• **Parametrické křivky**

→ mějme křivku $x = x(t)$
$y = y(t)$; $t \in \langle t_1; t_2 \rangle$.

→ předpokládejme, že lze odstranit parametr, tedy že existuje funkce f taková, že $y = f(x)$, pak $y(t) = f(x(t))$

• **Plocha pod parametrickou křivkou**

$$S = \int_a^b f(x)\,dx = \left| \begin{array}{l} x = x(t) \\ y = y(t) \\ dx = x'(t)\,dt \end{array} \right| = \int_{t_1}^{t_2} f(x(t))\,x'(t)\,dt \Rightarrow \boxed{S = \int_{t_1}^{t_2} y(t)\,x'(t)\,dt}$$

• **Délka parametrické křivky**

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \left| \begin{array}{l} x = x(t) \\ y = y(t) \\ dx = \frac{dx}{dt}\,dt \end{array} \right| = \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\frac{dx}{dt}\,dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 \left(\frac{dy}{dx}\right)^2}\,dt =$$

$$= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \Rightarrow \boxed{l = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt}$$

• **Objem tělesa vzniklého rotací parametrické křivky okolo osy x**

$$V = \pi \int_a^b f^2(x)\,dx = \left| \begin{array}{l} x = x(t) \\ y = y(t) \\ dx = x'(t)\,dt \end{array} \right| = \pi \int_{t_1}^{t_2} f^2(x(t))\,x'(t)\,dt \Rightarrow \boxed{V = \pi \int_{t_1}^{t_2} y^2(t)\,x'(t)\,dt}$$

• **Povrch tělesa vzniklého rotací parametrické křivky okolo osy x**

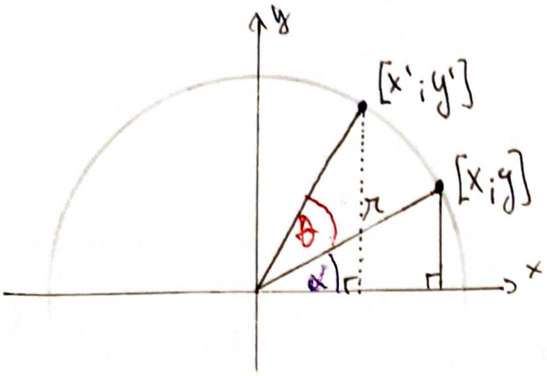
$$P = 2\pi \int_a^b |f(x)| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \left| \begin{array}{l} x = x(t) \\ y = y(t) \\ dx = \frac{dx}{dt}\,dt \end{array} \right| = 2\pi \int_{t_1}^{t_2} |f(x(t))| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\frac{dx}{dt}\,dt =$$

$$= 2\pi \int_{t_1}^{t_2} |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \Rightarrow \boxed{P = 2\pi \int_{t_1}^{t_2} |y(t)| \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt}$$

• Rotační matice

- bod $[x, y]$ můžeme rotovat o úhel θ rotační maticí $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

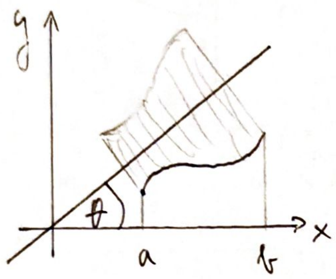
- důkaz:



$$x = r\cos\alpha \qquad x' = r\cos(\alpha+\theta) = r(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$
$$y = r\sin\alpha \qquad y' = r\sin(\alpha+\theta) = r(\cos\alpha\sin\theta + \sin\alpha\cos\theta)$$

$$\Rightarrow \boxed{\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}}$$

• Objem tělesa vzniklého rotací funkce okolo přímky procházející počátkem



$$y = f(x) \Rightarrow x(t) = t \ ; \ y(t) = f(t)$$
$$x \in \langle a; b\rangle \Rightarrow t \in \langle a; b\rangle$$

→ nyní tuto křivku budeme rotovat o $-\theta$

$$\Rightarrow x_0(t) = t\cos\theta + f(t)\sin\theta$$
$$y_0(t) = -t\sin\theta + f(t)\cos\theta \ ; \ t \in \langle a; b\rangle$$

$$\Rightarrow V = \pi\int_a^b y_0^2(t)\, x_0'(t)\, dt = \pi\int_a^b \left(f(t)\cos\theta - t\sin\theta\right)^2\left(\cos\theta + f'(t)\sin\theta\right) dt$$

$$\Rightarrow \boxed{V = \pi\int_a^b \left(f(x)\cos\theta - x\sin\theta\right)^2\left(f'(x)\sin\theta + \cos\theta\right) dx}$$

• Povrch tělesa vzniklého rotací funkce okolo přímky procházející počátkem

$$P = 2\pi\int_a^b |y_0(t)|\sqrt{[x_0'(t)]^2 + [y_0'(t)]^2}\, dt =$$

$$= 2\pi\int_a^b |f(t)\cos\theta - t\sin\theta|\sqrt{(\cos\theta + f'(t)\sin\theta)^2 + (-\sin\theta + f'(t)\cos\theta)^2}\, dt =$$

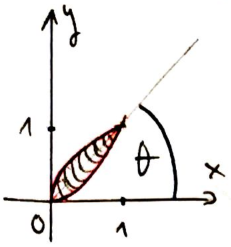
$$= 2\pi\int_a^b |f(t)\cos\theta - t\sin\theta|\sqrt{\cos^2\theta + [f'(t)]^2\sin^2\theta + \sin^2\theta + [f'(t)]^2\cos^2\theta}\, dt =$$

$$= 2\pi\int_a^b |f(t)\cos\theta - t\sin\theta|\sqrt{1 + [f'(t)]^2}\, dt$$

$$\Rightarrow \boxed{P = 2\pi\int_a^b |f(x)\cos\theta - x\sin\theta|\sqrt{1 + [f'(x)]^2}\, dx}$$

- **Příklad**

→ najdi povrch a objem tělesa vzniklého rotací funkce $y=x^2$ okolo přímky $y=x$; $x \in \langle 0;1 \rangle$



- $$V = \pi \int_a^b \left(f(x)\cos\theta - x\sin\theta \right)^2 \left(f'(x)\sin\theta + \cos\theta \right) dx$$

$$\theta = \frac{\pi}{4} \Rightarrow \cos\theta = \sin\theta = \frac{\sqrt{2}}{2}, \quad a=0, \; b=1$$

$$\Rightarrow V = \pi \int_0^1 \left[\frac{\sqrt{2}}{2}(x^2-x) \right]^2 \frac{\sqrt{2}}{2}(2x+1)\,dx =$$

$$= \pi \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \int_0^1 (x^4 - 2x^3 + x^2)(2x+1)\,dx =$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^1 (2x^5 - 4x^4 + 2x^3 + x^4 - 2x^3 + x^2)\,dx = \frac{\pi}{2\sqrt{2}} \int_0^1 (2x^5 - 3x^4 + x^2)\,dx =$$

$$= \frac{\pi}{2\sqrt{2}} \left[\frac{x^6}{3} - \frac{3x^5}{5} + \frac{x^3}{3} \right]_0^1 = \frac{\pi}{2\sqrt{2}} \left(\frac{1}{3} - \frac{3}{5} + \frac{1}{3} \right) = \frac{\pi}{2\sqrt{2}} \left(\frac{10}{15} - \frac{9}{15} \right) = \underline{\underline{\frac{\pi}{30\sqrt{2}}}}$$

- $$P = 2\pi \int_a^b \left| f(x)\cos\theta - x\sin\theta \right| \sqrt{1 + [f'(x)]^2}\,dx \to \left| x^2 \frac{\sqrt{2}}{2} - x\frac{\sqrt{2}}{2} \right| = \frac{\sqrt{2}}{2}|x^2 - x| = \frac{\sqrt{2}}{2}(x - x^2)$$

$x \in \langle 0;1 \rangle$

$$\Rightarrow P = 2\pi \int_0^1 \frac{\sqrt{2}}{2}(x-x^2)\sqrt{1+(2x)^2}\,dx = \pi\sqrt{2}\int_0^1 (x-x^2)\sqrt{1+(2x)^2}\,dx =$$

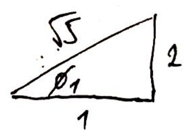
$2x = \mathrm{tg}\,\phi \qquad 0:\; 0 = \mathrm{tg}\,\phi \Rightarrow \phi = 0$

$2\,dx = \sec^2\phi\,d\phi \qquad 1:\; 2 = \mathrm{tg}\,\phi \Rightarrow \phi = \mathrm{arctg}(2) = \phi_1$

$$= \pi\sqrt{2} \int_0^{\phi_1} \left(\frac{1}{2}\mathrm{tg}\,\phi - \frac{1}{4}\mathrm{tg}^2\phi \right) \sqrt{1+\mathrm{tg}^2\phi}\; \frac{1}{2}\sec^2\phi\,d\phi =$$

$$= \pi\frac{\sqrt{2}}{8} \int_0^{\phi_1} \left(2\,\mathrm{tg}\,\phi - \mathrm{tg}^2\phi \right) \sec^3\phi\,d\phi =$$

$$= \pi\frac{\sqrt{2}}{8} \left[2\int_0^{\phi_1} \mathrm{tg}\,\phi \sec^3\phi\,d\phi - \int_0^{\phi_1} \mathrm{tg}^2\phi \sec^3\phi\,d\theta \right] =$$



$u = \sec\phi \qquad 0:\; u = \sec 0 = 1$

$du = \sec\phi\,\mathrm{tg}\,\phi\,d\phi \qquad \phi_1:\; u = \sec(\mathrm{arctg}\,2) = \sqrt{5}$

$$= \pi\frac{\sqrt{2}}{8} \left[2\int_1^{\sqrt{5}} u^2\,du - \int_0^{\phi_1} (\sec^2\phi - 1)\sec^3\phi\,d\phi \right] =$$

$$= \pi\frac{\sqrt{2}}{8} \left[2\frac{u^3}{3}\Big|_1^{\sqrt{5}} - \int_0^{\phi_1} \sec^5\phi\,d\phi + \int_0^{\phi_1} \sec^3\phi\,d\phi \right] =$$

$$= \pi\frac{\sqrt{2}}{8}\left(2\frac{5\sqrt{5}-1}{3} \right) - \pi\frac{\sqrt{2}}{8}\int_0^{\phi_1} \sec^5\phi\,d\phi + \pi\frac{\sqrt{2}}{8}\int_0^{\phi_1} \sec^3\phi\,d\phi =$$

$$= \pi\frac{\sqrt{2}}{12}(5\sqrt{5}-1) - \pi\frac{\sqrt{2}}{8}\int_0^{\phi_1} \sec^5\phi\,d\phi + \pi\frac{\sqrt{2}}{8}\int_0^{\phi_1} \sec^3\phi\,d\phi =$$

→ nyní určíme $\int_0^{\phi_1} \sec^3\phi \, d\phi$ $\quad \phi_1 = \operatorname{arctg}(2) \Rightarrow \sec\phi_1 = \sqrt{5}$

$$\int_0^{\phi_1} \sec^3\phi\, d\phi = \sec\phi\, \operatorname{tg}\phi \Big|_0^{\phi_1} - \int_0^{\phi_1} \sec\phi\, \operatorname{tg}^2\phi\, d\phi = \sqrt{5}\cdot 2 - \int_0^{\phi_1} \sec\phi\, (\sec^2\phi - 1)\, d\phi$$

$$\left.\begin{array}{ccc} & D & I \\ + & \sec\phi & \sec^2\phi \\ - & \sec\phi\,\operatorname{tg}\phi & \operatorname{tg}\phi \end{array}\right]$$

$$\Rightarrow 2\int_0^{\phi_1} \sec^3\phi\, d\phi = 2\sqrt{5} + \int_0^{\phi_1} \sec\phi\, d\phi = 2\sqrt{5} + \ln|\sec\phi + \operatorname{tg}\phi| \Big|_0^{\phi_1}$$

$$\Rightarrow \int_0^{\phi_1} \sec^3\phi\, d\phi = \underline{\underline{\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})}}$$

→ nyní určíme $\int_0^{\phi_1} \sec^5\phi\, d\phi$

$$\int_0^{\phi_1} \sec^5\phi\, d\phi = \sec^3\phi\, \operatorname{tg}\phi \Big|_0^{\phi_1} - 3\int_0^{\phi_1} \sec^3\phi\, \operatorname{tg}^2\phi\, d\phi = 5\sqrt{5}\cdot 2 - 3\int_0^{\phi_1} \sec^3\phi\, (\sec^2\phi - 1)\, d\phi$$

$$\left.\begin{array}{ccc} & D & I \\ + & \sec^3\phi & \sec^2\phi \\ - & 3\sec^2\phi\, \sec\phi\, \operatorname{tg}\phi & \operatorname{tg}\phi \end{array}\right]$$

$$\Rightarrow 4\int_0^{\phi_1} \sec^5\phi\, d\phi = 10\sqrt{5} + 3\int_0^{\phi_1} \sec^3\phi\, d\phi =$$

$$= 10\sqrt{5} + 3\sqrt{5} + \frac{3}{2}\ln(2+\sqrt{5})$$

$$\Rightarrow \int_0^{\phi_1} \sec^5\phi\, d\phi = \underline{\underline{\frac{13\sqrt{5}}{4} + \frac{3}{8}\ln(2+\sqrt{5})}}$$

$$\Rightarrow P = \pi\frac{\sqrt{2}}{12}(5\sqrt{5}-1) - \frac{\pi\sqrt{2}}{8}\int_0^{\phi_1} \sec^5\phi\, d\phi + \frac{\pi\sqrt{2}}{8}\int_0^{\phi_1} \sec^3\phi\, d\phi =$$

$$= \pi\frac{\sqrt{2}}{12}(5\sqrt{5}-1) - \frac{\pi\sqrt{2}}{8}\cdot\frac{13\sqrt{5}}{4} - \frac{\pi\sqrt{2}}{8}\cdot\frac{3}{8}\ln(2+\sqrt{5}) + \frac{\pi\sqrt{2}}{8}\sqrt{5} + \frac{\pi\sqrt{2}}{8}\cdot\frac{1}{2}\ln(2+\sqrt{5}) =$$

$$= \pi\cdot\frac{5}{12}\sqrt{10} - \pi\frac{\sqrt{2}}{12} - \pi\cdot\frac{13}{32}\sqrt{10} + \pi\cdot\frac{1}{8}\sqrt{10} - \frac{\pi\sqrt{2}}{8}\cdot\frac{3}{8}\ln(2+\sqrt{5}) + \frac{\pi\sqrt{2}}{8}\cdot\frac{4}{8}\ln(2+\sqrt{5}) =$$

$$= \pi\sqrt{10}\left(\frac{5}{12} - \frac{13}{32} + \frac{1}{8}\right) - \pi\frac{\sqrt{2}}{12} + \frac{\pi\sqrt{2}}{8}\cdot\frac{1}{8}\ln(2+\sqrt{5}) =$$

$$= \pi\sqrt{10}\left(\frac{5\cdot 8 - 3\cdot 13 + 12}{96}\right) - \pi\frac{\sqrt{2}}{12} + \frac{\pi\sqrt{2}}{64}\ln(2+\sqrt{5}) =$$

$$= \pi\sqrt{10}\,\frac{13}{96} - \pi\frac{\sqrt{2}}{12} + \pi\frac{\sqrt{2}}{64}\ln(2+\sqrt{5}) =$$

$$= \pi\frac{2\sqrt{5}\cdot 13}{96\sqrt{2}} - \pi\frac{2}{12\sqrt{2}} + \pi\frac{1}{32\sqrt{2}}\ln(2+\sqrt{5}) =$$

$$= \underline{\underline{\frac{\pi}{96\sqrt{2}}\left(26\sqrt{5} - 16 + 3\ln(2+\sqrt{5})\right)}}$$

⇒ objem tohoto tělesa je $\boxed{V = \frac{\pi}{30\sqrt{2}} \approx 0{,}074}$

⇒ povrch tohoto tělesa je $\boxed{P = \frac{\pi}{96\sqrt{2}}\left(26\sqrt{5} - 16 + 3\ln(2+\sqrt{5})\right) \approx 1{,}075}$