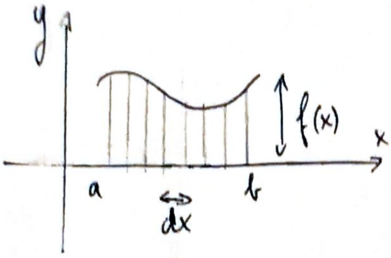
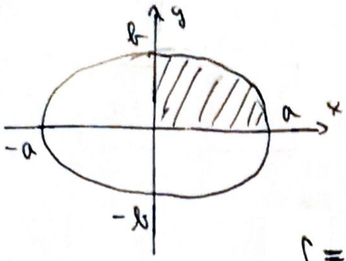
• Obsah elipsy



$$\boxed{S = \int_a^b f(x)\,dx}$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow y^2 = b^2 - \frac{b^2}{a^2}x^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

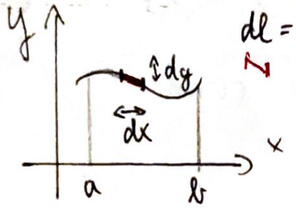
$$\Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$S = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \left|\begin{array}{ll} x = a\cdot\sin\theta & 0 = a\cdot\sin\theta \rightarrow \theta = 0 \\ dx = a\cos\theta\,d\theta & a = a\cdot\sin\theta \rightarrow \theta = \frac{\pi}{2} \end{array}\right.$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2(1-\sin^2\theta)}\,a\cos\theta\,d\theta = 4ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta =$$

$$= 2ab\int_0^{\frac{\pi}{2}}(1+\cos(2\theta))\,d\theta = 2ab\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}} \Rightarrow \boxed{S = \pi ab}$$

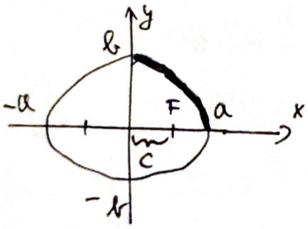
• Obvod elipsy



$$dl = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$dl = \sqrt{1 + [f'(x)]^2}\,dx$$

$$\boxed{O = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx}$$



$$f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$f'(x) = \frac{b}{a}\,\frac{-2x}{2\sqrt{a^2-x^2}}$$

$$e = \sqrt{a^2 - b^2}$$

$$\varepsilon = \frac{e}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$

$$O = 4\int_0^a \sqrt{1 + \frac{b^2}{a^2}\cdot\frac{x^2}{a^2-x^2}}\,dx = \left|\begin{array}{ll} x = a\cdot\sin\theta & 0 \rightarrow 0 \\ dx = a\cos\theta\,d\theta & a \rightarrow \frac{\pi}{2} \end{array}\right.$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{b^2}{a^2}\cdot\frac{a^2\sin^2\theta}{a^2 - a^2\sin^2\theta}}\,a\cos\theta\,d\theta =$$

$$= 4a\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta + \frac{b^2}{a^2}\cdot\frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta}\,d\theta = 4a\int_0^{\frac{\pi}{2}}\sqrt{1 - \sin^2\theta + \frac{b^2}{a^2}\sin^2\theta}\,d\theta =$$

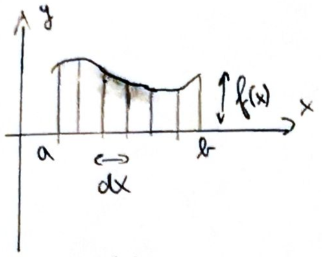
$$= 4a\int_0^{\frac{\pi}{2}}\sqrt{1 - \sin^2\theta\left(1 - \frac{b^2}{a^2}\right)}\,d\theta = 4a\int_0^{\frac{\pi}{2}}\sqrt{1 - \varepsilon^2\sin^2\theta}\,d\theta =$$

$$= 4a\,E(\varepsilon),$$

kde $E(\varepsilon) = \int_0^{\frac{\pi}{2}}\sqrt{1 - \varepsilon^2\sin^2\theta}\,d\theta$ je úplný eliptický integrál II. druhu.

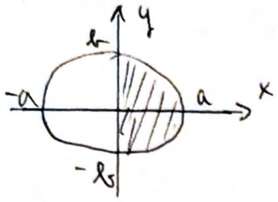
$$\Rightarrow \boxed{O = 4a\cdot E(\varepsilon)}$$

- Objem rotačního elipsoidu



$$dV = \pi r^2 v = \pi f^2(x)\,dx$$

$$\boxed{V = \pi \int_a^b f^2(x)\,dx}$$

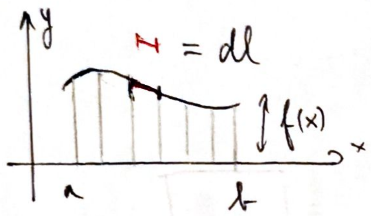


$$f(x) = \frac{b}{a}\sqrt{a^2-x^2}$$

$$V = 2\pi \int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx = 2\pi \frac{b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a =$$

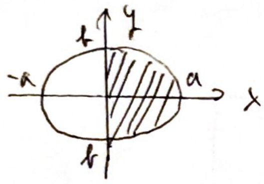
$$= 2\pi \frac{b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right] = \frac{4}{3}\pi b^2 a \Rightarrow \boxed{V = \frac{4}{3}\pi a b^2}$$

- Povrch rotačního elipsoidu



$$dP = 2\pi r\,dl = 2\pi f(x)\,dl = 2\pi f(x)\sqrt{(dx)^2 + (dy)^2} = 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\boxed{P = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx}$$



$$f(x) = \frac{b}{a}\sqrt{a^2-x^2}$$

$$f'(x) = \frac{b}{a}\frac{-2x}{2\sqrt{a^2-x^2}}$$

$$e = \sqrt{a^2-b^2}$$

$$\varepsilon = \frac{e}{a} = \sqrt{1-\frac{b^2}{a^2}}$$

$$\varepsilon^2 = 1 - \frac{b^2}{a^2}$$

$$\sqrt{1-\varepsilon^2} = \frac{b}{a}$$

$$P = 4\pi \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}}\,dx = \left| \begin{array}{ll} x = a\sin\theta & 0 \to 0 \\ dx = a\cos\theta\,d\theta & a \to \frac{\pi}{2} \end{array} \right.$$

$$= 4\pi \frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta\sqrt{1+\frac{b^2}{a^2}\frac{\sin^2\theta}{\cos^2\theta}}\,a\cos\theta\,d\theta =$$

$$= 4\pi ab \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta + \frac{b^2}{a^2}\sin^2\theta}\,\cos\theta\,d\theta =$$

$$= 4\pi ab \int_0^{\frac{\pi}{2}} \sqrt{1-\varepsilon^2\sin^2\theta}\,\cos\theta\,d\theta = \left| \begin{array}{l} \sin\varphi = \varepsilon\sin\theta \\ \cos\varphi\,d\varphi = \varepsilon\cos\theta\,d\theta \\ \cos\theta\,d\theta = \frac{1}{\varepsilon}\cos\varphi\,d\varphi \\ \theta = 0:\ \sin\varphi = 0 \Rightarrow \varphi = 0 \\ \theta = \frac{\pi}{2}:\ \sin\varphi = \varepsilon \Rightarrow \varphi = \arcsin\varepsilon \end{array} \right.$$

$$= 4\pi \frac{ab}{\varepsilon}\int_0^{\arcsin\varepsilon} \sqrt{1-\sin^2\varphi}\,\cos\varphi\,d\varphi =$$

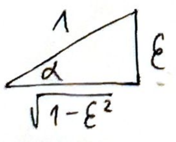
$$= 4\pi \frac{ab}{\varepsilon}\int_0^{\arcsin\varepsilon} \cos^2\varphi\,d\varphi =$$

$$= 2\pi \frac{ab}{\varepsilon}\int_0^{\arcsin\varepsilon} (1+\cos(2\varphi))\,d\varphi = 2\pi \frac{ab}{\varepsilon}\left[\varphi + \frac{1}{2}\sin(2\varphi)\right]_0^{\arcsin\varepsilon}$$

$$= 2\pi \frac{ab}{\varepsilon}\left[\arcsin\varepsilon + \frac{1}{2}\sin(2\arcsin\varepsilon)\right] =$$

$$\frac{1}{2}\sin(2\underbrace{\arcsin\varepsilon}_{\alpha}) = \sin\alpha\cos\alpha$$

$$\sin\alpha = \varepsilon = \frac{\varepsilon}{1}$$



$$\cos\alpha = \sqrt{1-\varepsilon^2}$$

$$= 2\pi \frac{ab}{\varepsilon}\left[\arcsin\varepsilon + \varepsilon\sqrt{1-\varepsilon^2}\right] = 2\pi \frac{ab}{\varepsilon}\left[\arcsin\varepsilon + \varepsilon\frac{b}{a}\right] =$$

$$= 2\pi ab\left[\frac{\arcsin\varepsilon}{\varepsilon} + \frac{b}{a}\right] = 2\pi b^2\left[\frac{a}{b}\cdot\frac{\arcsin\varepsilon}{\varepsilon} + 1\right]$$

$$\Rightarrow \boxed{P = 2\pi b^2\left(1 + \frac{a}{b}\cdot\frac{\arcsin\varepsilon}{\varepsilon}\right)}$$