rightarrow \frac{dy}{dx} = e^{2x\ln(2)}\cdot 2\ln(2)$

- $y = (e^2 - x^2)^{\sin(x)} = e^{\ln(e^2-x^2)^{\sin(x)}} = e^{\sin(x)\cdot\ln(e^2-x^2)}$

$$\frac{dy}{dx} = e^{\sin(x)\cdot\ln(e^2-x^2)}\cdot\left(\cos(x)\cdot\ln(e^2-x^2) + \sin(x)\cdot\frac{1}{e^2-x^2}\cdot(-2x)\right)$$

→ derivace složených funkcí

- $f'(x) = \left[\left(\sqrt{2x^3-1}+2\right)^8\right]' = (6x^2)\cdot\frac{1}{2\sqrt{2x^3-1}}\cdot 8\left(\sqrt{2x^3-1}+2\right)^7 =$

$$= \frac{24x^2\left(\sqrt{2x^3-1}+2\right)^7}{\sqrt{2x^3-1}}$$

- $f'(x) = \left(\sqrt{x+\sqrt{5x}}\right)' = \frac{1}{2\sqrt{x+\sqrt{5x}}}\cdot\left(1+(\sqrt{5x})'\right) =$

$$= \frac{1}{2\sqrt{x+\sqrt{5x}}}\cdot\left(1+\frac{1}{2\sqrt{5x}}\cdot 5\right) = \frac{1}{2\sqrt{x+\sqrt{5x}}} + \frac{5}{2\sqrt{5x}\cdot 2\sqrt{x+\sqrt{5x}}} =$$

$$= \frac{2\sqrt{5x}+5}{4\sqrt{5x^2+5x\sqrt{5x}}}$$

- $f'(x) = \left[(3x^4+x^2)^{-10}\right]' = -10(3x^4+x^2)^{-11}\cdot(12x^3+2x) = \frac{-20(6x^3+x)}{(3x^4+x^2)^{11}}$

- $f'(x) = \left[\cos(2x+4)\right]' = -\sin(2x+4)\cdot 2 = -2\sin(2x+4)$

- $f'(x) = \left[\sin^2(x)\right]' = 2\cdot\sin(x)\cdot\cos(x) = \sin(2x)$

- $f'(x) = \left[\sin(x^2)\right]' = \cos(x^2)\cdot 2x$

- $f'(x) = \left[\sqrt{\cos(2x)}\right]' = \frac{1}{2\sqrt{\cos(2x)}}\cdot(-1)\cdot\sin(2x)\cdot 2 = -\frac{\sin(2x)}{\sqrt{\cos(2x)}}$

- $f'(x) = \left[\sqrt[3]{\cos(2x)+2x}\right]'$ → $(\sqrt[3]{x})' = \frac{1}{3}\cdot x^{-\frac{2}{3}} = \frac{1}{3\cdot\sqrt[3]{x^2}}$

$$= \frac{1}{3\cdot\sqrt[3]{(\cos(2x)+2(x))^2}}\cdot\left((\cos(2x))'+2\right) = \frac{-\sin(2x)\cdot 2+2}{3\cdot\sqrt[3]{(\cos(2x)+2x)^2}}$$

- $f'(x) = \left[tg\left(3x-\frac{\pi}{4}\right)\right]' = \frac{1}{\cos^2\left(3x-\frac{\pi}{4}\right)}\cdot 3 = \frac{3}{\cos^2\left(3x-\frac{\pi}{4}\right)}$

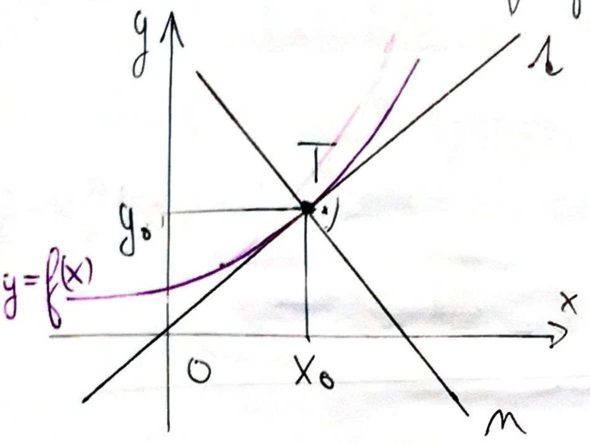
- $f'(x) = \left[\sqrt{\sin(3x)+5}\right]' = \frac{1}{2\sqrt{\sin(3x)+5}}\cdot\left(\cos(3x)\cdot 3+0\right) = \frac{3\cos(3x)}{2\sqrt{\sin(3x)+5}}$

- $f'(x) = \left[\ln(3\sin(x)-8)\right]' = \frac{1}{3\sin(x)-8}\cdot 3\cos(x) = \frac{3\cos(x)}{3\sin(x)-8}$

- $f'(x) = \frac{d}{dx}e^{\sin(x)} = e^{\sin(x)}\cdot\cos(x)$

VYUŽITÍ DERIVACE

• Tečna a normála grafu funkce



$t: y = k\cdot x + q = f'(x_0)\cdot x + q$

$T \in t: \quad y_0 = f'(x_0)\cdot x_0 + q$

$\Rightarrow q = f(x_0) - f'(x_0)\cdot x_0$

$t: y - y_0 = f'(x_0)(x - x_0)$

$$k_t \cdot k_n = -1 \quad \left\{ \begin{array}{l} k_t = k \\ k_n = -\frac{1}{k} \end{array} \right\} \quad n: y - y_0 = -\frac{1}{f'(x_0)}\cdot(x - x_0)$$

→ příklady

• $f: y = 2\sqrt{2}\cdot \sin(x) \wedge T[\frac{\pi}{4}; 2]$

$f': y = 2\sqrt{2}\cdot \cos(x) \Rightarrow f'(\frac{\pi}{4}) = 2 \rightarrow k_t = 2$, $\quad k_n = -\frac{1}{2}$

$t: y - 2 = 2(x - \frac{\pi}{4}) \Rightarrow y = 2x + 2 - \frac{\pi}{2}$

$4x - 2y + 4 - \pi = 0$

$n: y - 2 = -\frac{1}{2}(x - \frac{\pi}{4}) \Rightarrow y = -\frac{1}{2}x + 2 + \frac{\pi}{8}$

$4x + 8y - 16 - \pi = 0$

13) $f: y = 8(4 + x^2)^{-1} \wedge T[2; 1]$

$f': y = -8(4 + x^2)^{-2}\cdot 2x \Rightarrow f'(2) = \frac{-32}{8^2} = -\frac{1}{2}$ $\quad k_t = -\frac{1}{2}$, $\quad k_n = 2$

$t: y - 1 = -\frac{1}{2}(x - 2) = -\frac{1}{2}x + 1 \Rightarrow y = -\frac{1}{2}x + 2$

$x + 2y - 4 = 0$

$n: y - 1 = 2(x - 2) = 2x - 4 \Rightarrow y = 2x - 3$

$2x - y - 3 = 0$

• **Úhel protnutí dvou křivek**

1) najdu průsečík křivek

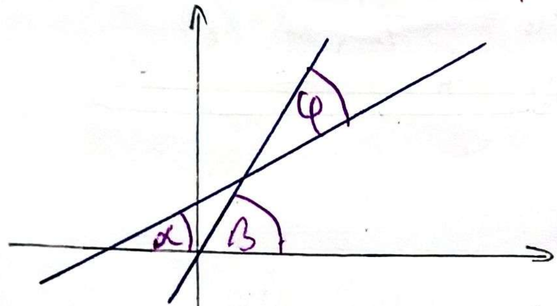
2) najdu směrnice tečen křivek v tom průsečíku

3) určím úhel sevřený těmi tečnami

⇒ využívám: $ax + by + c = 0 \Rightarrow by = -ax - c \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$

$$\left.\begin{array}{l} \Rightarrow k = -\frac{a}{b} \\ \Rightarrow \vec{v}(-b, a) \end{array}\right\} k = \frac{v_2}{v_1} \Leftrightarrow \vec{v}_h(v_1; v_2)$$

$$\cos(\varphi) = \frac{|\vec{v}_h \cdot \vec{v}_g|}{|\vec{v}_h| \cdot |\vec{v}_g|}$$



$$\varphi = |\alpha - \beta|$$

$$tg(\alpha - \beta) = \frac{tg(\alpha) - tg(\beta)}{1 + tg(\alpha)\, tg(\beta)}$$

$$\left.\begin{array}{l} tg(\alpha) = k_1 \\ tg(\beta) = k_2 \end{array}\right\} tg(\varphi) = \left|\frac{k_1 - k_2}{1 + k_1 \cdot k_2}\right|$$

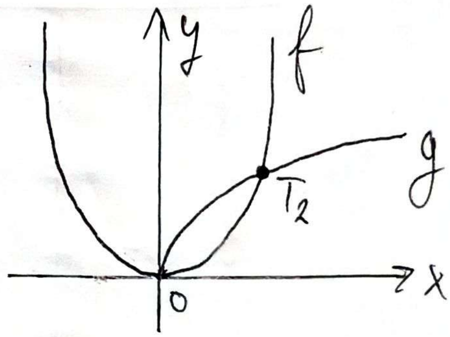
⇒ příklad: $f: y = \frac{x^2}{2} \wedge g: y = \sqrt{2x}$

1) průsečík $T \in f \cap g$

$\frac{x^2}{2} = \sqrt{2x} \Rightarrow \frac{x^4}{4} = 2x \Rightarrow x^4 = 8x$

$\Rightarrow x(x^3 - 8) = 0 \Rightarrow x_1 = 0 \wedge x_2 = 2$

$\Rightarrow T_1[0, 0] \wedge T_2[2, 2]$



2) T_2: $k_f = f'(2) \wedge f'(x) = x \Rightarrow \underline{k_f = 2}$

$k_g = g'(2) \wedge g'(x) = \frac{1}{2\sqrt{2x}} \cdot 2 = \frac{1}{\sqrt{2x}} \Rightarrow \underline{\underline{k_g = \frac{1}{2}}}$

3) $k_f = 2 = \frac{v_2}{v_1} \Rightarrow \vec{v}_f = (1, 2)$

$k_g = \frac{1}{2} = \frac{v_2}{v_1} \Rightarrow \vec{v}_g = (2, 1)$

$\Rightarrow \cos(\varphi) = \frac{|2 + 2|}{\sqrt{5} \cdot \sqrt{5}} = \frac{4}{5} \Rightarrow \underline{\underline{\varphi \doteq 37°}}$

→ příklady

8) $f: y = \frac{\cos(x)}{1+2\sin(x)} \wedge T\left[\frac{\pi}{6}; y_0\right] \rightarrow$ tečna $= ?$

$$T: y_0 = \frac{\cos\frac{\pi}{6}}{1+2\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{1+1} = \frac{\sqrt{3}}{4} \Rightarrow \underline{\underline{T\left[\frac{\pi}{6}; \frac{\sqrt{3}}{4}\right]}}$$

$$f': y = \frac{-\sin(x)\cdot(1+2\sin(x)) - \cos(x)2\cos(x)}{(1+2\sin(x))^2}$$

$$y = \frac{-\sin(x) - 2\sin^2(x) - 2\cos^2(x)}{(1+2\sin(x))^2} = \frac{-\sin(x)-2}{(1+2\sin(x))^2}$$

$$\Rightarrow k_1 = f'\left(\frac{\pi}{6}\right) = \frac{-\frac{1}{2}-2}{(1+1)^2} = \frac{-\frac{5}{2}}{4} = \underline{\underline{\frac{-5}{8}}}$$

$$\Rightarrow y - \frac{\sqrt{3}}{4} = -\frac{5}{8}\left(x - \frac{\pi}{6}\right) \Rightarrow \underline{\underline{y = -\frac{5}{8}x + \frac{\sqrt{3}}{4} + \frac{5\pi}{48}}}$$

$$\underline{\underline{48y + 30x - 12\sqrt{3} - 5\pi = 0}}$$

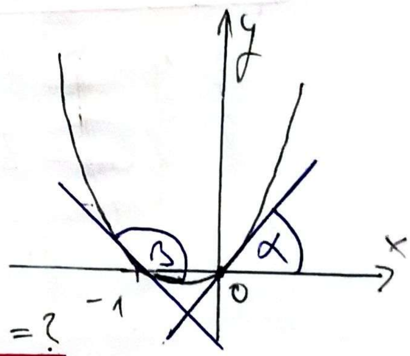
11) $f: y = x^2 + x \rightarrow$ úhel pod kterým protíná osu $x = ?$

$$P: 0 = x(x+1) \Rightarrow x = 0 \vee x = -1 \Rightarrow P_1[0;0] \quad P_2[0;-1]$$

$$f'(x) = 2x+1$$

$$\Rightarrow tg(\alpha) = f'(0) = 1 \rightarrow \underline{\underline{\alpha = 45^\circ}}$$

$$\Rightarrow tg(\beta) = f'(-1) = -1 \Rightarrow \underline{\underline{\beta = 135^\circ}}$$

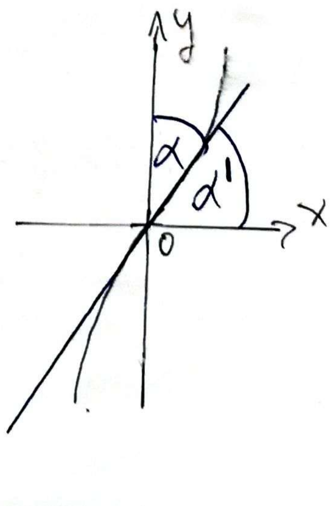


17) $f: y = tg(x)\sqrt{3} \rightarrow$ úhel pod kterým protíná $y = ?$

$$P[0,0] \wedge \alpha = 90 - \alpha'$$

$$f'(x) = \frac{\sqrt{3}}{\cos^2(x)}$$

$$\Rightarrow tg(\alpha') = f'(0) = \sqrt{3} \Rightarrow \alpha' = 60^\circ \Rightarrow \underline{\underline{\alpha = 30^\circ}}$$



- Tečna daná směrem

$$f: y = \frac{x-1}{x+1} \quad \wedge \quad t \parallel p: x - 2y + 7 = 0 \quad \rightarrow t = ?$$

$$p: 2y = x + 7 \Rightarrow y = \frac{x}{2} + \frac{7}{2} \Rightarrow k_p = k_t = \frac{1}{2}$$

$$f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} = k_t$$

$$\Rightarrow \frac{2}{(x_0+1)^2} = \frac{1}{2} \Rightarrow (x_0+1)^2 = 4 \Rightarrow x_0 + 1 = \pm 2 \quad \begin{cases} x_{01} = 1 \\ x_{02} = -3 \end{cases}$$

$$\Rightarrow y_{01} = 0 \quad \wedge \quad y_{02} = \frac{-4}{-2} = 2 \Rightarrow T_1[1,0] \wedge T_2[-3,2]$$

$$\Rightarrow t_1: y = \frac{1}{2}(x-1)$$

$$t_2: y - 2 = \frac{1}{2}(x+3)$$

- Tečna z bodu

$$f: y = x^2 + 3x + 2 \quad \wedge \quad A[1,-1] \in t \rightarrow t = ?$$

$$t: y - y_0 = k(x - x_0) \rightarrow k = f'(x_0) \qquad T[x_0, y_0]$$

$$k: f'(x) = 2x + 3 \Rightarrow f'(x_0) = 2x_0 + 3$$

$$A \in t: -1 - y_0 = (2x_0+3)(1-x_0) = 2x_0 - 2x_0^2 + 3 - 3x_0$$

$$-y_0 = -2x_0^2 - x_0 + 4$$

$$\left.\begin{array}{l} y_0 = 2x_0^2 + x_0 - 4 \\ T \in f: y_0 = x_0^2 + 3x_0 + 2 \end{array}\right\} \ominus \quad 0 = x_0^2 - 2x_0 - 6$$

$$D = 4 + 24 \Rightarrow x_{01,2} = \frac{2 \pm 2\sqrt{7}}{2}$$

$$\Rightarrow x_{01} = 1 + \sqrt{7} \Rightarrow y_{01} = (1+\sqrt{7})^2 + 3(1+\sqrt{7}) + 2$$

$$y_{01} = 1 + 2\sqrt{7} + 7 + 3 + 3\sqrt{7} + 2 = 13 + 5\sqrt{7}$$

$$\Rightarrow k_1 = 2(1+\sqrt{7}) + 3 = 5 + 2\sqrt{7}$$

$$\Rightarrow x_{02} = 1 - \sqrt{7} \Rightarrow y_{02} = 1 - 2\sqrt{7} + 7 + 3 - 3\sqrt{7} + 2 = 13 - 5\sqrt{7}$$

$$\Rightarrow k_2 = 2 - 2\sqrt{7} + 3 = 5 - 2\sqrt{7}$$

$$\Rightarrow t_1: y - 13 - 5\sqrt{7} = (5 + 2\sqrt{7})(x - 1 - \sqrt{7}) = 5x + 2\sqrt{7}x - 5 - 2\sqrt{7} - 5\sqrt{7} - 14$$

$$y = x(5 + 2\sqrt{7}) - 6 - 2\sqrt{7}$$

$$t_2: y - 13 + 5\sqrt{7} = (5 - 2\sqrt{7})(x - 1 + \sqrt{7}) = 5x - 2\sqrt{7}x - 5 + 2\sqrt{7} + 5\sqrt{7} - 14$$

$$y = x(5 - 2\sqrt{7}) - 6 + 2\sqrt{7}$$

→ príklady

1) $f: y = x - \ln(1+x^2) \wedge t \parallel p: y = x \rightarrow t, n = ?$

$$p: y = x \Rightarrow k_t = 1 \Rightarrow k_n = -1$$

$$f'(x) = 1 - \frac{1}{1+x^2} 2x = 1 - \frac{2x}{x^2+1} = k_t$$

$$\Rightarrow 1 = 1 - \frac{2x_0}{x_0^2+1} \Rightarrow \frac{2x_0}{x_0^2+1} = 0 \qquad \left.\begin{aligned} &\Rightarrow \underline{x_0 = 0} \\ &\Rightarrow y_0 = -\ln(1) = 0 \end{aligned}\right\} T[0,0]$$

$$\Rightarrow t: \quad y - 0 = 1(x-0)$$

$$\underline{\underline{x - y = 0}}$$

$$\Rightarrow n: \quad y - 0 = -1(x-0)$$

$$\underline{\underline{x + y = 0}}$$

2) $f: y = 4x - x^2 \wedge R[2,8] \rightarrow$ tečna z bodu $R = ?$

$$t: y - y_0 = k(x - x_0) \rightarrow k = f'(x_0) \qquad T[x_0, y_0]$$

$$k: f'(x) = -2x + 4 \Rightarrow f'(x_0) = -2x_0 + 4$$

$$R \in t: 8 - y_0 = (4 - 2x_0)(2 - x_0) = 8 - 4x_0 - 4x_0 + 2x_0^2$$

$$\left.\begin{aligned} &\underline{y_0 = 8x_0 - 2x_0^2} \\ T \in f: \; &y_0 = 4x_0 - x_0^2 \end{aligned}\right\} \begin{aligned} 8x_0 - 2x_0^2 &= -x_0^2 + 4x_0 \\ 4x_0 - x_0^2 &= 0 \\ x_0(4 - x_0) &= 0 \end{aligned} \quad \begin{cases} \underline{\underline{x_{0_1} = 0}} \\ \underline{\underline{x_{0_2} = 4}} \end{cases}$$

$$\Rightarrow k_1 = 4 \wedge k_2 = -8 + 4 = -4$$

$$\Rightarrow y_{0_1} = 0 \wedge y_{0_2} = 16 - 16 = 0$$

$$\Rightarrow \underline{\underline{t_1: y = 4x}}$$

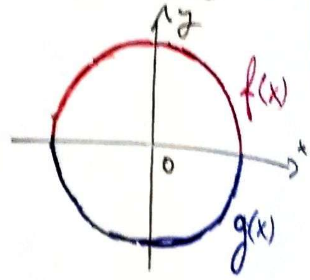
$$t_2: y = -4(x - 4) = -4x + 16$$

$$\underline{\underline{t_2: 4x + y - 16 = 0}}$$

IMPLICITNÍ FUNKCE A JEJÍ DERIVACE

→ implicitní fce = fce daná jako součást nějaké rovnice

→ příklad



$k: x^2 + y^2 = 25 \Rightarrow y = \pm\sqrt{25 - x^2}$ < $f(x) = \sqrt{25-x^2}$; $g(x) = -\sqrt{25-x^2}$

$\Rightarrow k: x^2 + [f(x)]^2 = 25$ ← implicitní fce

→ derivace: $[f(x)]^2$ – složená fce $\Rightarrow ([f(x)]^2)' = 2f(x)\cdot f'(x)$

$\Rightarrow k': 2x + 2f(x)\cdot f'(x) = 0$ ← byla to konstanta

$2f(x)\cdot f'(x) = -2x \Rightarrow f'(x) = -\frac{x}{f(x)}$ ← derivace implicitní fce

→ využití derivace implicitní fce pro sestrojení tečnice kružnice

- $k: x^2 + y^2 - 6x - 4y - 5 = 0 \wedge t \parallel p: x + y + 4 = 0 \rightarrow T[x_0, y_0]$

$p: y = -x - 4 \Rightarrow k = -1$

$\frac{d}{dx}: 2x + 2y\cdot y' - 6 - 4y' = 0 \Rightarrow y'(y-2) = 3 - x \Rightarrow y' = \frac{3-x}{y-2}$

$\Rightarrow k = y'(x_0) \Rightarrow -1 = \frac{3-x_0}{y_0-2} \Rightarrow -y_0 + 2 = 3 - x_0 \Rightarrow y_0 = x_0 - 1$

$T \in k: x_0^2 + y_0^2 - 6x_0 - 4y_0 - 5 = 0$

$x_0^2 + x_0^2 - 2x_0 + 1 - 6x_0 - 4x_0 + 4 - 5 = 0$

$2x_0^2 - 12x_0 = 0 \Rightarrow x_0(x_0 - 6) = 0$ < $x_{01} = 0 \rightarrow y_{01} = -1$; $x_{02} = 6 \rightarrow y_{02} = 5$

$\Rightarrow t_1: y + 1 = -1(x - 0) \Rightarrow y + x + 1 = 0$

$t_2: y - 5 = -1(x - 6) \Rightarrow y + x - 11 = 0$

→ Obecná rovnice tečny na kuželosečku

$K: A\cdot x^2 + B\cdot y^2 + Cx + Dy + E = 0$

$t: Ax\cdot x_0 + B\cdot y\cdot y_0 + \frac{C}{2}(x + x_0) + \frac{D}{2}(y + y_0) + E = 0$

→ příklad – tečna Geronovy lemniskáty

• $G: x^4 - x^2 + y^2 = 0 \quad \wedge \quad T\left[\frac{1}{2}, y_0\right] \quad \wedge \quad y_0 > 0$

$T \in G: \frac{1}{16} - \frac{1}{4} + y_0^2 = 0 \Rightarrow y_0^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16} \Rightarrow \underline{y_0 = \frac{\sqrt{3}}{4}} \Rightarrow T\left[\frac{1}{2}; \frac{\sqrt{3}}{4}\right]$

$\frac{d}{dx}: 4x^3 - 2x + 2y \cdot y' = 0 \Rightarrow y \cdot y' = x - 2x^3 \Rightarrow y' = \frac{x - 2x^3}{y}$

$\Rightarrow y'(x_0) = \frac{\frac{1}{2} - 2\frac{1}{8}}{\frac{\sqrt{3}}{4}} = \frac{\frac{1}{2} - \frac{1}{4}}{\frac{\sqrt{3}}{4}} = \frac{\frac{1}{4}}{\frac{\sqrt{3}}{4}} = \underline{\frac{\sqrt{3}}{3}}$

$\Rightarrow t: y - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right) \Rightarrow t: y - \frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{6} = 0$

$\Rightarrow \underline{t: 12y - 4\sqrt{3}x - \sqrt{3} = 0}$

$G: x = 0 \Rightarrow y = 0$

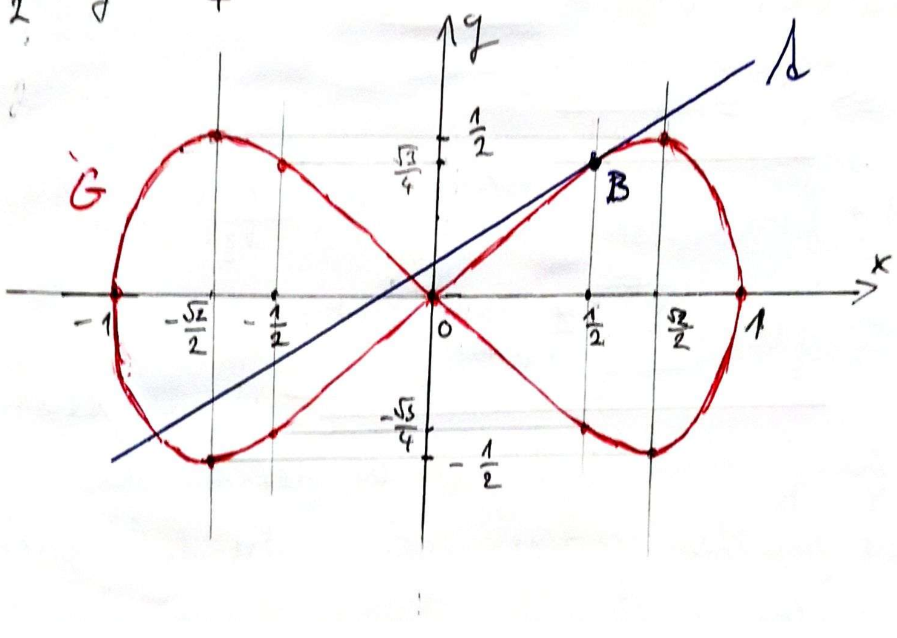
$x = \pm 1 \Rightarrow y = 0$

$x = \pm \frac{1}{2} \Rightarrow y = \pm \frac{\sqrt{3}}{4} \doteq \pm 0{,}43$

$y = \pm \frac{1}{2} \Rightarrow x^4 - x^2 + \frac{1}{4} = 0$

$D = 1 - 1 = 0$

$\Rightarrow x_{1,2}^2 = \frac{1}{2} \Rightarrow x = \pm \frac{\sqrt{2}}{2} \doteq \pm 0{,}71$



TAYLORŮV POLYNOM

→ pokud funkce f splňuje nějaké podmínky, pak sa funkci můžeme vyjádřit jako nekonečnou mocninnou řadu

- $f(x) = C_0 \cdot \underbrace{(x-a)^0}_{1} + C_1(x-a)^1 + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + \dots$

$\Rightarrow f(a) = C_0 \cdot 1 + C_1 \cdot (0) + C_2 \cdot (0) + \dots = \underline{\underline{C_0}}$

- $f'(x) = C_1 + 2 \cdot C_2(x-a) + 3 \cdot C_3(x-a)^2 + 4 \cdot C_4(x-a)^3 + \dots$

$\Rightarrow f'(a) = C_1 + 2 \cdot C_2 \cdot (0) + 3 \cdot C_3 \cdot (0) + \dots = \underline{\underline{C_1}}$

- $f''(x) = 2C_2 + 3 \cdot 2 \cdot C_3(x-a) + 4 \cdot 3 \cdot C_4(x-a)^2 + \dots$

$\Rightarrow f''(a) = 2C_2 + 0 + \dots = \underline{\underline{2C_2}} \Rightarrow C_2 = \frac{f^{(2)}(a)}{2}$

- $f^{(3)}(x) = 3 \cdot 2 \cdot 1 \cdot C_3 + 4 \cdot 3 \cdot 2 \cdot C_4(x-a) + \dots$

$\Rightarrow f^{(3)}(a) = 3 \cdot 2 \cdot 1 \cdot C_3 + 0 + \dots = \underline{\underline{3 \cdot 2 \cdot 1 \cdot C_3}} \Rightarrow C_3 = \frac{f^{(3)}(a)}{3 \cdot 2 \cdot 1} = \frac{f^{(3)}(a)}{3!}$

$$\Rightarrow C_n = \frac{f^{(n)}(a)}{n!}$$

$\Rightarrow f(x) = f(a) + f'(a) \cdot (x-a) + \frac{f^{2}(a)}{2!}(x-a)^2 + \frac{f^{3}(a)}{3!}(x-a)^3 + \dots$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

a si volím já

→ fce f musí být pro a definovaná!

→ využití pro počítání fcí sin, cos, ... na kalkulačce

→ $\sin(x)$, $a = 0$

n	$f^{(n)}(x)$	$f^{(n)}(a)$	C_n
0	$\sin(x)$	0	0
1	$\cos(x)$	1	1
2	$-\sin(x)$	0	0
3	$-\cos(x)$	-1	$-\frac{1}{3!}$
4	$\sin(x)$	0	0
5	$\cos(x)$	1	$\frac{1}{5!}$

$\Rightarrow \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$

podobně: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$

→ příklady - derivace

c) $f(x)=\sqrt{x\sqrt{x\sqrt{x}}} = x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = x^{\frac{7}{8}} \Rightarrow f'(x)=\frac{7}{8}x^{-\frac{1}{8}} = \frac{7}{8\sqrt[8]{x}}$

l) $f(x)=\frac{1}{4}\ln\left(\frac{x^2-1}{x^2+1}\right) \Rightarrow f'(x)=\frac{1}{4}\cdot\frac{x^2+1}{x^2-1}\cdot\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$

$\Rightarrow f'(x)=\frac{2x(2)}{4(x^2-1)(x^2+1)} = \frac{x}{x^4-1}$

k) $f(x)=\ln\left(\sqrt{\frac{1-\sin(x)}{1+\sin(x)}}\right) = \frac{1}{2}\ln\left(\frac{1-\sin x}{1+\sin x}\right)$

$1=\sin^2 x+\cos^2 x$

$\Rightarrow f'(x)=\frac{1}{2}\cdot\frac{1+\sin x}{1-\sin x}\cdot\frac{-\cos x(1+\sin x)-\cos x(1-\sin x)}{(1+\sin x)^2} =$

$=\frac{1}{2}\cdot\frac{-\cos x(2)}{(1-\sin x)(1+\sin x)} = -\frac{\cos(x)}{1-\sin^2(x)} = -\frac{1}{\cos x}$

m) $f(x)=\arcsin(\sin x-\cos x)$

$f'(x)=\frac{1}{\sqrt{1-(\sin x-\cos x)^2}}\cdot(\cos x+\sin x) = \frac{\sin x+\cos x}{\sqrt{1-1+2\sin x\cos x}} = \frac{\sin(x)+\cos(x)}{\sqrt{\sin(2x)}}$

l'Hospitalovo pravidlo

Věta 11 (l'Hospitalovo pravidlo). *Buď $x_0 \in \mathbb{R}^*$. Nechť je splněna jedna z podmínek*

- $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$,
- $\lim_{x\to x_0} |g(x)| = +\infty$.

Existuje-li (vlastní nebo nevlastní) $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$, *pak existuje také* $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ *a platí*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

V roce 1921 bylo dokázáno, že autorem tohoto pravidla je *Johann I. Bernoulli* (1667–1748), jehož byl *Guillaume Francois Antoine de l'Hospital* (1661–1704) žákem. Na základě poznámek z Bernoulliových přednášek vydal l'Hospital v roce 1696 první tištěnou učebnici diferenciálního počtu *Analýza nekonečně malých veličin.*

Výpočet limit s neurčitými výrazy pomocí l'Hospitalova pravidla:

- $$\infty - \infty \Rightarrow \lim_{x\to x_0} (f(x) - g(x)) = \lim_{x\to x_0} \left(\frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}} \right) = \lim_{x\to x_0} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}} \Rightarrow \frac{0}{0};$$
- $-\infty + \infty \Rightarrow$ analogicky jako předchozí úprava;
- $$0 \cdot \infty \Rightarrow \lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{g(x)}} \Rightarrow \frac{0}{0};$$
- $$0^0,\ \infty^0,\ 1^\infty \Rightarrow \lim_{x\to x_0} f(x)^{g(x)} = \lim_{x\to x_0} e^{g(x)\cdot \ln f(x)} = e^{\lim_{x\to x_0}(g(x) \ln f(x))}$$
  $$\Rightarrow \text{ předchozí případ } \Rightarrow \frac{0}{0}.$$

L'Hospitalovo pravidlo

10) a) $\lim\limits_{x\to 0} \frac{1-\cos 2x}{x\sin x} = \lim\limits_{x\to 0} \frac{1-\cos^2 x + \sin^2(x)}{x \sin x} = \lim\limits_{x\to 0} \frac{2\sin^2(x)}{x\sin(x)} = \lim\limits_{x\to 0} \frac{2\sin x}{x} = \underline{\underline{2}}$

LP: $\lim\limits_{x\to 0} \frac{1-\cos 2x}{x \sin x} = \left\| \frac{0}{0} \right\| = \lim\limits_{x\to 0} \frac{\sin(2x)\cdot 2}{\sin(x)+x\cos x} = \left\| \frac{0}{0} \right\| = \lim\limits_{x\to 0} \frac{2\cos(2x)\cdot 2}{\cos(x)+\cos(x)-x\sin(x)}$

$= \lim\limits_{x\to 0} \frac{4\cos(2x)}{2\cos x - x\sin x} = \frac{4}{2} = \underline{\underline{2}}$

d) $\lim\limits_{x\to 0} \frac{\sin 4x}{\sqrt{x+1}-1} = \lim\limits_{x\to 0} \frac{\sin 4x(\sqrt{x+1}+1)}{x+1-1} = 4\cdot \lim\limits_{x\to 0} \frac{\sin 4x(\sqrt{x+1}+1)}{4x} = 4\cdot 2 = \underline{\underline{8}}$

LP: $\left\| \frac{0}{0} \right\| = \lim\limits_{x\to 0} \frac{4\cos 4x}{\frac{1}{2\sqrt{x+1}}} = \lim\limits_{x\to 0} 8\cos 4x\sqrt{x+1} = \underline{\underline{8}}$

11) d) $\lim\limits_{x\to -\infty} \frac{2x^4-x^3+4}{5x^4+x^3+2} = \lim\limits_{x\to -\infty} \frac{2-0+0}{5+0+0} = \underline{\underline{\frac{2}{5}}}$

LP: $\left\| \frac{\infty}{\infty} \right\| = \lim\limits_{x\to -\infty} \frac{8x^3-3x^2}{20x^3+3x^2} = \lim\limits_{x\to -\infty} \frac{24x^2-6x}{60x^2+6x} = \lim\limits_{x\to -\infty} \frac{48x-6}{120x+6} = \lim\limits_{x\to -\infty} \frac{48}{120} = \underline{\underline{\frac{2}{5}}}$

l) $\lim\limits_{x\to \infty} \frac{2^{x+3}+4}{2^{x-1}+1} = \lim\limits_{x\to \infty} \frac{8\cdot 2^x+4}{\frac{1}{2}\cdot 2^x+1} = \lim\limits_{x\to \infty} \frac{8+0}{\frac{1}{2}+0} = \underline{\underline{16}}$

LP: $\left\| \frac{\infty}{\infty} \right\| = \lim\limits_{x\to \infty} \frac{\ln(x+3)\cdot 2^{x+3}}{\ln(x-1)\cdot 2^{x-1}} = \lim\limits_{x\to \infty} \frac{\ln(x+3)}{\ln(x-1)}\cdot 2^4 = 2^4\cdot \lim\limits_{x\to \infty} \frac{x-1}{x+3} = 2^4 \lim \frac{1}{1} = \underline{\underline{16}}$

13) g) $\lim\limits_{x\to \infty} \frac{\sqrt{x+2}+3\sqrt{x^2-6}}{2x+1} = \lim\limits_{x\to \infty} \frac{0+3\sqrt{1-0}}{2+0} = \underline{\underline{\frac{3}{2}}}$

LP: $\left\| \frac{\infty}{\infty} \right\| = \lim\limits_{x\to \infty} \frac{\frac{1}{2\sqrt{x+2}} + \frac{3}{2\sqrt{x^2-6}}\cdot 2x}{2} = \lim\limits_{x\to \infty} \frac{1}{4\sqrt{x+2}} + \lim\limits_{x\to \infty} \frac{3x}{2\sqrt{x^2-6}} =$

$= 0 + \lim\limits_{x\to \infty} \frac{3}{2\cdot \frac{1}{2\sqrt{x^2-6}}\cdot 2x} = \frac{3}{2}\cdot \lim\limits_{x\to \infty} \frac{\sqrt{x^2-6}}{x} = \frac{3}{2}\cdot \lim\limits_{x\to \infty} \frac{\sqrt{1-0}}{1} = \underline{\underline{\frac{3}{2}}}$

h) $\lim\limits_{x\to \infty} \sqrt{x}(\sqrt{x+2}-\sqrt{x}) = \lim\limits_{x\to \infty} \frac{\sqrt{x}(x+2-x)}{\sqrt{x+2}+\sqrt{x}} = \lim\limits_{x\to \infty} \frac{2\sqrt{x}}{\sqrt{x+2}+\sqrt{x}} = \lim\limits_{x\to \infty} \frac{2}{1+1} = \underline{\underline{1}}$

LP: $\lim\limits_{x\to \infty} \frac{2\sqrt{x}}{\sqrt{x+2}+\sqrt{x}} = \left\| \frac{\infty}{\infty} \right\| = \lim\limits_{x\to \infty} \frac{\frac{1}{\sqrt{x}}}{\frac{1}{2\sqrt{x+2}}+\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to \infty} \frac{\frac{1}{1}}{\frac{1}{2\cdot 1}+\frac{1}{2\cdot 1}} = \underline{\underline{1}}$

→ **důležité limity**

$$\lim_{x\to 0}\frac{n^x-1}{x}=\ln(n) \qquad \lim_{x\to 0}\frac{\log_n(x+1)}{x}=\frac{1}{\ln(n)}$$

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \qquad \lim_{x\to 0}\frac{e^x-1}{x}=1 \qquad \lim_{x\to 0}\frac{\ln(x+1)}{x}=1$$

$$\lim_{x\to 0}\frac{\mathrm{tg}(x)}{x}=1 \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e \qquad \lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=e^k$$

→ **příklady**

11) $$\lim_{x\to 0^+}\sqrt[3]{x}\ln x=\|0\cdot(-\infty)\|=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt[3]{x}}}=\left\|\frac{-\infty}{\infty}\right\|=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{3}x^{-\frac{4}{3}}}=$$

$$=\lim_{x\to 0^+}-3\cdot x^{\frac{1}{3}}=\underline{\underline{0}}$$

12) $$\lim_{x\to 1^-}\ln(1-x)\cdot\ln x=\|-\infty\cdot 0\|=\lim_{x\to 1^-}\frac{\ln(1-x)}{\frac{1}{\ln x}}=\left\|\frac{-\infty}{\infty}\right\|=\lim_{x\to 1^-}\frac{\frac{-1}{1-x}}{\frac{-1}{\ln^2 x}\cdot\frac{1}{x}}=$$

$$=\lim_{x\to 1^-}\frac{x\ln^2 x}{1-x}=\left\|\frac{0}{0}\right\|=\lim_{x\to 1^-}\frac{\ln^2 x+x\cdot 2\ln x\cdot\frac{1}{x}}{-1}=-\lim_{x\to 1^-}(\ln^2 x+2\ln x)=\underline{\underline{0}}$$

13) $$\lim_{x\to 0^+}(e^x-1)\cdot\mathrm{cotg}\,x=\lim_{x\to 0^+}\frac{e^x-1}{\mathrm{tg}\,x}=\lim_{x\to 0^+}\frac{e^x-1}{x}\cdot\frac{x}{\mathrm{tg}\,x}=\underline{\underline{1}}$$

$$=\lim_{x\to 0^+}\frac{e^x-1}{\mathrm{tg}\,x}=\left\|\frac{0}{0}\right\|=\lim_{x\to 0^+}\frac{e^x}{\frac{1}{\cos^2 x}}=e^0\cdot\cos^2(0)=\underline{\underline{1}}$$

23) $$\lim_{x\to 2}\frac{1-e^{x-2}}{x-2\sin(\frac{\pi}{x})}=-\lim_{x\to 2}\frac{e^{x-2}-1}{x-2}\Rightarrow \begin{matrix}y=x-2\\ x\to 2\end{matrix}\Rightarrow -\lim_{y\to 0}\frac{e^y-1}{y}=\underline{\underline{-}}$$

$$=\left\|\frac{0}{0}\right\|=\lim_{x\to 2}\frac{-e^{x-2}}{1-2\cos(\frac{\pi}{x})(-\pi x^{-2})}=\lim_{x\to 2}\frac{-e^{x-2}}{1+2\pi\frac{\cos(\frac{\pi}{x})}{x^2}}=$$

$$=\frac{-e^0}{1+0}=\underline{\underline{-1}}$$

27) $$\lim_{x\to 0^+}x(1-\ln x)=\|0\cdot\infty\|=\lim_{x\to 0^+}\frac{1-\ln x}{\frac{1}{x}}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to 0^+}\frac{-\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}x=\underline{\underline{0}}$$

22) $$\lim_{x\to 3}\frac{3\,\mathrm{arctg}\sqrt{x}-\pi}{\mathrm{tg}(x-3)}=\left\|\frac{3\cdot\frac{\pi}{3}-\pi}{0}\right\|=\left\|\frac{0}{0}\right\|=\lim_{x\to 3}\frac{3\cdot\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}}{\frac{1}{\cos^2(x-3)}}=$$

$$=\frac{3}{2}\cdot\lim_{x\to 3}\frac{\cos^2(x-3)}{\sqrt{x}(x+1)}=\frac{3}{2}\cdot\frac{\cos^2(0)}{\sqrt{3}\cdot 4}=\underline{\underline{\frac{\sqrt{3}}{8}}}$$

16) $\lim_{x\to\frac{\pi}{2}}\left(tg\,x + \frac{2}{2x-\pi}\right) = \lim_{x\to\frac{\pi}{2}}\left(tg\,x - \frac{1}{\frac{\pi}{2}-x}\right)$ $\Rightarrow x\to\frac{\pi}{2}^+ : \left\|-\infty - \frac{1}{0^-}\right\| = \|-\infty+\infty\|$

$x\to\frac{\pi}{2}^- : \left\|\infty - \frac{1}{0^+}\right\| = \|\infty-\infty\|$

$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\frac{1}{tg\,x}} - \frac{1}{\frac{\pi}{2}-x}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{\frac{\pi}{2}-x - \frac{1}{tg\,x}}{\frac{\frac{\pi}{2}-x}{tg\,x}}\right) = \left\|\frac{0}{0}\right\| = \lim_{x\to\frac{\pi}{2}} \frac{-1-\left(-\frac{1}{tg^2x}\cdot\frac{1}{\cos^2x}\right)}{\frac{-tg\,x - \left(\frac{\pi}{2}-x\right)\cdot\frac{1}{\cos^2 x}}{tg^2 x}} =$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\cos^2x}{\sin^2x}\cdot\frac{1}{\cos^2x}-1}{-\frac{1}{tg\,x}+\left(x-\frac{\pi}{2}\right)\cdot\frac{\cos^2 x}{\sin^2x}\cdot\frac{1}{\cos^2x}} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin^2x}-1}{\frac{x-\frac{\pi}{2}}{\sin^2x}-\frac{\cos x}{\sin x}} = \lim_{x\to\frac{\pi}{2}}\frac{\frac{1-\sin^2x}{\sin^2x}}{\frac{x-\frac{\pi}{2}-\cos x\sin x}{\sin^2x}} =$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos^2x}{x-\frac{\pi}{2}-\frac{1}{2}\sin2x} = \left\|\frac{0}{0}\right\| = \lim_{x\to\frac{\pi}{2}}\frac{-2\cos x\sin x}{1-\frac{1}{2}\cos(2x)\cdot2} = \lim_{x\to\frac{\pi}{2}}\frac{\sin2x}{\cos2x-1} = \frac{0}{-2} = \underline{0}$$

• $\lim_{x\to\frac{\pi}{2}}\left(\sin(x)^{tg(x)}\right) = \lim_{x\to\frac{\pi}{2}} e^{\ln\left(\sin(x)^{tg(x)}\right)} = e^{\lim_{x\to\frac{\pi}{2}}\left(tg(x)\cdot\ln(\sin x)\right)} =$

1) $x\to\frac{\pi}{2}^+ : e^{\|-\infty\cdot 0\|}$

2) $x\to\frac{\pi}{2}^- : e^{\|\infty\cdot 0\|}$

$$= e^{\lim_{x\to\frac{\pi}{2}}\left(\frac{\ln(\sin x)}{\frac{1}{tg\,x}}\right)} = e^{\lim_{x\to\frac{\pi}{2}}\frac{\ln(\sin x)}{cotg\,x}} \left\|\frac{0}{0}\right\| = e =$$

$$= e^{\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{\sin^2x}}} = e^{-\lim_{x\to\frac{\pi}{2}} \sin x\cdot\cos x} = e^{-1\cdot 0} = \underline{1}$$

• $\lim_{x\to0}\left(\frac{arctg\,x}{x}\right)^{x^{-2}} = e^{\lim_{x\to0}\left(x^{-2}\cdot\ln\frac{arctg\,x}{x}\right)} = e^{\lim_{x\to0}\frac{\ln\frac{arctg\,x}{x}}{x^2}} = e^{\left\|\frac{\ln\left(\frac{0}{0}\right)}{0}\right\|}$

1) $\lim_{x\to0}\frac{arctg\,x}{x} = \left\|\frac{0}{0}\right\| = \lim_{x\to0}\frac{\frac{1}{1+x^2}}{1} = \underline{1}$

2) $\lim_{x\to0}\left(\frac{\ln\frac{arctg\,x}{x}}{x^2}\right) = \left\|\frac{\ln 1}{0}\right\| = \left\|\frac{0}{0}\right\| = \lim_{x\to0}\frac{\frac{x}{arctg\,x}\cdot\frac{\frac{1}{1+x^2}\cdot x - arctg\,x}{x^2}}{2x} =$

$$= \lim_{x\to0}\frac{x\left(\frac{1}{1+x^2}x - arctg\,x\right)}{2x^3\cdot arctg\,x} = \lim_{x\to0}\frac{x-(1+x^2)\,arctg\,x}{2x^2 arctg\,x\cdot(1+x^2)} = \left\|\frac{0-0}{0}\right\| =$$

$$= \frac{1}{2}\lim_{x\to0}\frac{1-2x\cdot arctg\,x-(1+x^2)\frac{1}{1+x^2}}{2x\,arctg(x)\cdot(1+x^2)+x^2\cdot\frac{1}{1+x^2}\cdot(1+x^2)+x^2\cdot arctg(x)\cdot2x} =$$

$$= \frac{1}{2}\lim_{x\to0}\frac{-2x\,arctg\,x}{2x\,arctg(x)(1+x^2)+x^2+2x^3\,arctg\,x} =$$

$$= -\lim_{x\to0}\frac{arctg\,x}{2\,arctg(x)(1+x^2+x^2)+x} = \left\|\frac{0}{0}\right\| = -\lim_{x\to0}\frac{\frac{1}{1+x^2}}{2\cdot\frac{1+2x^2}{1+x^2}+2\cdot arctg\,x\cdot(1+4x)+1} =$$

$$= -\lim_{x\to0}\frac{\frac{1}{1+0}}{2\cdot\frac{1}{1}+2\cdot0+1} = -\frac{1}{3} \Rightarrow \lim_{x\to0}\left(\frac{arctg\,x}{x}\right)^{x^{-2}} = e^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{e}}$$