OPAKOVÁNÍ LINGEBRY

Def: Grupa $(G, \circ, e)$ je množina G, binární operace $\circ: G^2 \to G$ a prvek $e \in G$ splňující

1, $\forall a, b, c \in G: \ (a \circ b) \circ c = a \circ (b \circ c)$ - asociativita
2, $\forall a \in G: \ a \circ e = e \circ a = a$ - neutrální prvek
3, $(\forall a \in G)(\exists a^{-1} \in G): a \circ a^{-1} = a^{-1} \circ a = e$ - inverzní prvek

Def: Grupu nazýváme abelovská $\equiv \forall a, b \in G: \ a \circ b = b \circ a$. - komutativita

Def. Těleso je pětice $(\mathbb{K}, +, \cdot, 0, 1)$, kde

1, $(\mathbb{K}, +, 0)$, $(\mathbb{K}, \cdot, 1)$ jsou abelovské grupy
2, $\forall a, b, c \in K: \ a \cdot (b + c) = (a \cdot b) + (b \cdot c)$ - distributivita (zleva)
3, speciální podmínky:
  i, $0 \neq 1$
  ii, nevyžadujeme aby 0 měla inverzi vůči $\cdot$

Def: Vektorový prostor nad tělesem $(\mathbb{K}, +, \cdot, 0, 1)$ je trojice $(V, \oplus, \circ)$

1, V je množina, jejíž prvky nazýváme vektory
2, $\oplus: V \times V \to V$, $\circ: \mathbb{K} \times V \to V$
3, $(V, \oplus)$ je abelovská grupa
4, $\forall v \in V: \ 1 \circ v = v$
5, $(\forall a, b \in \mathbb{K})(\forall v \in V): \ (a \cdot b) \circ v = a \circ (b \circ v)$ ... asociativita
6, $(\forall a, b \in \mathbb{K})(\forall v \in V): \ (a + b) \circ v = (a \circ v) + (b \circ v)$ } distributivita
7, $(\forall a \in \mathbb{K})(\forall u, v \in V): \ a \circ (u \oplus v) = (a \circ u) \oplus (a \circ v)$ } (tak trochu)

Def: A set of vectors $U = \{v_i \mid i \in I\} \subseteq V$ is linearly dependent $\equiv$
$\exists \{v_1, \dots v_m\} \subseteq U$ & $\lambda_1, \dots \lambda_m \in \mathbb{K}$ s.t. $\sum_{i=1}^{m} \lambda_i v_i = 0$.

↳ finite set!

→ A set that is not lin. dep. is lin. independent.

Def: Let V be a vector space and $U = \{v_i \mid i \in I\} \subseteq V$.
We say that V is equal to the closure of the span of $U \equiv$

$(\forall v \in V): (\forall i \in I)(\exists \lambda_i): \ v = \sum_{i \in I} \lambda_i v_i$ ← vynechané detaily (kdy konverguje ta řada...)

Def: The set $U \subseteq V$ is called the (Schauder) basis $\equiv$
it is lin. ind & the closure of the span of U equals V.

Remark: Also exists Hamel basis, which may contain uncountably many vectors.

Def: A norm on a vector space V over $\mathbb{C}$ is a mapping $\|\cdot\|: V \to \mathbb{R}$ s.t.

i) $\forall v \in V: \|v\| \geq 0$ & $\|v\| = 0 \Leftrightarrow v = 0$ ... positivity

ii) $(\forall v \in V)(\forall \lambda \in \mathbb{C}): \|\lambda v\| = |\lambda| \cdot \|v\|$ ... scaling

iii) $\forall u, v \in V: \|u + v\| \leq \|u\| + \|v\|$. ... Δ-ineq

Examples:

- Vectors: $V = \mathbb{K}^n$
  - ℓ^p norm: $\|(v_1, \dots v_n)^T\|_p = (|v_1|^p + \dots + |v_n|^p)^{\frac{1}{p}}$
  - Euclidean norm: $\sqrt{v_1^2 + \dots + v_n^2}$ ... ℓ^2
  - Manhattan norm: $|v_1| + \dots + |v_n|$ ... ℓ^1
  - Maxim norm: $\max\{|v_1|, \dots |v_n|\}$ ... ℓ^∞
- Matrices: $V = \mathbb{K}^{m \times n}$
  - max col.: $\|A\|_1 = \max_j \left\{ \sum_{i=1}^{m} |a_{ij}| \right\}$
  - max row: $\|A\|_\infty = \max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$
  - in general: $\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$

  ↳ Intuition: Take the p-norm unit ball $V_{p,n} = \{x \in \mathbb{K}^n \mid \|x\|_p \leq 1\}$ and apply the linear map A to the ball. This will create a distorted convex shape $AV_{p,n} \subset \mathbb{K}^m$. $\|A\|_p$ measures the longest "radius" of this shape. The smallest ball to contain $AV_{p,n}$ is $\{x \in \mathbb{K}^m \mid \|x\|_p \leq \|A\|_p\}$.

- functions
  - $\|f\|_2 = \sqrt{\int_{-1}^{1} f(x)^2 dx}$ is a norm on the space of all integrable functions such that $\int_{-1}^{1} f(x)^2 dx$ is finite.
  - $\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p dx \right)^{\frac{1}{p}}$ ... in general

  ↳ there is nothing special about the range $(-1, 1)$

  → the vector space is not really a space of functions

  ↳ f and g have the same Lebesgue integral if they agree almost everywhere (the set where they disagree has measure zero)

  ⇒ the vectors are equivalence classes of functions that agree almost everywhere

  ↳ these norms would be denoted as $L^2[-1, 1]$ and $L^p[-1, 1]$

Def: An inner product on a vector space over $\mathbb{C}$ is a map $V \times V \to \mathbb{C}$ s.t.

1) $\forall u \in V$: $\langle u|u\rangle \geq 0$ & $\langle u|u\rangle = 0 \iff u = 0$
2) $\langle u|v\rangle = \overline{\langle v|u\rangle}$ ... symmetry
3) $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$ } linearity
4) $\forall \lambda \in \mathbb{C}$: $\langle \lambda u|v\rangle = \lambda\langle u|v\rangle$

$\langle u|v+w\rangle = \overline{\langle v+w|u\rangle} = \overline{\langle v|u\rangle + \langle w|u\rangle} = \langle u|v\rangle + \langle u|w\rangle$

$\langle u|\lambda v\rangle = \overline{\langle \lambda v|u\rangle} = \overline{\lambda\langle v|u\rangle} = \overline{\lambda}\langle u|v\rangle$

$\langle au|au\rangle = a\overline{a}\langle u|u\rangle = |a|^2\langle u|u\rangle$

$\langle \sum_i a_i u_i | \sum_j b_j v_j\rangle = \sum_i\sum_j a_i \overline{b_j}\langle u_i|v_j\rangle$

Def: A set of vectors V is <u>orthogonal</u> with respect to $\langle\cdot|\cdot\rangle \equiv \forall u,v \in V: \langle u|v\rangle = 0$.

Def: An <u>inner product space</u> is a vector space endowed with an inner product.

Examples:

* Hermit transpose
$(A^H)_{ij} = \overline{a_{ji}}$
$(AB)^H = B^H A^H$

- standard i.p. on $\mathbb{R}^n$: $\langle u|v\rangle = \sum u_i v_i = v^T u$
- standard i.p. on $\mathbb{C}^n$: $\langle u|v\rangle = \sum u_i \overline{v_i} = v^H u$
- inner product on $\mathbb{R}^{n\times n}$:

$$\langle A|B\rangle = tr(AB^T) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{ij} \quad \dots\ tr(A) = \sum_{i=1}^{n} a_{ii}$$

- inner product on the space of all real-valued functions continuous on $[0,1]$:

$$\langle f|g\rangle = \int_0^1 f(x)g(x)\,dx \quad \dots \text{ usually denoted } C[0,1]$$

Def: An orthogonal set of vectors V is <u>orthonormal</u> $\equiv \forall u \in V: \sqrt{\langle u|u\rangle} = 1$.

Theorem: If $\langle\cdot|\cdot\rangle$ is an <u>inner product</u> on V, then $\|\cdot\|: V \to \mathbb{R}_0^+$, $u \mapsto \sqrt{\langle u|u\rangle}$ is a <u>norm</u>.

Proof:

1) positivity: $\langle u|u\rangle \geq 0 \Rightarrow \sqrt{\langle u|u\rangle} \geq 0$ ✓
2) scaling: $\|\lambda u\| = \sqrt{\langle \lambda u|\lambda u\rangle} = \sqrt{|\lambda|^2\langle u|u\rangle} = |\lambda|\cdot\|u\|$ ✓
3) Δ-ineq: $\|u+v\| = \sqrt{\langle u+v|u+v\rangle} = \sqrt{\langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle v|v\rangle} \overset{*}{\leq}$

$*$: $\langle u|v\rangle = a+bi \Rightarrow \langle u|v\rangle + \langle v|u\rangle = 2a \leq 2\sqrt{a^2+b^2} = 2|\langle u|v\rangle|$

$\overset{*}{\leq} \sqrt{\|u\|^2 + 2|\langle u|v\rangle| + \|v\|^2} \leq \sqrt{\|u\|^2 + 2\|u\|\|v\| + \|v\|^2} = \|u\| + \|v\|$ ∎

Theorem (<u>Cauchy-Schwartz ineq.</u>): $|\langle u|v\rangle| \leq \|u\|\cdot\|v\|$

Proof: Proof only for real numbers (complex is only a bit more technical)

W.L.O.G. $v, u \neq 0$. $\forall a \in \mathbb{R}$: $0 \leq \|u+av\|^2 = \langle u+av|u+av\rangle = \|u\|^2 + 2a\langle u|v\rangle + a^2\|v\|^2$

$\to$ let $a := -\frac{\langle u|v\rangle}{\|v\|^2} \Rightarrow 0 \leq \|u\|^2 - 2\frac{\langle u|v\rangle^2}{\|v\|^2} + \frac{\langle u|v\rangle^2}{\|v\|^2} = \|u\|^2 - \frac{\langle u|v\rangle^2}{\|v\|^2}$

$\Rightarrow \langle u|v\rangle^2 = \|u\|^2\|v\|^2$ ∎

• Periodic functions

$f(x+n\cdot p) = f(x) \quad \forall n \in \mathbb{N}$

Def: A function $f: \mathbb{R} \to \mathbb{K}$ is <u>periodic</u> $\equiv (\exists p>0)(\forall x): f(x+p) = f(x)$.

The smallest period is called the <u>fundamental period</u>

EX: $\sin(x)$, $\cos(x)$

$f: \mathbb{R} \to \mathbb{C}$, $f(x) = e^{ix}$ $\quad \dots\ p = 2\pi \because f(x+2\pi) = e^{i(x+2\pi)} = e^{ix}\, e^{i2\pi} = e^{ix}$

Properties

1) f p-periodic $\Rightarrow -f$ and $1/f$ also p-periodic

2) f, g both p-periodic:

$h(x) := f(x) + g(x) \quad \dots\ h(x+p) = f(x+p) + g(x+p) = h(x) \quad \dots$ also $-$

$k(x) := f(x)\cdot g(x) \quad \dots\ k(x+p) = f(x+p)\cdot g(x+p) = k(x) \quad \dots$ also $\div$

3) $f \dots$ period p_f, $g \dots$ period p_g & $\frac{p_f}{p_g} \in \mathbb{Q}$

$\Rightarrow p_f = \frac{a}{b}\cdot p_g$, $a, b \in \mathbb{N}$

$\Rightarrow$ functions made from f and g will have period $b p_f = a p_g$

4) $f(x+p) = f(x) \Rightarrow g(x) := f(kx) \ \dots$ period p/k

$\hookrightarrow g(x + \frac{p}{k}) = f(kx+p) = f(kx) = g(x)$

EX: $f(x) = \sin(\frac{2}{3}x) \dots p_1 = 3\pi$
$g(x) = \cos(3x) \dots p_2 = \frac{2}{3}\pi$
$\Big\} \ \frac{p_1}{p_2} = \frac{3}{2/3} = \frac{9}{2} \Rightarrow 2p_1 = 9p_2 = 6\pi$

$\hookrightarrow \sin(\frac{2}{3}x)\cdot\cos(3x) \ \dots$ period 6π

• Odd and Even functions

Def: A function f is

- <u>odd</u> $\equiv \forall x \in D(f): f(-x) = -f(x)$
- <u>even</u> $\equiv \forall x \in D(f): f(-x) = f(x)$

$\dots \Rightarrow D_f$ is symmetric about 0.

f	g	$f+g$	$f\cdot g$
odd	odd	odd	$\times$
odd	even	$\times$	odd
even	even	even	even

$f(-x) + g(-x) = -f(x) - g(x)$, $\quad f(-x)g(-x) = f(x)g(x)$

$-f(x) + g(x) \quad | \quad -f(x)g(x)$

$f(x) + g(x) \qquad f(x)\cdot g(x)$

Real Fourier Series

- we will first need some standard integrals.
- note that the period of $\sin\left(\frac{2\pi x}{l}\right)$ and $\cos\left(\frac{2\pi x}{l}\right)$ is l
- assume $m, n \in \mathbb{N}^+$

↳ $\sin(\pi n) = 0, \quad \cos(\pi n) = (-1)^n$

① $\displaystyle\int_{x_0}^{x_0+l} \sin\left(\frac{2\pi n x}{l}\right) dx = 0$ $\quad n \neq 0$

$$= \frac{-l}{2\pi n} \cos\left(\frac{2\pi n x}{l}\right)\Big|_{x_0}^{x_0+l} = \frac{-l}{2\pi n}\left[\cos\left(\frac{2\pi n x_0}{l} + 2\pi n\right) - \cos\left(\frac{2\pi n x_0}{l}\right)\right] = 0$$

② $\displaystyle\int_{x_0}^{x_0+l} \cos\left(\frac{2\pi n x}{l}\right) dx = 0$ $\quad$ similar

③ $\displaystyle\int_{x_0}^{x_0+l} \sin\left(\frac{2\pi n x}{l}\right)\cos\left(\frac{2\pi m x}{l}\right) dx = 0$

	D	I
$+$	$\sin\left(\frac{2\pi n x}{l}\right)$	$\cos\left(\frac{2\pi m x}{l}\right)$
$-$	$\frac{2\pi n}{l}\cos\left(\frac{2\pi n x}{l}\right)$	$\frac{l}{2\pi m}\sin\left(\frac{2\pi m x}{l}\right)$
$+$	$-\left(\frac{2\pi n}{l}\right)^2 \sin\left(\frac{2\pi n x}{l}\right)$	$-\left(\frac{l}{2\pi m}\right)^2 \cos\left(\frac{2\pi m x}{l}\right)$

$$I = \frac{l}{2\pi m}\sin(\dots)\sin(\dots)\Big|_{x_0}^{x_0+l} + \frac{n}{m}\cdot\frac{l}{2\pi m}\cos(\dots)\cos(\dots)\Big|_{x_0}^{x_0+l} + \frac{n^2}{m^2} I$$

$$\Rightarrow I\left(1 - \frac{n^2}{m^2}\right) = \frac{l}{2\pi m}\left[☺ - ☺\right]$$

$\sin\left(\frac{2\pi n(x_0+l)}{l}\right) = \sin\left(\frac{2\pi n x_0}{l} + 2\pi n\right) = \sin\left(\frac{2\pi n x_0}{l}\right)$ ⊛

Def: The Kronecker delta function is defined as $\delta_{m,n} := \begin{cases} 1, & \text{if } m = n \\ 0, & \text{else} \end{cases}$

$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
$\Rightarrow \sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$
$\Rightarrow \cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$

④ $\displaystyle\int_{x_0}^{x_0+l} \sin\left(\frac{2\pi n x}{l}\right)\sin\left(\frac{2\pi m x}{l}\right) dx = \frac{l}{2}\delta_{m,n}$

→ let $N := n - m$ and $M := n + m$

a) $n \neq m$: $\displaystyle\frac{1}{2}\int_{x_0}^{x_0+l} \cos\left(\frac{2\pi N x}{l}\right) - \cos\left(\frac{2\pi M x}{l}\right) dx =$

$$= \frac{1}{2}\left[-\sin(\dots N) + \sin(\dots M)\right]_{x_0}^{x_0+l} \overset{⊛}{=} 0$$

$N = 0$
⇑
b) $n = m$: $\displaystyle\frac{1}{2}\int_{x_0}^{x_0+l} 1 - \cos\left(\frac{2\pi M x}{l}\right) dx = \frac{1}{2}x\Big|_{x_0}^{x_0+l} + \frac{1}{2}\left[☺ - ☺\right] = \frac{1}{2}l.$

⑤ $\displaystyle\int_{x_0}^{x_0+l} \cos\left(\frac{2\pi n x}{l}\right)\cos\left(\frac{2\pi m x}{l}\right) dx = \frac{l}{2}\delta_{m,n}$

↳ the same integral, just $\frac{1}{2}\int \cos(\dots N) + \cos(\dots M)\, dx$

Note: if $x_0 = -\frac{1}{2}l \Rightarrow x_0 + l = \frac{1}{2}l$

↳ in this case, the integrals hold for any $m, n \in \mathbb{R}$
↳ its also much easier to show, since one can utilize the fact that sin is odd and cos is even.

Def: The Fourier series of a l-periodic function f is the following:

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right)$$

→ lets assume that this expansion exists and find a_n and b_n

- → integrate both sides $\quad \cos(0)=1, \ \sin(0)=0$

$$\int_{x_0}^{x_0+l} f(x)\,dx = \frac{1}{2}\int_{x_0}^{x_0+l} a_0\,dx + \sum_{n=1}^{\infty} a_n \int_{x_0}^{x_0+l} \cos\left(\frac{2\pi n x}{l}\right) dx + \sum_{n=1}^{\infty} b_n \int_{x_0}^{x_0+l} \sin\left(\frac{2\pi n x}{l}\right) dx$$

→ note that we will need to impose conditions for swapping Σ and $\int$

→ using 1 and 2

$$\int_{x_0}^{x_0+l} f(x)\,dx = \frac{1}{2} a_0 l + 0 + 0 \Rightarrow a_0 = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\,dx$$

actually [illegible]

- → to find a_m for $m \neq 0$, first multiply by $\cos\left(\frac{2\pi m x}{l}\right)$

$$\int_{x_0}^{x_0+l} f(x)\cos\left(\frac{2\pi m x}{l}\right) dx = \int_{x_0}^{x_0+l} \frac{1}{2} a_0 \cos(\ldots) + \sum_{n=1}^{\infty} a_n \cos(\ldots)\cos(\ldots) + \sum_{n=1}^{\infty} b_n \sin(\ldots)\cos(\ldots)\,dx$$

→ using 2, 3 and 5

$$\int_{x_0}^{x_0+l} f(x)\cos\left(\frac{2\pi m x}{l}\right) dx = 0 + \sum_{n=1}^{\infty} a_n \frac{l}{2}\delta_{m,n} + 0 = a_m \frac{l}{2} \Rightarrow a_m = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\cos\left(\frac{2\pi m x}{l}\right) dx$$

- → to find b_m for $m \neq 0$, first multiply by $\sin\left(\frac{2\pi m x}{l}\right)$, similarly:

$$\int_{x_0}^{x_0+l} f(x)\sin\left(\frac{2\pi m x}{l}\right) dx = 0 + 0 + b_m \frac{l}{2} \Rightarrow b_m = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\sin\left(\frac{2\pi m x}{l}\right) dx$$

Theorem: Suppose f is l-periodic and has the following properties:

i) f is bounded

ii) f has finitely many discontinuities within one period

iii) f has finitely many minima and maxima within one period

iv) $\int_{x_0}^{x_0+l} |f(x)|\,dx$ is finite ... $f(x)$ is absolutely integrable over one period.

If f is continuous at x, then

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right)$$

where

$$a_0 = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\,dx, \quad a_n = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\cos\left(\frac{2\pi n x}{l}\right) dx, \quad b_n = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\sin\left(\frac{2\pi n x}{l}\right) dx$$

If f is not continuous at $x=a$, then $\mathcal{F}(a) = \frac{1}{2}\left(\lim_{x\to a^-} f(x) + \lim_{x\to a^+} f(x)\right)$.

- Fourier in the language of vector spaces

The vector space = space of all functions that obey the conditions of the theorem

→ basis = $\{\cos(\frac{2\pi m x}{\ell}) \mid m \in \mathbb{N}_0\} \cup \{\sin(\frac{2\pi m x}{\ell}) \mid m \in \mathbb{N}^+\}$

→ inner product: $\langle f | g \rangle := \int_{x_0}^{x_0+\ell} f(x) g(x)\, dx$

using the integrals on the previous pag

↳ this basis is orthogonal with respect to this inner product

→ the complex Fourier series will use a different orthogonal basis

→ the so-called Laguerre polynomials give an orthogonal polynomial basis

↳ with a different inner product

→ note that the Tylor series does not give an orthogonal basis .. $\{1, x, x^2 ...\}$

- Fourier of odd and even functions

Theorem: If f has a Fourier series, has period ℓ and

1) f is odd $\Rightarrow f(x) = \sum_{m=1}^{\infty} b_m \sin(\frac{2\pi m x}{\ell})\, dx$, $b_m = \frac{2}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x) \sin(\frac{2\pi m x}{\ell})\, dx$.

2) f is even $\Rightarrow f(x) = \frac{1}{2} a_0 + \sum_{m=1}^{\infty} a_m \cos(\frac{2\pi m x}{\ell})\, dx$, $a_m = \frac{2}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x) \cos(\frac{2\pi m x}{\ell})\, dx$.

Proof:

1) $a_m = \frac{2}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x) \cos(\frac{2\pi m x}{\ell})\, dx = \frac{2}{\ell} \int_{-a}^{a} \text{odd} = 0$

odd · even = odd

2) $b_m = \frac{2}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x) \sin(\frac{2\pi m x}{\ell})\, dx = 0$

even · odd = odd

Note: If we are interested only in the behavior of a function on a certain interval $[0, \frac{\ell}{2})$, we can define an odd or even extension:

- even extension: $f_e(x) := \begin{cases} f(x), & x \in [0, \frac{\ell}{2}) \\ f(-x), & x \in [-\frac{\ell}{2}, 0) \end{cases}$, $f_e(x+\ell) = f_e(x)$

↳ only a_m

- odd extension: $f_o(x) := \begin{cases} f(x), & x \in [0, \frac{\ell}{2}) \\ -f(-x), & x \in [-\frac{\ell}{2}, 0) \end{cases}$, $f_o(x+\ell) = f_o(x)$

↳ only b_m

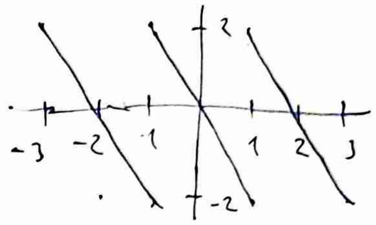
period ℓ.

Exercises:

① **Find the real Fourier series of** $f(x) = -2x$ for $x \in [-1,1)$, $f(x+2) = f(x)$.

$\downarrow$ $l = 2$

$\rightarrow f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi n x) + \sum_{n=1}^{\infty} b_n \sin(\pi n x)$



$\rightarrow f$ is odd $\Rightarrow a_n = 0$

$\rightarrow b_n = \frac{2}{2} \int_{-1}^{1} f(x) \sin(\pi n x)\, dx = -2 \int_{-1}^{1} x \sin(\pi n x)\, dx$

$= -2 \left[-\frac{x}{\pi n} \cos(\pi n x) + \frac{1}{(\pi n)^2} \sin(\pi n x) \right]_{-1}^{1}$

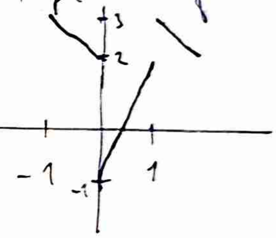
$= -2 \left[-\frac{1}{\pi n}(-1)^n - \frac{1}{\pi n}(-1)^n \right] = \frac{4}{\pi n}(-1)^n$!

	D	I
+	x	$\sin(\pi n x)$
−	1	$-\frac{1}{\pi n}\cos(\pi n x)$
+	0	$-\frac{1}{(\pi n)^2}\sin(\pi n x)$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n} (-1)^n \sin(\pi n x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(\pi n x)$

② **Find the Fourier series of** $f(x) := \begin{cases} 2 - x, & x \in [-1, 0) \\ 3x - 1, & x \in [0, 1) \end{cases}$, $f(x+2) = f(x)$

$\downarrow$ $l = 2$



$\rightarrow f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi n x) + \sum_{n=1}^{\infty} b_n \sin(\pi n x)$.

$\Rightarrow a_0 = \frac{2}{2} \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} (2-x)\, dx + \int_{0}^{1} (3x-1)\, dx$

$= \left[2x - \frac{x^2}{2} \right]_{-1}^{0} + \left[\frac{3}{2}x^2 - x \right]_{0}^{1} = -\left(-2 - \frac{1}{2}\right) + \left(\frac{3}{2} - 1\right) = \frac{5}{2} + \frac{1}{2} = 3$

$\Rightarrow a_n = \int_{-1}^{0} (2-x) \cos(\pi n x)\, dx + \int_{0}^{1} (3x-1) \cos(\pi n x)\, dx$

	D	I
+	$2-x$	$\cos(\pi n x)$
−	-1	$\frac{1}{\pi n}\sin(\pi n x)$
+	0	$\frac{-1}{(\pi n)^2}\cos(\pi n x)$

	D	I
+	$3x-1$	$\cos(\pi n x)$
−	3	$\frac{1}{\pi n}\sin(\pi n x)$
+	0	$-\frac{1}{(\pi n)^2}\cos(\pi n x)$

$= \left[\frac{2-x}{\pi n} \sin(\pi n x) - \frac{1}{(\pi n)^2} \cos(\pi n x) \right]_{-1}^{0} + \left[\frac{3x-1}{\pi n} \sin(\pi n x) + \frac{3}{(\pi n)^2} \cos(\pi n x) \right]_{0}^{1}$

$= -\frac{1}{(\pi n)^2} + \frac{1}{(\pi n)^2}(-1)^n + \frac{3}{(\pi n)^2}(-1)^n - \frac{3}{(\pi n)^2} = \frac{4}{(\pi n)^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ \frac{-8}{(\pi n)^2}, & n \text{ odd} \end{cases}$

$\Rightarrow b_n = \int_{-1}^{0} (2-x) \sin(\pi n x)\, dx + \int_{0}^{1} (3x-1) \sin(\pi n x)\, dx$

	D	I
+	$2-x$	$\sin(\pi n x)$
−	-1	$-\frac{1}{\pi n}\cos(\pi n x)$
+	0	$-\frac{1}{(\pi n)^2}\sin(\pi n x)$

	D	I
+	$3x-1$	$\sin(\pi n x)$
−	3	$-\frac{1}{\pi n}\cos(\pi n x)$
+	0	$-\frac{1}{(\pi n)^2}\sin(\pi n x)$

$= \left[\frac{x-2}{\pi n} \cos(\pi n x) - \frac{1}{(\pi n)^2} \sin(\pi n x) \right]_{-1}^{0} + \left[\frac{1-3x}{\pi n} \cos(\pi n x) + \frac{3}{(\pi n)^2} \sin(\pi n x) \right]_{0}^{1}$

$= \frac{-2}{\pi n} + \frac{3}{\pi n}(-1)^n - \frac{2}{\pi n}(-1)^n - \frac{1}{\pi n} = \frac{(-1)^n}{\pi n} - \frac{3}{\pi n}$

$\Rightarrow f(x) = \frac{3}{2} + \sum_{k=0}^{\infty} \frac{-8}{\pi^2 (2k+1)^2} \cos(\pi(2k+1)x) + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\pi n} - \frac{3}{\pi n} \right) \sin(\pi n x)$

③ Let $f(x)=x^3-3x^2+x-2$ for $0\le x<1$.

a) find an even extension of f with period 3

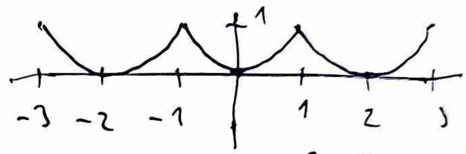
$$f_e(x):=\begin{cases} f(x)=x^3-3x^2+x-2, & x\in[0,1.5)\\ f(-x)=-x^3-3x^2-x-2, & x\in[-1.5,0)\end{cases}, \quad f_e(x+3)=f_e(x)$$

b) find an odd extension of f with period 5

$$f_o(x):=\begin{cases} f(x)=x^3-3x^2+x-2, & x\in[0,2.5)\\ -f(-x)=x^3+3x^2+x+2, & x\in[-2.5,0)\end{cases}, \quad f_o(x+5)=f_o(x)$$

④ Find the Fourier series of $f(x)=x^2$, $x\in[-1,1)$, $f(x+2)=f(x)$.

a) $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(\pi n x)+\sum_{n=1}^{\infty}b_n\sin(\pi n x)$



$l=2$

$\to f$ is even $\Rightarrow b_n=0$

- $a_0=\frac{2}{2}\int_{-1}^{1}x^2\,dx=2\int_0^1 x^2\,dx=2\frac{x^3}{3}\Big|_0^1=\frac{2}{3}$

- $a_n=\int_{-1}^{1}x^2\cos(\pi n x)\,dx=2\int_0^1 x^2\cos(\pi n x)\,dx=$

	D	I
		$\cos(\pi n x)$
$+$	x^2	$\frac{1}{\pi n}\sin(\pi n x)$
$-$	$2x$	$\frac{-1}{(\pi n)^2}\cos(\pi n x)$
$+$	2	$\frac{-1}{(\pi n)^3}\sin(\pi n x)$
$-$	0	

$$=2\left[\frac{x^2}{\pi n}\sin(\pi n x)+\frac{2x}{(\pi n)^2}\cos(\pi n x)-\frac{2}{(\pi n)^3}\sin(\pi n x)\right]_0^1$$

$$=2\left[\frac{2}{(\pi n)^2}(-1)^n-0\right]=\frac{4}{(\pi n)^2}(-1)^n$$

$$\Rightarrow f(x)=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{(\pi n)^2}(-1)^n\cos(\pi n x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(\pi n x)$$

b) find $\zeta(2)$ and $\eta(2)$

- $f(0)=0=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{1}{3}+\frac{4}{\pi^2}(-\eta(2))\Rightarrow \eta(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

- $f(1)=1=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n=\frac{1}{3}+\frac{4}{\pi^2}\zeta(2)\Rightarrow \zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

⑤ Find the Fourier series of $f(x)=x^2$, $x\in[-\pi,\pi)$, $f(x+2\pi)=f(x)$

$l=2\pi$

$\to$ same function as ④, just different limits

- $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{1}{\pi}\int_0^{\pi}x^2\,dx=\frac{2}{\pi}\cdot\frac{x^3}{3}\Big|_0^{\pi}=\frac{2}{3}\pi^2$

- $a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos(nx)\,dx=\frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx)+\frac{2x}{n^2}\cos(nx)-\frac{2}{n^3}\sin(nx)\right]_0^{\pi}=$

$$=\frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos(\pi n)-0\right]=\frac{4}{n^2}(-1)^n$$

$$\Rightarrow f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

- Complex Fourier Series

→ using Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ ⤳ $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$

→ suppose $f: \mathbb{R} \to \mathbb{C}$ has a Fourier series, where we allow the coefficients to be complex | f is ℓ-periodic

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2\pi n x}{\ell}\right) + b_n\sin\left(\frac{2\pi n x}{\ell}\right)\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(\exp\left(\frac{2\pi i n x}{\ell}\right) + \exp\left(\frac{-2\pi i n x}{\ell}\right)\right) + \frac{b_n}{2i}\left(\exp\left(\frac{2\pi i n x}{\ell}\right) - \exp\left(\frac{-2\pi i n x}{\ell}\right)\right)\right]$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\frac{a_n - i b_n}{2}\exp\left(\frac{2\pi i n x}{\ell}\right) + \sum_{n=1}^{\infty}\frac{a_n + i b_n}{2}\exp\left(\frac{-2\pi i n x}{\ell}\right)$$

$$= \underline{\underline{\sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{\ell}\right)}}, \quad c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - i b_n}{2}, \quad c_{-n} = \frac{a_n + i b_n}{2},$$

→ to find a nice expression for c_m, multiply both sides by $\exp\left(\frac{-2\pi i m x}{\ell}\right)$ and $\int$

$$\int_{x_0}^{x_0+\ell} f(x)\exp\left(\frac{-2\pi i m x}{\ell}\right) = \int_{-x_0}^{x_0+\ell}\sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{\ell}\right)\exp\left(\frac{-2\pi i m x}{\ell}\right)dx =$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{x_0}^{x_0+\ell}\exp\left(\frac{2\pi i (n-m)x}{\ell}\right)dx$$

a) $m = n$: $\exp(\dots) = \exp(0) = 1 \Rightarrow \int_{x_0}^{x_0+\ell} 1\,dx = \ell$

exp is periodic

b) $m \neq n$: $\int = \frac{\ell}{2\pi i(n-m)}\exp\left(\frac{2\pi i(n-m)x}{\ell}\right)\Big|_{x_0}^{x_0+\ell} = K\cdot\left[\exp\left(\frac{2\pi i(n-m)x_0}{\ell} + 2\pi i(n-m)\right) - e(\dots)\right]$

$= K\cdot[☺ - ☺] = 0$

$$\Rightarrow \int_{x_0}^{x_0+\ell} f(x)\exp\left(\frac{-2\pi i m x}{\ell}\right)dx = \sum_{n=-\infty}^{\infty} c_n \ell\,\delta_{m,n} = c_m \ell \Rightarrow \underline{\underline{c_m = \frac{1}{\ell}\int_{x_0} f(x)\exp\left(-\frac{2\pi i m x}{\ell}\right)dx}}$$

Theorem: Suppose $f: \mathbb{R} \to \mathbb{C}$ is a ℓ-periodic function which satisfies

i) f is bounded

ii) f has finitely many discontinuities within one period

iii) f has finitely many minima and maxima within one period

iv) $\int_{x_0}^{x_0+\ell} |f(x)|\,dx$ is finite ... f is absolutely integrable over one period

If f is continuous at x, then

$$\underline{f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{\ell}\right)}, \quad \underline{c_n = \frac{1}{\ell}\int_{x_0}^{x_0+\ell} f(x)\exp\left(\frac{-2\pi i n x}{\ell}\right)dx}$$

If f is not continuous at $x = a$, then the Fourier series at a coverges to

$$F(a) = \frac{1}{2}\left[\lim_{x\to a^-} f(x) + \lim_{x\to a^+} f(x)\right]$$

- **Vector space of Complex Fourier Series**

Intuice: Fourier series = Součet nascalovaných bodů na kružnicích, co se všechny točí s různou frekvencí

→ basis $= \left\{ \exp\left(\frac{2\pi i n x}{\ell}\right) \,\middle|\, n \in \mathbb{Z} \right\}$ ... functions of period ℓ

→ inner product: $\langle f | g \rangle := \int_{x_0}^{x_0+\ell} f(x)\, \overline{g(x)}\, dx$

→ on the previous page, we have shown that $\int_{x_0}^{x_0+\ell} \exp\left(\frac{2\pi i (n-m) x}{\ell}\right) = \ell\, \delta_{m,n}$

$$\left.\begin{array}{l} f(x) = \exp\left(\frac{2\pi i n x}{\ell}\right) \\ g(x) = \exp\left(\frac{2\pi i m x}{\ell}\right) \end{array}\right\} \langle f | g \rangle = \int_{x_0}^{x_0+\ell} f(x)\, \overline{g(x)}\, dx = \int_{x_0}^{x_0+\ell} \exp\left(\frac{2\pi i n x}{\ell}\right) \exp\left(-\frac{2\pi i m x}{\ell}\right) dx$$

$\Rightarrow$ this basis is orthogonal with respect to $\langle \cdot | \cdot \rangle$

$$\overline{\exp(iz)} = \exp(-iz)$$

Proposition: If f is real valued, then $\overline{C_n} = C_{-n}$.

Proof: Since $\overline{f(x)} = f(x)$, we have

$$\overline{C_n} = \overline{\frac{1}{\ell}\int_{x_0}^{x_0+\ell} f(x) \exp\left(-\frac{2\pi i n x}{\ell}\right) dx} = \frac{1}{\ell}\int_{x_0}^{x_0+\ell} f(x) \exp\left(\frac{2\pi i n x}{\ell}\right) dx = C_{-n} \quad \blacksquare$$

Theorem (Parsevals): Suppose that $A: \mathbb{R} \to \mathbb{C}$ and $B: \mathbb{R} \to \mathbb{C}$ are ℓ-peridic functions with Fourier series

$$A(x) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{2\pi i n x}{\ell}\right) \Rightarrow \hat{a} := \{(n, a_n) \mid n \in \mathbb{Z}\}$$

$$B(x) = \sum_{n=-\infty}^{\infty} b_n \exp\left(\frac{2\pi i n x}{\ell}\right) \Rightarrow \hat{b} := \{(n, b_n) \mid n \in \mathbb{Z}\}$$

Then

$$\langle \hat{a} | \hat{b} \rangle_{\mathbb{C}} = \frac{1}{\ell} \langle A | B \rangle \quad \ldots \quad \sum_{n=-\infty}^{\infty} a_n \overline{b_n} = \frac{1}{\ell} \int_{x_0}^{x_0+\ell} A(x)\, \overline{B(x)}\, dx$$

Proof:

$$A(x)\,\overline{B(x)} = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{2\pi i n x}{\ell}\right) \overline{\sum_{m=-\infty}^{\infty} b_m \exp\left(\frac{2\pi i m x}{\ell}\right)} \to$$

$$= \sum_{n,m=-\infty}^{\infty} a_n \overline{b_m} \exp\left(\frac{2\pi i (n-m) x}{\ell}\right)$$

$$\Rightarrow \int_{x_0}^{x_0+\ell} A(x)\,\overline{B(x)}\, dx = \int \sum \cdots dx = \sum_{n,m=-\infty}^{\infty} a_n \overline{b_m} \int_{x_0}^{x_0+\ell} \exp\left(\frac{2\pi i (n-m) x}{\ell}\right) dx =$$

$$= \sum_{n,m=-\infty}^{\infty} a_n \overline{b_m}\, \ell \cdot \delta_{m,n} = \ell \cdot \sum_{n=-\infty}^{\infty} a_n \overline{b_n} \quad \blacksquare$$

Corollary: If $f: \mathbb{R} \to \mathbb{C}$ has a fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \exp\left(\frac{2\pi i n x}{\ell}\right) \Rightarrow \sum_{n=-\infty}^{\infty} |C_n|^2 = \frac{1}{\ell} \int_{x_0}^{x_0+\ell} |f(x)|^2\, dx$$

Corollary: If $f: \mathbb{R} \to \mathbb{R}$ has a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{\ell}\right)$$

then

$$\frac{1}{\ell}\int_{x_0}^{x_0+\ell} |f(x)|^2 dx = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

Proof: When deriving the form of the complex Fourier series, we noted that

$$c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n - i b_n), \quad c_{-n} = \frac{1}{2}(a_n + i b_n).$$

If we substitute this to the Parseval's theorem formula, we get

$$\frac{1}{\ell}\int_{x_0}^{x_0+\ell} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = |c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_{-n}|^2$$

Since f is real valued, we have $c_{-n} = \overline{c_n} \Rightarrow |c_{-n}| = |c_n|$

$$= |c_0|^2 + 2\sum_{n=1}^{\infty} |c_n|^2 = \left(\frac{a_0}{2}\right)^2 + 2\sum_{n=1}^{\infty} \left|\frac{1}{2}(a_n - i b_n)\right|^2 = \left(\frac{a_0}{2}\right)^2 + 2\sum_{n=1}^{\infty} \frac{1}{4}(a_n^2 + b_n^2) \quad \blacksquare$$

Exercises:

(1) Earlier (5), we have shown that $f(x) = x^2$, $x \in [-\pi, \pi)$, $f(x+2\pi) = f(x)$ has the Fourier series $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$. Find $\zeta(4)$.

→ to find $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}$, apply Parcerals theorem to $f(x)$:

$$\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |f(x)|^2 dx = \left(\frac{\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(4\frac{(-1)^n}{n^2}\right)^2, \quad \ell = 2\pi, \quad b_n = 0$$

LH: $\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{1}{\pi}\int_0^{\pi} x^4 dx = \frac{1}{\pi}\frac{x^5}{5}\Big|_0^{\pi} = \frac{\pi^4}{5}$

RH: $\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{\pi^4}{9} + 8\zeta(4)$

$$\zeta(4) = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}$$

(2) Find the complex Fourier series of $f(x) = x^3$, $x \in [-\pi, \pi)$, $f(x+2\pi) = f(x)$.

$\hookrightarrow \ell = 2\pi$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 \exp\left(-\frac{2\pi i n x}{2\pi}\right) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 \exp(-inx)\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-x^3}{in}e^{-inx} + \frac{3x^2}{n^2}e^{-inx} + \frac{6x}{in^3}e^{-inx} - \frac{6}{n^4}e^{-inx}\right]_{-\pi}^{\pi}$$

D	I
$+\ x^3$	$\exp(-inx)$
$-\ 3x^2$	$\frac{-1}{in}\exp(-inx)$
$+\ 6x$	$\frac{1}{(in)^2}e = \frac{-1}{n^2}e$
$-\ 6$	$\frac{-1}{(in)^3}e = \frac{1}{in^3}e$
$+\ 0$	$\frac{1}{(in)^4}e = \frac{1}{n^4}e$

$e^{in\pi} = e^{-in\pi} = (-1)^n \quad \forall n \in \mathbb{Z}$

$$= \frac{1}{2\pi}(-1)^n\left[\frac{-x^3}{in} + \frac{6x}{in^3}\right]_{-\pi}^{\pi} = \frac{(-1)^n}{2\pi}\left[\frac{-2\pi^3}{in} + \frac{12\pi}{in^3}\right]$$

$$= i\cdot(-1)^n\left[\frac{\pi^2}{n} - \frac{6}{n^3}\right]$$

$n = 0$: $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 dx = 0$

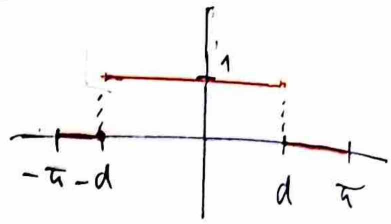
$$\Rightarrow f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left(\frac{\pi^2}{n} - \frac{6}{n^3}\right)(-1)^n\, i\, \exp(inx)$$

③ Find the sum of the series $\sum_{n=1}^{\infty} \frac{\sin^2(nd)}{n^2}$, $d \in (0,\pi)$

We will apply Parsevals theorem to the function

$$f(x) = \begin{cases} 1, & 0 < |x| < d \\ 0, & d < |x| < \pi \end{cases}, \quad f(x+2\pi) = f(x)$$

$\hookrightarrow l = 2\pi$



$\rightarrow f(x)$ is even $\Rightarrow b_n = 0$

$\rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-d}^{d} 1\,dx = \frac{2d}{\pi}$

$\rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-d}^{d}\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{1}{n}\sin(nx)\right]_0^d = \frac{2}{\pi n}\sin(dn)$

Parseval: $\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} |f(x)|^2\,dx = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{a_n^2+b_n^2}{2}$

RH: $\frac{d^2}{\pi^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\sin^2(dn) = \frac{d^2}{\pi^2} + \frac{2}{\pi^2}\cdot\sum$

LH: $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-d}^{d}1\,dx = \frac{1}{2\pi}\cdot 2d = \frac{d}{\pi}$

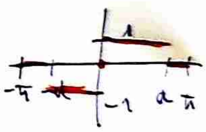
$$\underline{\underline{\sum = \frac{\pi^2}{2}\left(\frac{d}{\pi} - \frac{d^2}{\pi^2}\right) = \frac{d}{2}(\pi - d)}}$$

④ Find the sum of the series $\sum_{n=1}^{\infty}\frac{\cos^2(nd)}{n^2}$, $d \in (0,\pi)$

$\rightarrow$ last time we used  to create sin out of $a_n \sim \int \cos$

$\Rightarrow$ need to create cos out of $b_n \sim \int \sin$

$\Rightarrow$ need odd function $\Rightarrow$ try $f(x) := \begin{cases} \mathrm{sign}(x), & 0 < |x| < d \\ 0, & d < |x| < \pi \end{cases}$



$\rightarrow f(x)$ is odd $\Rightarrow a_n = 0$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-d}^{d}\mathrm{sign}(x)\sin(nx)\,dx =$

$= \frac{1}{\pi}\int_{-d}^{d}\sin(n|x|)\,dx \overset{\text{even}}{=} \frac{2}{\pi}\int_0^d \sin(n|x|)\,dx = \frac{2}{\pi}\int_0^d \sin(nx)\,dx$

$= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^d = \frac{-2}{\pi n}\left[\cos(dn) - 1\right]$

$\rightarrow$ squaring will give us $\cos^2$ but also $\cos$ $\Rightarrow$ this is a dead end

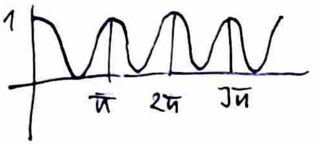
$\rightarrow$ use result from ③

$$\sum_{n=1}^{\infty}\frac{\cos^2(nd)}{n^2} = \sum_{n=1}^{\infty}\frac{1-\sin^2(nd)}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{\sin^2(nd)}{n^2} = \underline{\underline{\frac{\pi^2}{6} - \frac{d}{2}(\pi - d)}}$$

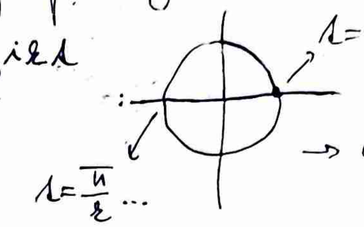
Motivation behind the Fourier transform

→ goal: We have a signal wave  and want to decompose it to a sum of sin / cos wave

→ lets take a look at a pure frequency

$f(x) =$ 

e^{ikt}



→ as t goes from 0 to $\frac{2\pi}{k}$ e^{ikt} revolves 1 rotation

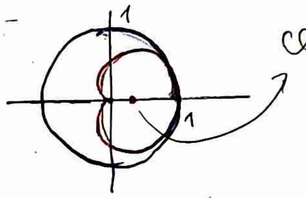
→ define $g(t) = f(t)\, e^{ikt}$

↳ $g(t)$ traces a complex curve using $f(t)$ as radius

↳ $f(x)$ has period π ⇒ we want 1 rotation in $t \in (0, \pi)$

$\Rightarrow \frac{2\pi}{k} = p = \pi$

$\Rightarrow g(t) = f(t)\, e^{2it}$:



center of mass

→ if we chose a different k, the curve would be much more complicated and the center of mass would be near the origin

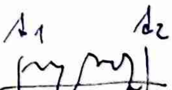
⇒ so if we know that f is a pure frequency, but dont know the period, we can try different values of k, look at the distance of "the center of mass" from the origin and max dist $\Rightarrow \frac{2\pi}{k} =$ period

→ now consider a more complicated function:

$$f(x) = f_1(x) + f_2(x) \Rightarrow g(t) = (f_1(t) + f_2(t))\, e^{ikt} = g_1(t) + g_2(t)$$

→ we can approximate the center of mass by sampling N points from g and taking an average $\Rightarrow C \approx \sum_{i=1}^{N} g(t_i) = \sum_i [g_1(t_i) + g_2(t_i)] = C_1 + C_2$

⇒ the center of mass is linear with respect to the functions which make up f ⇒ if we find the spikes in distance, we get the frequencies which make up f

→ if the time when the signal is measured goes from t_1 to t_2, 

then the center of mass equals

$$C = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} g(t)\, dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t)\, e^{ikt}$$

→ the Fourier transform assumes $t_1 = -\infty$, $t_2 = +\infty$ and omits the $t_2 - t_1$ term. Some definitions also slightly modify the integral.

- Fourier Transforms

Def: The Fourier transform of a function $f: \mathbb{R} \to \mathbb{C}$ is the function

$$\hat{f}: \mathbb{R} \to \mathbb{C} \quad , \quad \hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

Note: Not all functions have a Fourier transform, since we require $\lim_{x \to \pm\infty} f(x) = 0$, else the integral will not converge.

EX: $f(x) = 1$: $\int_{-\infty}^{\infty} e^{-ikx} dx = -\frac{1}{ik} e^{-ikx} \Big|_{-\infty}^{\infty}$

↳ but we cant evaluate e^{-ikx} at ∞ or $-\infty$, because its just some random point on the unit circle, but we dont know which one

Corollary: Periodic functions dont have a Fourier transform.

Note: The Fourier transform is sometimes defined without the $\sqrt{2\pi}$ and $\exp(-2\pi i k x)$ is sometimes also used.

Def: The inverse Fourier transform of a function $\hat{f}: \mathbb{R} \to \mathbb{C}$ is the function

$$f: \mathbb{R} \to \mathbb{C}, \quad f(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk.$$

Notation: The Fourier transform and inverse F.t. are denoted as

$$\mathcal{F}: f \mapsto \hat{f} \quad \text{and} \quad \mathcal{F}^{-1}: \hat{f} \mapsto f.$$

Theorem (Fourier integral): It holds that $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$.

Proposition: The Fourier transform of $f(x) = e^{-\frac{1}{2}x^2}$ is $f(x)$ itself.

Proof: $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2}) \exp(-ikx)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-(\frac{x^2}{2} + ikx))\, dx$

→ we want to utilize the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

$$\Rightarrow \frac{x^2}{2} + ikx = \frac{1}{2}\left(x^2 + 2x(ik) + (ik)^2 - (ik)^2\right) = \frac{1}{2}\left[(x+ik)^2 + k^2\right] = \left(\frac{x+ik}{\sqrt{2}}\right)^2 + \frac{k^2}{2}$$

$$\Rightarrow I(k) = \int_{-\infty}^{\infty} \exp\left[-\left(\frac{x+ik}{\sqrt{2}}\right)^2 - \frac{k^2}{2}\right] dx = \exp\left(-\frac{k^2}{2}\right) \int_{-\infty}^{\infty} \exp\left[-\left(\frac{x+ik}{\sqrt{2}}\right)^2\right] dx$$

→ substitute $u := \frac{x+ik}{\sqrt{2}} \Rightarrow du = \frac{1}{\sqrt{2}} dx$ → u is complex

↳ this is no longer rigorous but gives an intuition why it should hold

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \int_{-\infty}^{\infty} \exp(-u^2) \sqrt{2}\, du = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot (\sqrt{\pi}\sqrt{2}) = e^{-\frac{k^2}{2}} = f(k)$$

∎

- Properties of Fourier transforms

→ suppose that f and g are functions with Fourier transforms $\hat{f}$ and $\hat{g}$

① Linearity: $\alpha f(x) + \beta g(x) \overset{\mathcal{F}}{\longleftrightarrow} \alpha \hat{f}(k) + \beta \hat{g}(k)$

$$\hookrightarrow \mathcal{F}(\alpha f + \beta g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x))\, e^{-ikx} dx = \frac{\alpha}{\sqrt{2\pi}} \int f(x) e^{-ikx} dx + \frac{\beta}{\sqrt{2\pi}} \int g(x) e^{-ikx} dx$$
$$= \alpha \hat{f}(k) + \beta \hat{g}(k)$$

② Scaling rule: $f(\alpha x) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|\alpha|} \hat{f}\left(\frac{k}{\alpha}\right)$, $\alpha \neq 0$

$$h(x) := f(\alpha x) \Rightarrow \mathcal{F}(h)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha x) e^{-ikx} dx$$

$u := \alpha x$, $du = \alpha dx$

- $\alpha > 0$: $\frac{1}{\alpha} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\left(\frac{k}{\alpha}\right) u} dx = \frac{1}{|\alpha|} \hat{f}\left(\frac{k}{\alpha}\right)$
- $\alpha < 0$: $\frac{1}{\alpha} \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(u)\, e^{-i\left(\frac{k}{\alpha}\right) u} dx = \frac{1}{-\alpha} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \dots dx = \frac{1}{|\alpha|} \hat{f}\left(\frac{k}{\alpha}\right)$

③ Translation along x: $f(x-a) \overset{\mathcal{F}}{\longleftrightarrow} e^{-ika} \hat{f}(k)$

$$\mathcal{F}(f(x-a))(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{-ikx} dx = \left| \begin{matrix} u = x-a \\ du = dx \end{matrix} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-ik(u+a)} du$$
$$= \frac{1}{\sqrt{2\pi}} e^{-ika} \int_{-\infty}^{\infty} f(u) e^{-iku} du = e^{-ika} \hat{f}(k)$$

④ Translation along k: $e^{iax} f(x) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(k-a)$

$$\mathcal{F}^{-1}(\hat{f}(k-a))(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k-a) e^{ikx} dk = \left| \begin{matrix} u = k-a \\ du = dk \end{matrix} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(u) e^{i(u+a)x} du$$
$$= \frac{1}{\sqrt{2\pi}} e^{iax} \int_{-\infty}^{\infty} \hat{f}(u) e^{iux} du = e^{iax} f(x)$$

⑤ Inversion rule: $\hat{f}(x) \overset{\mathcal{F}}{\longleftrightarrow} f(-k)$ ... ⟶ $\mathcal{F}[\mathcal{F}[f]](x) = f(-x)$

$\mathcal{F}^{-1}[\mathcal{F}^{-1}[\hat{f}]](k) = \hat{f}(-k)$

$$\mathcal{F}(\hat{f})(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{i(-k)x} dx = f(-k)$$

Fourier integral theorem

⑥ Even symmetry: f even $\Leftrightarrow$ $\hat{f}$ even

$$\Rightarrow: \quad \hat{f}(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(-k)x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ik(-x)} dx = \left| \begin{matrix} u = -x \\ du = -dx \end{matrix} \right.$$
$$= \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-u) e^{-iku} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-iku} du = \hat{f}(k)$$

$\Leftarrow$: similar but for $f(-k) = f(k)$

⑦ Odd symmetry: f odd $\Leftrightarrow$ $\hat{f}$ odd

$$\Rightarrow: \hat{f}(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(-k)x} dx \quad \left| \begin{matrix} u = -x \\ du = -dx \end{matrix} \right| = \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-u) e^{-iku} du$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-iku} du = -\hat{f}(k)$$

$\Leftarrow$: similar but for $f(-k) = -f(k)$

8. **Derivative rule:** $\frac{d^n}{dx^n} f(x) \xleftrightarrow{\mathcal{F}} (ik)^n \hat{f}(k)$

$n=0$: $f(x) \xleftrightarrow{\mathcal{F}} \hat{f}(x)$ ✓

$n-1 \to n$: $\mathcal{F}\left(\frac{d^n}{dx^n} f(x)\right)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} f(x)\, e^{-ikx}\, dx =$

	D	I
+	e^{-ikx}	$\frac{d^n}{dx^n} f(x)$
−	$-ik\, e^{-ikx}$	$\frac{d^{n-1}}{dx^{n-1}} f(x)$

Assume it holds for $n-1$
$\Rightarrow \frac{d^{n-1}}{dx^{n-1}} f(x)$ has a Fourier transform
$\Rightarrow \lim_{x\to\pm\infty} \left(\frac{d^{n-1}}{dx^{n-1}} f(x)\right) = 0$

$= \frac{1}{\sqrt{2\pi}} \left[e^{-ikx} \frac{d^{n-1}}{dx^{n-1}} f(x) \right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{d^{n-1}}{dx^{n-1}} f(x)\, dx =$

$= (\text{something on } \oplus) \cdot 0 + ik \cdot (ik)^{n-1} \hat{f}(k) = \underline{(ik)^n \hat{f}(k)}$

$\mathcal{F}^{-1}\left((ik)^n \hat{f}(k)\right) = \mathcal{F}^{-1}\left(\mathcal{F}\left(\frac{d^n}{dx^n} f(x)\right)\right) = \underline{\frac{d^n}{dx^n} f(x)}$

9. **Monomial rule:** $x^n f(x) \xleftrightarrow{\mathcal{F}} (i)^n \frac{d^n}{dk^n} \hat{f}(k)$

$n=0$: $f(x) \xleftrightarrow{\mathcal{F}} \hat{f}(k)$ ✓

$n-1 \to n$: $\mathcal{F}^{-1}\left((i)^n \frac{d^n}{dk^n} \hat{f}(k)\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i)^n \frac{d^n}{dk^n} \hat{f}(k)\, e^{ikx}\, dk =$

	D	I
+	e^{ikx}	$\frac{d^n}{dk^n} \hat{f}(k)$
−	$ix e^{ikx}$	$\frac{d^{n-1}}{dk^{n-1}} \hat{f}(k)$

$= \frac{i^n}{\sqrt{2\pi}} \left[e^{ikx} \frac{d^{n-1}}{dx^{n-1}} \hat{f}(k) \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i)^{n+1} x\, e^{ikx} \frac{d^{n-1}}{dk^{n-1}} \hat{f}(k)\, dk$

$= 0 + \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i)^{n-1} \frac{d^{n-1}}{dx^{n-1}} \hat{f}(k)\, e^{ikx}\, dk = x \cdot (x^{n-1} f(x)) = \underline{x^n f(x)}$

👁 If $\mathcal{F}(f) = \hat{f}$, then $\lim_{k\to\pm\infty} \hat{f}(k) = 0$

↳ intuitively, if $k \to \infty$, then the circle is rotating really fast, so the center of mass will be at the origin

Exercises:

1. Find the F.T. of the function $f(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & x<0 \end{cases}$ and express it as an integral

$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x(ik+1)}\, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{1+ik} e^{-x(1+ik)} \right]_0^{\infty} = \underline{\frac{1}{\sqrt{2\pi}(1+ik)}}$

$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \underline{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+ik}\, dk}$

2. Using 1, find the F.T. of the function $h(x) = \begin{cases} 4e^{-5x}, & x>0 \\ 0, & x<0 \end{cases}$

→ $h(x) = 4 \cdot f(5x)$

→ using linearity: $\hat{h}(k) = 4 \cdot \mathcal{F}(f(5x))(k)$

→ using scaling: $\hat{h}(k) = 4 \cdot \frac{1}{|5|} \hat{f}\left(\frac{k}{5}\right) = \frac{4}{5} \hat{f}\left(\frac{k}{5}\right) = \frac{4}{5} \cdot \frac{1}{\sqrt{2\pi}(1+ik/5)} = \underline{\frac{4}{\sqrt{2\pi}(5+ik)}}$

• The Dirac Delta Function

Def: Denote $f_n(x) := \begin{cases} 1/n, & 0 \le x \le n \\ 0, & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f_n(x)\, dx = \int_0^n \frac{1}{n}\, dx = \frac{n}{n} = \underline{1}$$

Def: The Dirac delta function is defined by $\delta(x) := \lim_{n \to 0^+} f_n(x)$

Note: This is not a function but rather something called a distribution.

Note: Alternatively, we can define δ as an object with the following properties:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \quad \& \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Properties:

① Sifting property: $\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a)$ ... for any function f

② Composition with a function

We want to define $\delta(g(x))$ in a way s.t. substitution works

$$\int_{\mathbb{R}} \delta(g(x))\, f(g(x))\, |g'(x)|\, dx = \int_{g(\mathbb{R})} \delta(u)\, f(u)\, du$$

→ if g is nonzero everywhere, then clearly $\delta(g(x)) = 0$, so lets assume that g has a real root x_0, then the integral on the RHS evaluates to $f(0)$ in $g(\mathbb{R})$, meaning $f(g(x_0))$ in $\mathbb{R}$

⇒ if we set $\delta(g(x)) := \frac{\delta(x - x_0)}{|g'(x_0)|}$, then

$$LHS = \int_{\mathbb{R}} \delta(x - x_0)\, f(g(x)) \left|\frac{g'(x)}{g'(x_0)}\right| dx = f(g(x_0)) \left|\frac{g'(x_0)}{g'(x_0)}\right| = f(g(x_0)) \quad ✓$$

Def: Let $g(x)$ be a continuously differentiable function with isolated zeroes $x_1, x_2, \dots$ with non-zero derivatives at these points. Define

$$\delta(g(x)) := \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} = \begin{cases} 0, & g(x) \neq 0 \\ \frac{\delta(0)}{|g'(x)|}, & g(x) = 0 \end{cases}$$

Corollary: $\int_{-\infty}^{\infty} f(x)\, \delta(g(x))\, dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$, where x_i are zeroes of g

③ $\mathcal{F}(\delta) = \frac{1}{\sqrt{2\pi}}$ ... $\hat{\delta}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^{-ik0} = \frac{1}{\sqrt{2\pi}}$

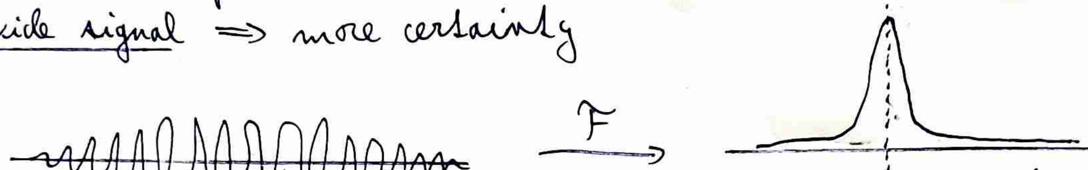
④ $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$ ... $\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\delta}(k) e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$

⑤ $\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) \exp(-ikx') dk = \delta(x-x')$ ... $\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk$

↳ similar to $\frac{1}{l} \int_{x_0}^{x_0+l} \exp\left(\frac{2\pi i n x}{l}\right) \exp\left(-\frac{2\pi i m x}{l}\right) dx = \delta_{m,n}$

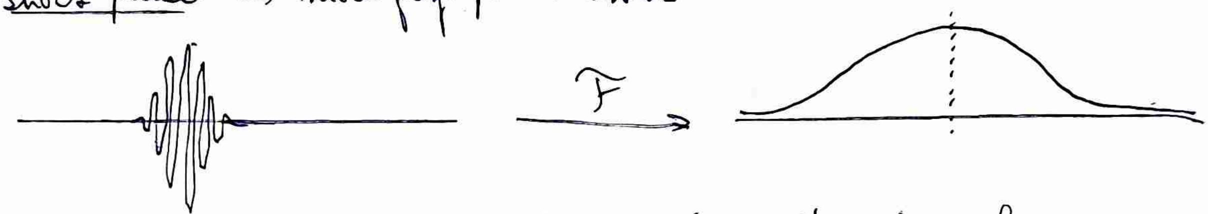
<u>Intuition</u>: The Fourier transform helps us identify the frequencies of sines, from which a given signal is composed

- <u>Wide signal</u> ⇒ more certainty

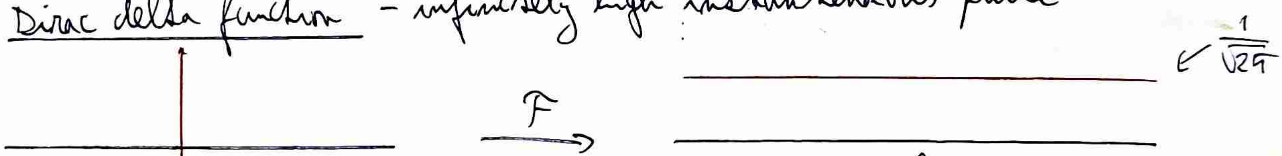


↳ max ~ frequency

- <u>Short pulse</u> ⇒ more freq. possibilities



- <u>Dirac delta function</u> – infinitely high instantaneous pulse



↳ it could have been any frequency

→ when we find the inverse F.T. of the constant function $y(x) = 1$, we expect to get something with the Dirac delta function

⑥ $\delta(-x) = \delta(x)$ ... δ is even

→ using the composition property with $g(x) := -x$ ⇒ $y(x) = 0 \Leftrightarrow x = 0$

$$\delta(-x) = \delta(g(x)) = \frac{\delta(x-0)}{|g'(0)|} = \frac{\delta(x)}{|-1|} = \delta(x)$$

⑦ $\mathcal{F}(1)(k) = \sqrt{2\pi}\, \delta(k)$

using inversion $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$

↳ $\mathcal{F}(\delta) = \frac{1}{\sqrt{2\pi}}$ ⇒ $\delta(-x) = \mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\right)$

↳ using linearity: $\delta(-x) = \delta(x) = \mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\right) = \frac{1}{\sqrt{2\pi}} \mathcal{F}(1)$ ⇒ $\mathcal{F}(1) = \sqrt{2\pi}\, \delta(x)$

<u>Note</u>: If we distributed the $\sqrt{2\pi}$ constants differently in the definition of the F.T. and I.F.T., we would get $\mathcal{F}(\delta) = 1$ and $\mathcal{F}(1) = \delta$.

The Tophat and Sinc functions

Def: The <u>tophat</u> and <u>sinc</u> functions are defined as follows:

$$\chi(x) = \mathrm{rect}(x) := \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \quad \searrow \text{even} \qquad \mathrm{sinc}(x) := \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \rightsquigarrow \text{even}$$

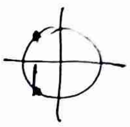
Note: The tophat function the <u>characteristic function</u> of the set $[-1,1]$.
In general, for $S \subseteq \mathbb{R}$, define $\chi_S(x) := 1$ if $x \in S$, else 0.

Note: The tophat/rect function is sometimes defined as $\begin{cases} 1, & |x| \leq \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$

Properties:

(1) $\mathcal{F}(\mathrm{rect})(\xi) = \sqrt{\frac{2}{\pi}}\,\mathrm{sinc}(\xi)$

$$\hat{\chi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(x)\, e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x}\, dx$$



- $\xi = 0$: $\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1\, dx = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$
- $\xi \neq 0$: $\frac{1}{\sqrt{2\pi}} \left[-\frac{1}{i\xi} e^{-i\xi x} \right]_{-1}^{1} = \frac{-1}{i\xi\sqrt{2\pi}} \left[e^{-i\xi} - e^{i\xi} \right] = -\frac{1}{i\xi\sqrt{2\pi}} \left[-2i\sin(\xi) \right] = \frac{2}{\sqrt{2\pi}} \frac{\sin(\xi)}{\xi}$

(2) $\mathrm{rect}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{sinc}(\xi)\, e^{i\xi x}\, d\xi$

(3) $\mathcal{F}(\mathrm{sinc})(\xi) = \sqrt{\frac{\pi}{2}}\, \mathrm{rect}(\xi)$

$$\rightarrow \mathcal{F}(\chi)(\xi) = \sqrt{\frac{2}{\pi}}\, \mathrm{sinc}(\xi) \Rightarrow \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\, \mathrm{sinc}(\xi)\right)(x) = \chi(-x) = \chi(x) \Rightarrow \mathcal{F}(\mathrm{sinc})(x) = \sqrt{\frac{\pi}{2}}\, \chi(x)$$

(4) $\mathrm{sinc}(\pi x) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$

$\mathrm{sinc}(\pi x) = 0 \iff \sin(\pi x) = 0 \ \&\ x \neq 0 \iff x \in \mathbb{Z} \setminus \{0\}$

$\hookrightarrow$ sinc is analytical $\Rightarrow$ can be written as a polynomial

$\hookrightarrow$ $\mathrm{sinc}(0) = 1 \Rightarrow$ absolute term will be 1

$$\Rightarrow \mathrm{sinc}(\pi x) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)\left(1 + \frac{x}{n}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

compare coefficients of x^2 to solve the Basel problem

(5) $\mathrm{sinc}(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$ Taylor of $\sin / x$

(6) $\int_{-\infty}^{\infty} \mathrm{sinc}(x)\, dx = \pi$ .. plug in $x = 0$ to (2)

(7) $\sum_{n=-\infty}^{\infty} \mathrm{sinc}(n) = \pi$

(8) $\sum_{n=-\infty}^{\infty} \mathrm{sinc}(n - x) = \pi$ ... for $\forall x \in \mathbb{R}$

(9) $\sum_{n=-\infty}^{\infty} \mathrm{sinc}(\pi n - x) = 1$ ... for $\forall x \in \mathbb{R}$

We will show this shortly

- Poisson Summation Formula

Def (little o notation): Let $f, g: \mathbb{R} \to \mathbb{C}$ be complex valued functions defined on a neighborhood of $+\infty$ and let $g(x) \neq 0$ for x sufficiently large. Then

$$f \in o(g) \equiv \lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = 0$$

... f grows slower than g

... $|\cdot|$ is the complex modulus

Def: A function $f: \mathbb{R} \to \mathbb{C}$ is called a Schwarz function $\equiv f \in C^\infty$ and

↳ infinite continuous derivs

$$\forall n \in \mathbb{N}_0, \forall c \in \mathbb{R}: \quad f^{(n)}(x) \in o(x^c)$$

Intuition: The function (and all of its derivatives) grows slower than polynomials.

Ex: $\ln(x)$, $\sin(x)$, $\cos(x)$ are all clearly Schwarz

$\sinc(x)$ is also Schwarz

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

... further derivatives will grow even slower

Theorem: Let $f: \mathbb{R} \to \mathbb{C}$ be a Schwarz function with Fourier transform $\hat{f}$. Then

$$\forall x \in \mathbb{R}: \quad \sum_{n\in\mathbb{Z}} f(n+x) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k) \exp(2\pi i k x)$$

we are reordering the series here
⇒ we should show that it is absolutely convergent

Proof: Let $F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$

$$F(x+1) = \sum_{n\in\mathbb{Z}} f(x+(n+1)) = \sum_{n\in\mathbb{Z}} f(x+n) = F(x)$$

we should check if F satisfies the necessary conditions

$\Rightarrow F$ is 1-periodic ⇒ lets find its Fourier series

$$F(x) = \sum_{k\in\mathbb{Z}} c_k \exp(2\pi i k x)$$

$$\rightarrow c_k = \int_0^1 F(x) \exp(-2\pi i k x)\,dx = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n) \exp(-2\pi i k x)\,dx$$

→ can swap $\sum$ and $\int$ because f is Schwarz

$$= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) \exp(-2\pi i k x)\,dx = \left| \begin{array}{ll} u = x+n, & 1 \to n+1 \\ du = dx; & 0 \to n \end{array} \right.$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(u) \exp(-2\pi i k(u-n))\,du = \left| \text{note } k, n \in \mathbb{Z} = \exp(2\pi i k n) = 1 \right.$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(u) \exp(-2\pi i k u)\,du = \int_{-\infty}^{\infty} f(u) \exp(-2\pi i k u)\,du$$

$$= \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i(2\pi k)u}\,du = \sqrt{2\pi}\, \hat{f}(2\pi k)$$

$$\Rightarrow F(x) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k) \exp(2\pi i k x)$$

Note: By plugging in $x=0$ we get $\sum_{n\in\mathbb{Z}} f(n) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k)$

$$\sum_{n=-\infty}^{\infty} \operatorname{sinc}(n+x) = \pi \quad \text{for } \forall x \in \mathbb{R}$$

→ using the Poisson summation formula we have

$$\sum_{n\in\mathbb{Z}} \operatorname{sinc}(n+x) = \sqrt{2\pi}\sum_{k\in\mathbb{Z}} \mathcal{F}(\operatorname{sinc})(2\pi k)\cdot \exp(2\pi i k x) \quad \dots \quad \mathcal{F}(\operatorname{sinc}) = \sqrt{\tfrac{\pi}{2}}\,\operatorname{rect}$$

$\operatorname{rect}(2\pi k) = 1 \iff |2\pi k| < 1$

$|k| < \frac{1}{2\pi}$

$$= \sqrt{2\pi}\sum_{k\in\mathbb{Z}} \sqrt{\tfrac{\pi}{2}}\,\operatorname{rect}(2\pi k)\exp(2\pi i k x) =$$

$$= \sqrt{2\pi}\cdot\sqrt{\tfrac{\pi}{2}}\cdot 1\cdot \exp(0) = \pi$$

→ it can be similarly shown that $\sum_{n\in\mathbb{Z}} \operatorname{sinc}(n\pi + x) = 1$ for $\forall x \in \mathbb{R}$

Exercises

① Find the F.T. of $g(x) = \begin{cases} 0, & x\in(-\infty,-1) \\ 2, & x\in(-1,0) \\ 2-e^{-x}, & x\in(0,1) \\ -e^{-x}, & x\in(1,\infty) \end{cases}$

↳ g can be written as

$$g(x) = 2\cdot\operatorname{rect}(x) - f(x), \quad \text{where } f(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x>0 \end{cases}$$

→ earlier (Ex. 1.) we have shown that $\hat{f}(k) = \frac{1}{\sqrt{2\pi}(1+ik)}$

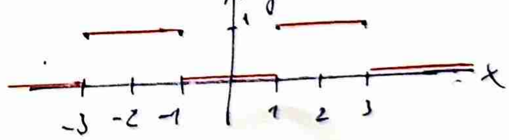
$$\Rightarrow \hat{g}(k) = 2\,\mathcal{F}(\operatorname{rect})(k) - \hat{f}(k) = 2\cdot\sqrt{\tfrac{2}{\pi}}\operatorname{sinc}(k) - \frac{1}{\sqrt{2\pi}(1+ik)}$$

② Find the F.T. of $g(x) = \begin{cases} 0, & x<0 \\ e^{-bx}, & x>0, \end{cases} \quad b\in\mathbb{R}^+$

$$g(x) = f(bx) \Rightarrow \hat{g}(k) = \frac{1}{|b|}\hat{f}\left(\frac{k}{b}\right) = \frac{1}{b}\hat{f}\left(\frac{k}{b}\right) = \frac{1}{b}\frac{1}{\sqrt{2\pi}(1+ik/b)} = \frac{1}{\sqrt{2\pi}(b+ik)}$$

Note: We need $b>0$, otherwise $\lim_{x\to\infty} g(x) = +\infty$ and the integral would diverge.

③ Find the F.T. of $f(x) = \begin{cases} 1, & 1<|x|<3 \\ 0, & |x|<1 \text{ or } |x|>3 \end{cases}$



↳ this can be expressed using the tophat function:

$$f(x) = \operatorname{rect}(x+2) + \operatorname{rect}(x-2), \quad \text{recall } f(x-a) \xrightarrow{\mathcal{F}} e^{-ika}\hat{f}(k)$$

$$\mathcal{F}(\operatorname{rect})(k) = \sqrt{\tfrac{2}{\pi}}\operatorname{sinc}(k)$$

$$\Rightarrow \hat{f}(k) = e^{-ik(-2)}\sqrt{\tfrac{2}{\pi}}\operatorname{sinc}(k) + e^{-ik(2)}\sqrt{\tfrac{2}{\pi}}\operatorname{sinc}(k)$$

$$= \sqrt{\tfrac{2}{\pi}}\operatorname{sinc}(k)\left(e^{2ik} + e^{-2ik}\right) = \sqrt{\tfrac{2}{\pi}}\operatorname{sinc}(k)\cdot 2\cos(2k)$$

④ Show $\cos(\alpha x) f(x) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left(\hat{f}(k-\alpha) + \hat{f}(k+\alpha)\right)$

$$\mathcal{F}(\cos(\alpha x) f(x)) = \mathcal{F}\left(\tfrac{1}{2}(e^{i\alpha x} + e^{-i\alpha x}) f(x)\right) = \tfrac{1}{2}\mathcal{F}(e^{i\alpha x} f(x)) + \tfrac{1}{2}\mathcal{F}(e^{-i\alpha x} f(x))$$

recall: $e^{i\alpha x} f(x) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(x-\alpha)$

$$= \tfrac{1}{2}\hat{f}(x-\alpha) + \tfrac{1}{2}\hat{f}(x+\alpha)$$

5) Find the F.T. of $g(x) = x\, e^{-\frac{x^2}{2}}$

→ recall that $e^{-\frac{x^2}{2}}$ is the F.T. of itself

a) use the monomial rule $\quad x^n f(x) \xleftrightarrow{\mathcal{F}} (i)^n \frac{d^n}{dk^n} \hat{f}(k)$

$$\mathcal{F}\left(x e^{-\frac{x^2}{2}}\right)(k) = i \frac{d}{dk}\left(\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)\right) = i \frac{d}{dk}\left(e^{-\frac{k^2}{2}}\right)$$

$$= i(-k)\, e^{-\frac{k^2}{2}} = \underline{\underline{-ik\, e^{-\frac{k^2}{2}}}}$$

b) use the derivative rule $\quad \frac{d^n}{dx^n} f(x) \xleftrightarrow{\mathcal{F}} (ik)^n \hat{f}(k)$

$$\frac{d}{dx} e^{-\frac{x^2}{2}} = -x\, e^{-\frac{x^2}{2}} \Rightarrow g(x) = \frac{d}{dx}\left(-e^{-\frac{x^2}{2}}\right)$$

$$\Rightarrow \mathcal{F}(g)(k) = ik \cdot \mathcal{F}\left(-e^{-\frac{x^2}{2}}\right)(k) = ik\left(-e^{-\frac{k^2}{2}}\right) = \underline{\underline{-ik\, e^{-\frac{k^2}{2}}}}$$

6) Evaluate the integrals

a) $\int_{\mathbb{R}} x^2 \delta(x-3)\, dx = 3^2 = 9$

b) $\int_{\mathbb{R}} \delta(x^2+x)\, dx$ ... we have $\delta(g(x))$ with $g(x) = x^2 + x$

$$\left.\begin{aligned} &g(x) = 0 \iff x = 0 \lor x = -1 \\ &g'(x) = 2x+1 \Rightarrow g'(0) = 1,\ g'(-1) = -1 \end{aligned}\right\} \delta(g(x)) = \frac{\delta(x-0)}{|1|} + \frac{\delta(x+1)}{|-1|}$$

$$\Rightarrow \int_{\mathbb{R}} \delta(x^2+x)\, dx = \int_{\mathbb{R}} (\delta(x) + \delta(x+1)) \cdot 1\, dx = 1(0) + 1(-1) = 1 + 1 = \underline{\underline{2}}$$

↑ function $1 : x \mapsto 1$

c) $\int_0^2 e^x \delta'(x-1)\, dx$

	D	I
+	e^x	$\delta'(x-1)$
−	e^x	$\delta(x-1)$

$$= \left[e^x \delta(x-1)\right]_0^2 - \int_0^2 e^x \delta(x-1)\, dx = 0 - 0 - e^1 = \underline{\underline{-e}}$$

d) $\int_0^\infty e^{-ax} \delta(\cos x)\, dx \quad , a \in \mathbb{R}^+$

$g(x) := \cos(x) = 0 \iff x = \frac{\pi}{2} + k\pi \quad , k \in \mathbb{N}_0$ ... interal is over $\mathbb{R}^+$

$g'(x) = -\sin x \Rightarrow g'(x_0) = -\sin\left(\frac{\pi}{2} + k\pi\right) = \pm 1 \Rightarrow |g'(x_0)| = 1$

$$\Rightarrow \int_0^\infty e^{-ax} \sum_{k=0}^\infty \delta\left(x - \frac{\pi}{2} - k\pi\right) dx = \sum_{k=0}^\infty \exp\left(-a\left(\frac{\pi}{2} + k\pi\right)\right)$$

$$= \sum_{k=0}^\infty \exp\left(-\frac{a\pi}{2}\right) \exp(-ak\pi) = e^{-\frac{a\pi}{2}} \sum_{k=0}^\infty \left(e^{-a\pi}\right)^k \quad \ldots a > 0 \Rightarrow e^{-a\pi} < 1$$

$$= e^{-\frac{a\pi}{2}} \frac{1}{1 - e^{-a\pi}} = \frac{1}{\exp\left(\frac{a\pi}{2}\right) - \exp\left(-\frac{a\pi}{2}\right)} = \frac{1}{2\left(\frac{e^{\cdots} - e^{\cdots}}{2}\right)} =$$

$$= \frac{1}{2 \sinh\left(\frac{a\pi}{2}\right)} = \underline{\underline{\frac{1}{2} \operatorname{csch}\left(\frac{a\pi}{2}\right)}}$$

Plancharel's Theorem

Theorem: Suppose f and g are complex-valued functions with Fourier transforms $\hat{f}$ and $\hat{g}$. Then, provided the integral exists,

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \hat{f}(k)\,\overline{\hat{g}(k)}\,dk$$

Proof:

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx = \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)\,e^{ikx}\,dk\right)\overline{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(t)\,e^{itx}\,dt\right)}\,dx$$

$\overline{e^{itx}} = e^{-itx}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(k)\,e^{ikx}\,\overline{\hat{g}(t)}\,e^{-itx}\,dk\,ds\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(k)\,\overline{\hat{g}(s)}\,e^{i(k-s)x}\,dx\,ds\,dk$$

using Fubini's theorem

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)\int_{-\infty}^{\infty}\overline{\hat{g}(s)}\int_{-\infty}^{\infty}e^{i(k-s)x}\,dx\,ds\,dk$$

... recall $\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx$

$$= \int_{-\infty}^{\infty}\hat{f}(k)\int_{-\infty}^{\infty}\overline{\hat{g}(s)}\,\delta(k-s)\,ds\,dk = \int_{-\infty}^{\infty}\hat{f}(k)\,\overline{\hat{g}(k)}\,dk$$

... use the sifting property of δ ▨

Note: This means that $\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|\hat{f}(k)|^2\,dk$

Example: Use Plancharel's theorem to calculate $\int_{-\infty}^{\infty}\frac{1}{x^2+a^2}\,dx$, $a\in\mathbb{R}^+$

→ we need a function g s.t. $g(x)\,\overline{g(x)} = \frac{1}{x^2+a^2} \Rightarrow g(x) = \frac{1}{x+ia}$

→ earlier we have shown that the F.T. of

$$f(x) = \begin{cases} e^{-ax}, & x>0 \\ 0, & x<0 \end{cases} \quad \text{is} \quad \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{a+ik}$$

very similar

$$\Rightarrow \int_{-\infty}^{\infty}|\hat{f}(k)|^2\,dk = \int_{-\infty}^{\infty}\frac{1}{2\pi}\,\frac{1}{a^2+k^2}\,dk$$

$$\Rightarrow \int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{0}^{\infty}e^{-2ax}\,dx = \frac{-1}{2a}e^{-2ax}\Big|_0^{\infty} = \frac{1}{2a}$$

$$\int_{-\infty}^{\infty}\frac{1}{x^2+a^2}\,dx = \frac{\pi}{a}$$

- Convolutions

→ we want to be able to find the F.T. of a product of functions

Def: The convolution of two functions f and g is the function

$$(f*g)(x) := \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

Note: The functions f and g need to tend to 0 as $x \to \pm\infty$ sufficiently rapidly in order for the integral to converge. But this is not an issue since we need that for the F.T. to exist anyway.

Properties

① $f*g = g*f$ ... commutativity

$$(g*f)(x) = \int_{-\infty}^{\infty} g(x-t)f(t)\,dt = \left|\begin{matrix} u = x-t \\ du = -dt \end{matrix}\right| = -\int_{\infty}^{-\infty} g(u)f(x-u)\,du = \int_{-\infty}^{\infty} f(x-u)g(u)\,du = (f*$$

② $(f*g)*h = f*(g*h)$ ... associativity

$$(f*(g*h))(x) = \int_{-\infty}^{\infty} f(x-t)(g*h)(t)\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-t)\,g(t-u)\,h(u)\,du\,dt$$

$$((f*g)*h)(x) = (h*(f*g))(x) = \dots = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-t)f(t-u)g(u)\,du\,dt$$

$$\left.\begin{matrix} v = x-t \\ dv = -dt \end{matrix}\right| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(v)f(x-v-u)g(u)\,du\,dv$$

$$\left.\begin{matrix} w = v+u \\ dw = du \end{matrix}\right| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(v)f(x-w)g(w-v)\,dw\,dv$$

$t = w$
$u = v$

③ $f*(g+h) = (f*g)+(f*h)$ ... $*$ is distributive over $+$

$$(f*(g+h))(x) = \int_{-\infty}^{\infty} f(x-t)(g+h)(t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt + \int_{-\infty}^{\infty} f(x-t)h(t)\,dt$$

④ $(\alpha f)*(\beta g) = \alpha\beta(f*g)$ ... linear with respect to scalar mult.

$$((\alpha f)*(\beta g))(x) = \int_{-\infty}^{\infty} \alpha f(x-t)\beta g(t)\,dt = \alpha\beta\int_{-\infty}^{\infty} f(x-t)g(t)\,dt = \alpha\beta(f*g)$$

⑤ $f(x+\alpha)*g(x+\beta) = (f*g)(x+\alpha+\beta)$ ... shifting property

$$(f(x+\alpha)*g(x+\beta))(x) = \int_{-\infty}^{\infty} f(x+\alpha-t)g(t+\beta)\,dt = \left|\begin{matrix} u = t+\beta \\ du = dt \end{matrix}\right| = \int_{-\infty}^{\infty} f(x+\alpha+\beta-u)g(u)\,du = (f*g)(x+\alpha+\beta)$$

⑥ $f(\alpha x)*g(\alpha x) = \frac{1}{|\alpha|}(f*g)(\alpha x)$

$$(f(\alpha x)*g(\alpha x))(x) = \int_{-\infty}^{\infty} f(\alpha(x-t))g(\alpha t)\,dt = \left|\begin{matrix} u = \alpha t \\ du = \alpha\,dt \end{matrix}\right. \quad \begin{matrix} ,\ \alpha>0:\ \infty\to\infty,\ -\infty\to-\infty \\ ;\ \alpha<0:\ \infty\to-\infty,\ -\infty\to\infty \end{matrix}$$

$$= \frac{1}{|\alpha|}\int_{-\infty}^{\infty} f(\alpha x-u)g(u)\,du = \frac{1}{|\alpha|}(f*g)(\alpha x)$$

⑦ $\underline{f * \delta = \delta * f = f}$ ... δ is the neutral element

$$(f*\delta)(x) = \int_{-\infty}^{\infty} f(x-t)\,\delta(t)\,dt = f(x) \quad \& \quad (\delta * f)(x) = \int_{-\infty}^{\infty} \delta(x-t)\,f(t)\,dt = f(x)$$

The set S of all functions f for which exists a function f^{-1} s.t. $f * f^{-1} = \delta$ forms a field $(S, +, *, 0, \delta)$, where $0: \mathbb{R} \to \mathbb{C}$, $0: x \mapsto 0$.

⑧ $\underline{(f*g)(x) \xleftrightarrow{\mathcal{F}} \sqrt{2\pi}\,\hat{f}(\xi)\,\hat{g}(\xi)}$ ... convolution theorem

$$\mathcal{F}(f*g)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x)\, e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt\, e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} f(x-t)\, e^{-i\xi x}\, dx\, dt = \left| \begin{matrix} u = x-t \\ du = dx \end{matrix} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} f(u)\, e^{-i\xi(u+t)}\, du\, dt$$

$$= \int_{-\infty}^{\infty} g(t)\, e^{-i\xi t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\xi u}\, du\, dt = \int_{-\infty}^{\infty} g(t)\, e^{-i\xi t}\, \hat{f}(\xi)\, dt = \underline{\sqrt{2\pi}\, \hat{f}(\xi)\, \hat{g}(\xi)}$$

⑨ $\underline{f(x)\,g(x) \xleftrightarrow{\mathcal{F}} \frac{1}{\sqrt{2\pi}} (\hat{f} * \hat{g})(\xi)}$ ... convolution theorem

use ⑧: $\mathcal{F}\left(\sqrt{2\pi}\,\hat{f}(\xi)\,\hat{g}(\xi)\right) = (f*g)(-x) \Rightarrow \mathcal{F}\left(\hat{f}(\xi)\,\hat{g}(\xi)\right) = \frac{1}{\sqrt{2\pi}} (f*g)(-x)$

$\to$ we have $\mathcal{F}(\hat{f})(x) = f(-x) =: \hat{\hat{f}}(x)$ and $\mathcal{F}(\hat{g})(x) = g(-x) =: \hat{\hat{g}}(x)$

$$\to (f*g)(-x) = \int_{-\infty}^{\infty} f(-x-t)\, g(t)\, dt = \int_{-\infty}^{\infty} \hat{\hat{f}}(x+t)\, \hat{\hat{g}}(-t)\, dt = \left| \begin{matrix} u = -t & , \infty \to -\infty \\ du = -dt & , -\infty \to \infty \end{matrix} \right.$$

$$= \int_{-\infty}^{\infty} \hat{\hat{f}}(x-u)\, \hat{\hat{g}}(u)\, du = (\hat{\hat{f}} * \hat{\hat{g}})(x)$$

$$\Rightarrow \mathcal{F}\left(\hat{f}(\xi)\,\hat{g}(\xi)\right)(x) = \frac{1}{\sqrt{2\pi}} (\hat{\hat{f}} * \hat{\hat{g}})(x)$$

<u>Note</u>: The constants $\sqrt{2\pi}$ may and may not appear here based on the definition of the F.T.

<u>Lemma</u>: The constant function $1: x \mapsto 1$ is not the neutral element.
That is, in general $f * 1 \neq f$.

<u>Pf</u>: Let's try $f(x) = \operatorname{sinc}(x)$

$$(f*1)(x) = \int_{-\infty}^{\infty} \operatorname{sinc}(x-t)\, 1(t)\, dt = \int_{-\infty}^{\infty} \operatorname{sinc}(x-t)\, dt = \left| \begin{matrix} u = x-t & , \infty \to -\infty \\ du = -dt & , -\infty \to \infty \end{matrix} \right.$$

$$= \int_{-\infty}^{\infty} \operatorname{sinc}(u)\, du = \pi \neq 1.$$

Exercises

① Find $f * g$, where $f(x) = e^{-\frac{x^2}{2}}$ and $g(x) = e^{-3\frac{x^2}{2}}$

$$(f*g)(x) = \int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}(x-t)^2\right) \exp\left(-\tfrac{3}{2}t^2\right) dt = \int_{-\infty}^{\infty} \exp\left(-\frac{4t^2-2xt+x^2}{2}\right) dt$$

$$4t^2-2xt+x^2 = 4\left(t^2-\tfrac{1}{2}xt+\left(\tfrac{1}{4}x\right)^2-\tfrac{x^2}{16}+\tfrac{x^2}{4}\right) = 4\left(t-\tfrac{1}{4}x\right)^2+\tfrac{3}{4}x^2 = \left(2t-\tfrac{1}{2}x\right)^2+\tfrac{3}{4}x^2$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{(2t-\frac{1}{2}x)^2}{2}-\frac{3}{8}x^2\right) dt = \exp\left(-\tfrac{3}{8}x^2\right)\int_{-\infty}^{\infty} \exp\left(-\frac{(2t-\frac{1}{2}x)^2}{2}\right) dt = \left|\begin{matrix} u = 2t-\frac{1}{2}x \\ du = 2dt \end{matrix}\right.$$

$$= \frac{1}{2}\exp\left(-\tfrac{3}{8}x^2\right)\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2}\right) du = \left|\begin{matrix} v = \frac{u}{\sqrt{2}} \\ dv = \frac{1}{\sqrt{2}}du \end{matrix}\right| = \frac{\sqrt{2}}{2}\exp\left(-\tfrac{3}{8}x^2\right)\int_{-\infty}^{\infty} e^{-v^2} dv = \underline{\underline{\sqrt{\tfrac{\pi}{2}}\exp\left(-\tfrac{3}{8}x^2\right)}}$$

② Use the convolution theorem to solve ①

$$\mathcal{F}^{-1}\left(\sqrt{2\pi}\,\hat{f}\,\hat{g}\right)(x) = (f*g)(x) \quad \to \text{ need to find } \hat{f}, \hat{g}$$

- $\hat{f} = f$ ... gauss
- $\hat{g}(k) = \mathcal{F}\left(e^{-3\frac{x^2}{2}}\right)(k) = \mathcal{F}\left(f(\sqrt{3}x)\right)(k) = \frac{1}{\sqrt{3}}\hat{f}\left(\frac{k}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\exp\left(-\frac{k^2}{6}\right)$

$$\Rightarrow (f*g)(x) = \mathcal{F}^{-1}\left[\sqrt{2\pi}\exp\left(-\tfrac{k^2}{2}\right)\tfrac{1}{\sqrt{3}}\exp\left(-\tfrac{k^2}{6}\right)\right](x) = \mathcal{F}^{-1}\left[\sqrt{\tfrac{2\pi}{3}}\exp\left(-\tfrac{2}{3}k^2\right)\right](x)$$

know: $f\left(\frac{x}{\alpha}\right) \xleftrightarrow{\mathcal{F}} |\alpha|\hat{f}(\alpha k) \Rightarrow \frac{(\alpha k)^2}{2} = \frac{2k^2}{3} \Rightarrow \alpha^2 = \frac{4}{3} \Rightarrow \alpha = \pm\frac{2}{\sqrt{3}}$

$$= (f*g)(x) = \sqrt{\tfrac{2\pi}{3}}\cdot\tfrac{1}{|\alpha|}\mathcal{F}^{-1}\left[|\alpha|\exp\left(-\tfrac{(\alpha k)^2}{2}\right)\right](x) = \sqrt{\tfrac{2\pi}{3}}\,\tfrac{\sqrt{3}}{2}\exp\left(-\tfrac{(x/\alpha)^2}{2}\right)$$

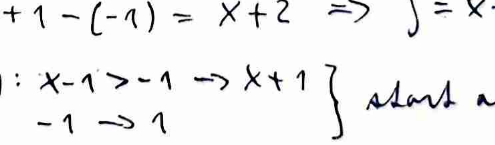
$$= \sqrt{\tfrac{\pi}{2}}\exp\left(-\tfrac{x^2}{2}\cdot\tfrac{3}{4}\right) = \underline{\underline{\sqrt{\tfrac{\pi}{2}}\exp\left(-\tfrac{3}{8}x^2\right)}}$$

③ Find $\chi * \chi$, where χ is the top-hat function

$\chi(x-t) = \chi(t-x)$

$\downarrow$ centered at x

$$(\chi*\chi)(x) = \int_{-\infty}^{\infty}\chi(x-t)\chi(t)dt$$



1, $|x| > 2 \Rightarrow \chi(x-t)\chi(t) = 0$

2, $x \in [-2,0] \Rightarrow$ overlap: $\chi(x-t)$: $x-1 \to x+1 < 1$ start at -1, end at $x+1$

$\chi(t)$: $-1 \to 1$

$\Rightarrow$ length $= x+1-(-1) = x+2 \Rightarrow \int = x+2$

3, $x \in [0,2] \Rightarrow$ overlap: $\chi(x-t)$: $x-1 > -1 \to x+1$ } start at $x-1$, end 1

$\chi(t)$: $-1 \to 1$

$\Rightarrow$ length $= 1-(x-1) = 2-x \Rightarrow \int = 2-x$

$$\Rightarrow (\chi*\chi)(x) = \begin{cases} 0, & |x| > 2 \\ 2-|x|, & |x| \le 2 \end{cases}$$

④ Calculate the F.T. of $\chi * \chi$.

$$\mathcal{F}(\chi*\chi)(k) = \sqrt{2\pi}\,\hat{\chi}(k)\,\hat{\chi}(k) = \sqrt{2\pi}\left(\sqrt{\tfrac{2}{\pi}}\operatorname{sinc}(k)\right)^2 = \underline{2\sqrt{\tfrac{2}{\pi}}\operatorname{sinc}^2(k)}$$

⑤ Evaluate the following integrals

a) $\int_{-\infty}^{\infty} \sin(x)\,\delta(x-5)\,dx = \underline{\underline{\sin(5)}}$

b) $\int_{-\infty}^{\infty} e^{2x}\,\delta(x^2-1)\,dx \quad \rightarrow x^2-1=0 \iff x=\pm 1$

$\frac{d}{dx}(x^2-1) = 2x \Rightarrow g'(-1)=-2\,,\; g'(1)=2 \Rightarrow |g'(x_0)|=2$

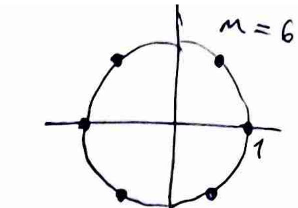
$= \int_{-\infty}^{\infty} e^{2x}\left[\frac{\delta(x-1)}{2} + \frac{\delta(x+1)}{2}\right] dx = \frac{1}{2}e^{2\cdot 1} + \frac{1}{2}e^{2\cdot(-1)} = \frac{e^{2}+e^{-2}}{2} = \underline{\underline{\cosh(2)}}$

c) $\int_{\frac{1}{2}}^{\infty} \frac{1}{x^2}\,\delta(\sin(\pi x))\,dx$

$\rightarrow g(x) := \sin(\pi x) = 0 \iff x \in \mathbb{N} \quad \ldots \int$ is from $\frac{1}{2}$

$g'(x) = \pi\cos(\pi x) \Rightarrow g'(x_0) = \pi(-1)^{x_0} \Rightarrow |g'(x_0)| = \pi$

$\Rightarrow \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{\delta(x-n)}{\pi}\,dx = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\cdot\frac{\pi^2}{6} = \underline{\underline{\frac{\pi}{6}}}$



• The Discrete Fourier Transform

→ complex square roots:

$x = \sqrt[n]{1} \Rightarrow |x| = 1 \Rightarrow x = e^{i\varphi}$ for some φ

$\rightarrow x^n = 1 \Rightarrow e^{in\varphi} = \cos(n\varphi) + i\sin(n\varphi) = 1 \Rightarrow n\varphi = k\cdot 2\pi \Rightarrow \varphi = \frac{2k\pi}{n}, \; k = 0, 1, \dots n-1$

if $|x| = 1$, then $x^{-1} = e^{-i\varphi} = \overline{x}$

Def: $\omega \in \mathbb{C}$ is an n^{th} primitive root of unity $\equiv \omega^n = 1$ & $\omega^1, \omega^2, \dots \omega^{n-1} \neq 1$

↳ $\omega = \exp(\frac{2\pi}{n} i)$, $\overline{\omega} = \exp(-\frac{2\pi}{n} i)$ are both primitive roots of unity.

The problem: Suppose we know the value of the function f at N regularly spaced points:

$$x_k = \frac{2\pi k}{N} \rightsquigarrow f_k = f(x_k) \qquad \forall k = 0, 1, \dots n-1$$

There are two tasks we might want to do

1, evaluate the polynomial $p_f(x) = \sum_{k=0}^{N-1} f_k x^k$ at N roots of unity

↳ this allows $\Theta(n \log n)$ polynomial multiplication using FFT and unlocks $\Theta(n \log n)$ scalar multiplication algorithms

2, approximate the (unknown) function f as

$$\hat{f}(x) = \sum_{n=0}^{N-1} c_n \exp(inx) \quad \text{s.t.} \quad \forall k: \hat{f}(x_k) = f_k$$

These two tasks are in fact the same task.

Def: The discrete Fourier transform (DFT) is a linear map $\mathcal{F}: \mathbb{C}^N \to \mathbb{C}^N$ s.t.

$$\mathcal{F}: (f_0, \dots f_{N-1}) \mapsto (c_0, \dots c_{N-1}) \equiv \forall n: c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k \, \omega^{-nk}, \quad \omega = e^{\frac{2\pi}{N} i}$$

Note: The DFT evaluates the polynomial defined by $f_0, \dots f_{N-1}$ as $p_f(x) = \sum_{k=0}^{N-1} f_k x^k$ in N distinct points $c_n = \frac{1}{n} p_f(\omega^{-n}) \rightarrow 1, \omega^{-1}, \omega^{-2}, \dots \omega^{-N+1}$

↳ since ω is a primitive root of unity, these points lie on 

$\mathcal{F}$ is a linear map $\Rightarrow \mathcal{F}(\vec{f} + \vec{g}) = \mathcal{F}(\vec{f}) + \mathcal{F}(\vec{g})$ & $\mathcal{F}(\alpha \cdot \vec{f}) = \alpha \cdot \mathcal{F}(\vec{f})$

$\mathcal{F}$ is a l.m. $\Rightarrow \vec{c} = \mathcal{F}(\vec{f}) \Leftrightarrow \vec{c} = \Omega \vec{f}$ where $\Omega \in \mathbb{C}^{N \times N}$, $\Omega_{nk} = \frac{1}{N} \omega^{-nk}$

↳ index from 0

To get $\vec{f}$ back from $\vec{c}$, use $\vec{f} = \Omega^{-1} \vec{c}$

↑ inverse DFT $\mathcal{F}^{-1}$

Lemma. $\Omega^{-1} = N\overline{\Omega}$

Pf: $(\Omega\overline{\Omega})_{mj} = \sum_k \Omega_{mk}\overline{\Omega}_{kj} = \sum_k \frac{1}{N}\omega^{-mk}\,\overline{\frac{1}{N}\omega^{-kj}} = \frac{1}{N^2}\sum_k \omega^{-mk}\omega^{kj} = \frac{1}{N^2}\sum_{k=0}^{N-1}\omega^{k(j-m)}$

1) $m=j$: $\omega^{k(j-m)} = 1 \Rightarrow (\Omega\overline{\Omega})_{mj} = \frac{1}{N^2}\cdot N = \frac{1}{N}$

2) $m\neq j$: $\omega^{k(j-m)} \neq 1 \Rightarrow (\Omega\overline{\Omega})_{mj} = \frac{1}{N^2}\cdot\frac{(\omega^{j-m})^N-1}{\omega^{j-m}-1} = \frac{1}{N^2}\cdot\frac{0}{\neq 0} = 0$ because $\omega^N = 1$

$\Rightarrow \Omega\cdot\overline{\Omega} = \frac{1}{N}I_n \Rightarrow \Omega^{-1} = N\overline{\Omega}$

Corollary: To calculate the inverse DFT $\mathcal{F}^{-1}(c_0,\dots c_{N-1}) = (f_0,\dots f_{N-1})$, calculate the standard DFT with the root of unity $\omega' = \overline{\omega} = e^{-\frac{2\pi}{N}i}$ and multiply the result by N

$$\vec{f} = \Omega^{-1}\vec{c} = N\cdot\overline{\Omega}\vec{c}$$

Note: We are only using the fact that ω is an N^{th} primitive root of unity in $\mathbb{C}$.
- → if our goal isn't to use the DFT to approximate f as $\hat{f}$, but we only care about evaluating $f_f(x)$ at N distinct points, then the field $\mathbb{C}$ is not important
- ⇒ if $(f_0,\dots f_{N-1}) \in \mathbb{Z}$, then we can find a suitable field $\mathbb{Z}_p$ s.t. ω is an n^{th} root of unity in $\mathbb{Z}_p$ and do calculations there
  - ↳ this is what is used in practice
- → we need to be careful to choose a sufficiently large p so that the result doesn't overflow when multiplying numbers using FFT

Theorem: We know $f_0,\dots f_{N-1}$ and want a function

$$\hat{f}(x) = \sum_{m=0}^{N-1} c_m \exp(imx) \quad \text{s.t.} \quad \forall k = 0,\dots N-1: \hat{f}\left(\frac{2\pi k}{N}\right) = f_k$$

The coefficients are $\vec{c} = \mathcal{F}(\vec{f})$ i.e. $c_m = \frac{1}{N}\sum_{k=0}^{N-1} f_k\omega^{mk}$, $\omega = e^{\frac{2\pi}{N}i}$

Proof:

$$\hat{f}\left(\frac{2\pi k}{N}\right) = \sum_{m=0}^{N-1} c_m \exp\left(im\frac{2\pi k}{N}\right) = \sum_{m=0}^{N-1} c_m\omega^{mk} = \sum_{m=0}^{N-1}\left(\frac{1}{N}\sum_{j=0}^{N-1} f_j\omega^{-mj}\right)\omega^{mk}$$

$$= \frac{1}{N}\sum_{j=0}^{N-1} f_j\sum_{m=0}^{N-1}\omega^{m(k-j)} = \frac{1}{N}\cdot f_k N = f_k$$

1) $k=j$: $\omega^{m(k-j)} = 1 \Rightarrow \sum \omega = N$

2) $k\neq j$: $\omega^{m(k-j)} \neq 0 \Rightarrow \sum \omega = 0$ as proven at the top of the page ∎

Note: We have also just shown that the set of vectors $\{(1,\omega^k,\omega^{2k},\dots\omega^{(N-1)k}) \mid k\in\{0,\dots N-1\}\}$ is an orthogonal basis of $\mathbb{C}^N$ with respect to the i.p. $\langle u|v\rangle = v^H u$.

• Parcevals theorem for the DFT

Theorem: If $\vec{X} = \mathcal{F}(\vec{x})$ and $\vec{Y} = \mathcal{F}(\vec{y})$, then $\langle \vec{x} | \vec{y} \rangle_{\mathbb{C}} = N \cdot \langle \vec{X} | \vec{Y} \rangle_{\mathbb{C}}$

$y^H x = N \cdot Y^H X$

Proof: Denote $\omega = e^{\frac{2\pi}{N} i}$

$$Y^H X = \sum_{m=0}^{N-1} X_m \overline{Y_m} := \sum_{m=0}^{N-1} \overline{Y_m} \cdot \left(\frac{1}{N} \sum_{k=0}^{N-1} x_k \omega^{-mk} \right) =$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x_k \underbrace{\sum_{m=0}^{N-1} \overline{Y_m} \cdot \omega^{-mk}}_{*} = \frac{1}{N} \sum_{k=0}^{N-1} x_k \overline{y_k} = \frac{1}{N} y^H x$$

$* = \overline{\sum_{m=0}^{N-1} Y_m \cdot \overline{\omega}^{mk}}$ → inverse DFT ↗ multiplying by N ↘ using $\overline{\omega}$ $\Rightarrow * = \overline{y_k}$ ∎

Corollary: If $\vec{c} = \mathcal{F}(\vec{f})$, then $\sum |f_k|^2 = N \cdot \sum |c_k|^2$.

↳ we can use this as a check if we have obtained the correct DFT

• The Fast Fourier Transform Algorithm

→ calculating DFT: $\vec{c} = \Omega \cdot \vec{f}$ takes $O(N^2)$ operations

→ FFT uses only $O(N \log N)$ operations

👁 WLOG $N = 2^k$, else fill $\vec{f}$ with zeros

↳ notice that the first $|\vec{f}|$ elements of $\vec{c}$ will not change

Input: $N = 2^k$, $(f_0, \dots f_{N-1}) \in \mathbb{C}^N$, $\omega = n^{th}$ primitive root of unity

Output: $(y_0, \dots y_{N-1}) \in \mathbb{C}^N$ ... evaluation of $p_f(x)$ at $(1, \omega, \omega^2, \dots \omega^{N-1})$

$$p_f(x) = f_0 x^0 + f_1 x^1 + f_2 x^2 + \dots + f_{N-1} x^{N-1}$$
$$= \underbrace{(f_0 x^0 + f_2 x^2 + \dots + f_{N-2} x^{N-2})}_{E(x^2)} + \underbrace{(f_1 x^1 + f_3 x^3 + \dots + f_{N-1} x^{N-1})}_{x \cdot O(x^2)}$$

👁 $p(x) = E(x^2) + x\, O(x^2)$
$p(-x) = E(x^2) - x\, O(x^2)$ → we want to use this

👁 $\omega^{N/2} = -1 \Rightarrow \omega^{N/2+k} = -\omega^k$

	0	1	⋯	N/2−1	N/2	N/2+1	⋯	N−1
ω	1	ω		$\omega^{N/2-1}$	-1	$-\omega$		$-\omega^{N/2-1}$

→ remember that we are evaluating at $(1, \omega, \omega^2, \dots \omega^{N-1})$ ⇒ first half = − second half

→ we will recursively use FFT to evaluate $E(y)$ and $O(y)$ at $y = (1, \omega^2, \omega^4, \dots \omega^{N-2})$ ⇝ $E_0, \dots E_{\frac{N}{2}-1}$, $O_0, \dots O_{\frac{N}{2}-1}$

→ note: ω^2 is an $(\frac{N}{2})^{th}$ prim root of 1

→ $m = 0, \dots \frac{N}{2} - 1$: $y_m = p(\omega^m) = E_m + \omega^m O_m$
$y_{\frac{N}{2}+m} = p(-\omega^m) = E_m - \omega^m O_m$

Algorithm FFT:

Input: $N = 2^\ell$, ω, $(f_0, \dots f_{N-1})$

Output: $(y_0, \dots y_{N-1})$ s.t. $y_k = p_f(\omega^k)$ ... $p_f(x) = \sum_{k=0}^{N-1} f_k x^k$

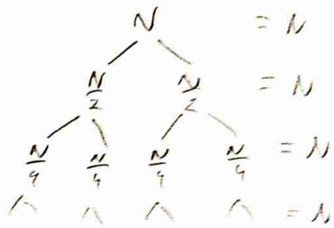
0. If $N = 1$: $y_0 \leftarrow f_0$ and we are done
1. $(E_0, \dots E_{N/2-1}) \leftarrow FFT(N/2, \omega^2, (f_0, f_2, \dots f_{N-2}))$
2. $(O_0, \dots O_{N/2-1}) \leftarrow FFT(N/2, \omega^2, (f_1, f_3, \dots f_{N-1}))$

} 2x recursion of size $\frac{N}{2}$

3. For $m = 0, \dots N/2 - 1$:

$$y_m \longleftarrow E_m + \omega^m O_m$$
$$y_{\frac{N}{2}+m} \longleftarrow E_m - \omega^m O_m$$

} $\Theta(N)$



$\Rightarrow$ total $= \underbrace{\text{height}}_{\log_2 N} \cdot N$

$\Theta(N \log N)$

Calculating DFT using FFT:

$\mathcal{F}: (f_0, \dots f_{N-1}) \mapsto (c_0, \dots c_{N-1})$

$$c_m = \frac{1}{N} \sum_{k=0}^{N-1} f_k \omega^{-mk}, \quad \omega = e^{\frac{2\pi}{N} i}$$

$\rightarrow$ calculate $\vec{y} = FFT(N, e^{\frac{-2\pi}{N} i}, \vec{f}) \Rightarrow \vec{c} = \frac{1}{N} \vec{y}$

Multiplying polynomials

Def: A polynomial of degree $N-1$ is $P(x) = \sum_{k=0}^{N-1} p_k x^k$, $p \in \mathbb{C}^N$, $p_{N-1} \neq 0$.

Def: A graph of the polynomial P of degree $N-1$ at $(x_0, \dots x_{N-1})$ is $(P(x_0), \dots P(x_{N-1}))$

if all x_i are distinct, then Graph(P) uniquely identifies P.

Polynomial multiplication

$\rightarrow$ we have polynomials P and Q and want to find the coefficients of $R = P \cdot Q$

$\forall x: R(x) = P(x) Q(x) \Rightarrow$ we can find the graph of R using Graph(P) and Graph(Q)

$\hookrightarrow$ degree(R) = degree(P) + degree(Q) $\Rightarrow$ use graph with at least deg(R) points to uniquely identify R

$$\begin{array}{l} P \xrightarrow{FFT} \text{Graph}(P) \searrow \\ \qquad\qquad\qquad\qquad\quad \text{Graph}(R) \xrightarrow{\text{Inverse FFT}} R \\ Q \xrightarrow{FFT} \text{Graph}(Q) \nearrow \end{array}$$

Note: Because $\vec{y} = \Omega \vec{x} \Leftrightarrow \vec{x} = N \cdot \overline{\Omega} \vec{y}$ we have

$$\vec{y} = FFT(N, \omega, \vec{x}) \Leftrightarrow \vec{x} = N \cdot FFT(N, \overline{\omega}, \vec{y})$$

Note: As mentioned before, ω doesn't have to be complex and it might be better to chose a finite field $\mathbb{Z}_p$. But be careful, p must be larger then the absolute value of the max. coefficient of R $\rightarrow$ based on P, Q we can get an upper bound.

Example: Given $N=4$ and $(f_0, f_1, f_2, f_3) = (0, 1, 4, 9)$

find the interpolated function $\hat{f}: \mathbb{C} \to \mathbb{C}$, $\hat{f}\left(\frac{2\pi}{4} k\right) = f_k$ for $\forall k \in \{0,1,2,3\}$

$$\hat{f}(x) = \sum_{n=0}^{3} c_n \exp(inx) \quad \to \text{ we need to find } \vec{c} \in \mathbb{C}^4$$

a) using the formula for c_n

$$c_n = \frac{1}{4} \sum_{k=0}^{3} f_k \omega^{nk}, \quad \omega = e^{-\frac{2\pi}{4} i} = e^{-\frac{\pi}{2} i} = -i$$

$$\omega^2 = -1, \quad \omega^3 = -\omega = i, \quad \omega^4 = 1$$

$$c_0 = \frac{1}{4} \sum_{k=0}^{3} f_k \omega^0 = \frac{1}{4}(0+1+4+9) = \frac{7}{2}$$

$$c_1 = \frac{1}{4} \sum_k f_k \omega^k = \frac{1}{4}(0+\omega+4\omega^2+9\omega^3) = \frac{1}{4}(\omega - 4 - 9\omega) = \frac{1}{4}(-4-8\omega) = -1-2\omega = -1+2i$$

$$c_2 = \frac{1}{4}(0+\omega^2+4\omega^4+9\omega^6) = \frac{1}{4}(-1+4+\overbrace{9\omega^2}^{-9}) = \frac{1}{4}(-6) = -\frac{3}{2}$$

$$c_3 = \frac{1}{4}(\omega^3+4\omega^6+9\omega^9) = \frac{1}{4}(i+4(-1)+9(-i)) = \frac{1}{4}(-4-8i) = -1-2i$$

$$\vec{c} = \begin{bmatrix} 7/2 \\ -1+2i \\ -3/2 \\ -1-2i \end{bmatrix}$$

b) using the Fourier matrix $\vec{c} = \Omega \vec{f}$

$$\Omega_{nk} = \frac{1}{N} \omega^{nk}, \quad \omega = e^{-\frac{2\pi}{4} i}$$

↳ index from 0

$$\Omega = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$\Rightarrow \vec{c} = \Omega \vec{f} = \Omega \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \frac{1}{4}\left(\begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} + 4 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + 9 \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \right) = \frac{1}{4} \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix} = \begin{bmatrix} 7/2 \\ -1+2i \\ -3/2 \\ -1-2i \end{bmatrix}$$

c) using FFT → divide to odd and even part and combine

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & -1 & -\omega \\ 1 & -1 & 1 & -1 \\ 1 & -\omega & -1 & \omega \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix}$$

E: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

O: $\begin{bmatrix} 1 & 1 \\ \omega & -\omega \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & \\ & \omega \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ -8\omega \end{bmatrix} = \begin{bmatrix} 10 \\ 8i \end{bmatrix}$

Top half: [ ] + [ ]

Bot half: [ ] − [ ]

$$\Rightarrow \frac{1}{4} \begin{bmatrix} 4+10 \\ -4+8i \\ 4-10 \\ -4-8i \end{bmatrix} = \begin{bmatrix} 7/2 \\ -1+2i \\ -3/2 \\ -1-2i \end{bmatrix}$$

d) reality check using Parcevals theorem : $x^H x = N \cdot X^H X$

$$\hookrightarrow \sum_k |f_k|^2 = N \cdot \sum_k |c_k|^2$$

- $\sum_k |f_k|^2 = 0+1+16+81 = 98$
- $4 \sum_k |c_k|^2 = 4\left(\frac{49}{4} + (1+4) + \frac{9}{4} + (1+4)\right) = 49+20+9+20 = 98$ ✓

Example: Given $N=4$ and $(c_0, c_1, c_2, c_3) = \left(\frac{7}{2}, -1+2i, -\frac{3}{2}, -1-2i\right)$

find (f_0, f_1, f_2, f_3) where $f_k = \hat{f}\left(\frac{2\pi}{4} k\right)$

→ inverse DFT

a) using the formula for $\hat{f}$ $\qquad \hat{f}(x) = \sum_{n=0}^{3} c_n \exp(inx)$

↳ $f_k = \hat{f}\left(\frac{2\pi}{4}k\right) = \sum_{n=0}^{3} c_n \exp\left(\frac{2\pi k n}{4} i\right) = \sum_{n=0}^{3} c_n \omega^{kn}, \quad \omega = e^{\frac{2\pi}{4} i}$ $\Rightarrow$ $\omega = i$, $\omega^2 = -1$, $\omega^3 = -i$, $\omega^4 = 1$

$$f_0 = \sum c_n \omega^0 = \frac{7}{2} - 1 + 2i - \frac{3}{2} - 1 - 2i = 0$$

$$f_1 = \sum c_n \omega^n = \frac{7}{2} + (-1+2i)i - \frac{3}{2}(-1) + (-1-2i)(-i) = 5 - i - 2 + i - 2 = 1$$

$$f_2 = \sum c_n \omega^{2n} = \frac{7}{2} + (-1+2i)(-1) - \frac{3}{2}(1) - (1+2i)(-1) = 2 + 1 - 2i + 1 + 2i = 4$$

$$f_3 = \sum c_n \omega^{3n} = \frac{7}{2} + (-1+2i)(-i) - \frac{3}{2}(-1) - (1+2i)(i) = 5 + i + 2 - i + 2 = 9$$

b) using the inverse Fourier matrix $\qquad \Omega^{-1} = N\bar{\Omega} \Rightarrow \Omega^{-1}_{nk} = \omega^{nk}, \quad \omega = e^{\frac{2\pi}{4} i}$

$$\vec{f} = \Omega^{-1}\vec{c} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 7/2 \\ -1+2i \\ -3/2 \\ -1-2i \end{bmatrix} = \underbrace{\cdots}_{\text{ugly}} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix}$$

c) using FFT

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 7/2 \\ -1+2i \\ -3/2 \\ -1-2i \end{bmatrix}$$

$E \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$O \rightarrow \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} -1+2i \\ -1-2i \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

not ugly

$$\Rightarrow \quad \begin{matrix} + \\ - \end{matrix} \quad = \begin{bmatrix} 2-2 \\ 5-4 \\ 2+2 \\ 5+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix}$$

- Solving ODEs using Power Series

Def: A power series about the point x_0 is a series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad , \quad a_i \in \mathbb{R}$$

Def: The radius of convergence of a power series is $R \in \mathbb{R}^+ \cup \{\infty\}$ s.t.

$|x-x_0| < R \Rightarrow$ the power series converges at x — we say that x is in the radius of convergence
$|x-x_0| > R \Rightarrow$ the power series diverges at x

Fact: Power series converge absolutely within the radius of convergence
$\Rightarrow$ they can be added, multiplied, differentiated and integrated term by term and it will hold inside of the radius

→ we can find the radius of convergence using $\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

Def: The function $f: \mathbb{R} \to \mathbb{R}$ is analytic at the point $x_0 \equiv \exists F$, power series about x_0 with radius of convergence $R > 0$ s.t. $|x-x_0| < R \Rightarrow F(x) = f(x)$.

Convergent power series about x_0 form a vector space with basis $\{(x-x_0)^n \mid n \in \mathbb{N}_0\}$

Polynomials are a special type of power series. Since polynomials converge everywhere, they are analytic and have ∞ radius of convergence.
↳ degree of a polynomial $= d \equiv a_d \neq 0$ & $n > d \Rightarrow a_n = 0$.

Example: Find a power series solution of the ODE $(4-x^2)y'' + 6y = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$
↳ $n=0,1 \Rightarrow \sum \overset{0}{\cdots}$

$$\Rightarrow (4-x^2) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

coefficients:

$\forall n \geq 0$: x^n: $4(n+2)(n+1)a_{n+2} - \overbrace{n(n-1)a_n}^{0 \text{ if } n \in \{0,1\}} + 6a_n = 0$

$$\Rightarrow a_{n+2} = a_n \frac{n(n-1)-6}{4(n+2)(n+1)} = a_n \frac{n^2-n-6}{4(n+2)(n+1)} = a_n \frac{(n-3)(n+2)}{4(n+2)(n+1)} = a_n \frac{n-3}{4(n+1)}$$

→ we can use generating functions to find an exact formula for a_n using a_0 and a_1
↳ and from this perhaps an exact formula for y

→ but we can easily get a particular solution of the ODE:

$a_5 = a_3 \cdot \frac{3-3}{4\cdot(4)} = 0 \Rightarrow a_5, a_7, a_9, \ldots = 0$ → $a_3 = a_1 \frac{1-3}{4\cdot 2} = -\frac{1}{4}a_1$

if we let $a_0 = 0$, we have $a_2, a_4, \ldots = 0$

⇒ a non-zero polynomial solution is given by

$$y(x) = a_1 x + a_3 x^3 = a_1 x - \frac{1}{4}a_1 x^3 = \underline{\underline{a_1 x\left(1 - \frac{x^2}{4}\right)}}$$

→ if this was an IVP and we knew $y(0)$, we could figure out a_0 and $y'(0)$ would give us a_1

⇒ consider $y(0) = 1$ & $y'(0) = 0$

$y(0) = 1 = \sum_{n=0}^{\infty} a_n x^n \Rightarrow a_0 = 1$

$y'(0) = 0 = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow a_1 = 0 \Rightarrow$ all odd terms $= 0$

→ $a_2 = -\frac{3}{4}a_0 = -\frac{3}{4} \Rightarrow a_4 = \frac{2-3}{4(2+1)}a_2 = -\frac{1}{12}\cdot\left(-\frac{3}{4}\right) = \frac{1}{16}$ ...

⇒ $y(x) = 1 - \frac{3}{4}x^2 + \frac{1}{16}x^4 + \ldots$

• Ordinary and singular points

→ the forms of a powerseries solution depends on the type of point the expansion is about

Def: Consider the linear 2nd order ODE in standard form

$$y'' + py' + qy = g \quad , \quad p, q, g : I \subseteq \mathbb{R} \to \mathbb{R}$$

The point $x_0 \in I$ is an

i) ordinary point ≡ p, q, r are analytic at x_0

ii) singular point ≡ otherwise

If this is a homogeneous eq. $r(x) = 0$, then x_0 is a

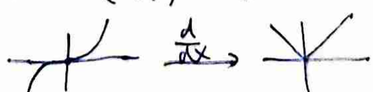
iii) regular singular point ≡ x_0 is a singular point, but $(x-x_0)p$ and $(x-x_0)^2 q$ are analytic at x_0

iv) essential singularity ≡ otherwise

If $f: \mathbb{R} \to \mathbb{R}$ is analytic at x_0, then it is infinitely differentiable at x_0
↳ it can be expressed as a powerseries

⇒ if f is not differentiable at x_0, it is not analytical there

Ex: $|x|$ at 0, $\frac{1}{x}$ at 0, $\frac{x}{(x-2)^2}$ at 2, $f: x \mapsto \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ at 0



no derivative at $x = 0$

Series Solutions about an Ordinary Point

$y'' + py' + q = r$, x_0 is an ordinary point

1) Substitute $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, y' and y'' into the equation
2) If p, q or r are something like $\frac{x^2}{1-x}$, then multiply it out so there are no fractions, it will make everything easier
3) Expand the resulting coefficients to be powerseries about x_0 (Taylor series)
4) Group the terms by powers of $(x-x_0)$ and find a formula for the coefficients
   ⇒ there will be two undetermined coefficients and the rest expressed recursively
   ⇒ we can use generating functions to find an expression for them
   ⇒ the two unknown coefficients can be found if given $y(x_0)$ and $y'(x_0)$
5) The radius of convergence of the resulting series is the largest radius which avoids any singular points → $(x_0-R, x_0+R) \not\ni$ sing. point

Series Solutions about a Regular Singular Point

Theorem (Fuch): Consider the second order linear ODE

$$(x-x_0)^2 y'' + (x-x_0) p y' + q y = 0, \qquad p, q \text{ analytic at } x_0$$

Divide by x^2 to obtain

$$y'' + \frac{p}{x-x_0} y' + \frac{q}{(x-x_0)^2} y = 0 \qquad \Rightarrow x_0 \text{ is a regular singular point}$$

There exists a solution of the form

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n, \qquad r \in \mathbb{C}, \quad a_0 \neq 0$$

Def: An Euler equation is an equation which can be written in the form

$$ax^2 y'' + bxy' + cy = 0, \qquad a, b, c \in \mathbb{R}, \quad a \neq 0$$

Method: We will focus on finding solutions defined on the interval $(0, \infty)$
→ they will have the form x^r, which is defined for $x \in (0, \infty)$

guess: $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$\Rightarrow ax^2 r(r-1) x^{r-2} + bxrx^{r-1} + cx^r = x^r(ar(r-1) + br + c) = 0$$

$$\Leftrightarrow p(r) := ar(r-1) + br + c = 0$$

→ The polynomial $p(r)$ is called the indicial polynomial

$\Rightarrow x^r$ is a solution $\Leftrightarrow$ $ar(r-1)+br+c=0$

Theorem: Suppose the roots of the indicial equation $ar(r-1)+br+c=0$ are r_1 and r_2. Then the general solution of the Euler equation

$$\circledast \quad ax^2y''+bxy'+cy=0$$

on $(0,\infty)$ is

i) $y=C_1x^{r_1}+C_2x^{r_2}$ ... $r_1 \neq r_2$ distinct real roots

ii) $y=(C_1+C_2\ln x)x^r$ ... $r_1=r_2$ repeated root

iii) $y=x^\lambda(C_1\cos(\mu\ln x)+C_2\sin(\mu\ln x))$ ... $r_{1,2}=\lambda\pm i\mu$ complex roots

Proof: Consider $Y(t):=y(e^t)=y(x)$ where $x=e^t$.

$\Rightarrow Y'(t)=y'(x)x'=xy'(x)$

$\Rightarrow Y''(t)=x'y'(x)+x\cdot y''(x)x'=xy'(x)+x^2y''(x)=Y'(t)+x^2y''(x)$

Substitute to $\circledast$:

$$a(Y''-Y')+bY'+cY=0 \quad \Rightarrow aY''+(b-a)Y'+cY=0 \quad \boxtimes$$

This is a 2^{nd} order linear ODE with constant coefficients.

char eq: $ar^2+(b-a)r+c=0 \Leftrightarrow ar(r-1)+br+c=0$ indicial eq.

We solve $\boxtimes$ for $Y(t)=y(e^t)$ $\Rightarrow$ substitute $t=\ln x$ to find $y(x)$

i) $Y(t)=C_1e^{r_1t}+C_2e^{r_2t} \Rightarrow y(x)=C_1e^{r_1\ln x}+C_2e^{r_2\ln x}=C_1x^{r_1}+C_2x^{r_2}$

ii) $Y(t)=(C_1+C_2t)e^{rt} \Rightarrow y(x)=(C_1+C_2\ln x)e^{r\ln x}=(C_1+C_2\ln x)x^r$

iii) $Y(t)=e^{\lambda t}(C_1\cos(\mu t)+C_2\sin(\mu t)) \Rightarrow y(x)=x^\lambda(C_1\cos(\mu\ln x)+C_2\sin(\mu\ln x))$

Examples:

① $x^2y''-xy'-8y=0$ → $y=x^r \Rightarrow r(r-1)-r-8=r^2-2r-8=(r-4)(r+2)=0$

$\Rightarrow r_1=4,\ r_2=-2 \Rightarrow y(x)=C_1x^4+C_2x^{-2}$ $x>0$

② $x^2y''-5xy'+9y=0$ → $y=x^r \Rightarrow r(r-1)-5r+9=r^2-6r+9=(r-3)^2=0$

$\Rightarrow$ repeated root $r=3 \Rightarrow y(x)=(C_1+C_2\ln x)x^3$ $x>0$

③ $x^2y''+3xy'+2y=0$ → $y=x^r \Rightarrow r(r-1)+3r+2=r^2+2r+2=(r+1)^2+1=0$

$\Rightarrow (r+1)^2=-1 \Rightarrow r+1=\pm i \Rightarrow r_{1,2}=-1\pm i$

$Y(t)=e^{-t}(C_1\cos t+C_2\sin t) \Rightarrow y(x)=\frac{1}{x}(C_1\cos(\ln x)+C_2\sin(\ln x))$ $x>0$

- <u>The Frobenius method</u>

$(x-x_0)^2 y'' + (x-x_0) p y' + q y = 0$, $\quad p, q$ analytic at $x_0 \Rightarrow x_0$ regular singular point

$\Rightarrow$ change variables $z := x - x_0 \Rightarrow y(x) = y(z + x_0) \Rightarrow$ WLOG: $x_0 = 0$

1) Fuch's theorem sais that $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ gives a solution

$\Rightarrow$ substitute $y(x)$, $y'(x)$ and $y''(x)$ into the equation

$$0 = x^2 y'' + x p y' + q y = x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + x p \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + q \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} a_n \left[(n+r)(n+r-1) + p(x)(n+r) + q(x) \right] x^{n+r}$$

$$= \left[r(r-1) + r p(x) + q(x) \right] a_0 x^r + \sum_{n=1}^{\infty} a_n \left[(n+r)(n+r-1) + p(x)(n+r) + q(x) \right] x^{n+r}$$

2) Expand $p(x)$ and $q(x)$ as power series about 0 – Taylor expansion

3) Group terms by powers of x. Since the RHS $= 0$, all of the coefficients $= 0$.

- we get a recurrent relation for a_n with a_0 undetermined
  - $\Rightarrow$ the general solution needs two independent solutions
- the lowest power of x after expanding $p(x)$ and $q(x)$ will be x^r

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(0)}{k!} x^k, \quad q(x) = \sum_{k=0}^{\infty} \frac{q^{(k)}(0)}{k!} x^k$$

$\Rightarrow$ coefficient of x^r: $\left[r(r-1) + r p(0) + q(0) \right] a_0 = 0$

$\Rightarrow$ indicial equation $r(r-1) + r p(0) + q(0) = 0$

$\rightarrow \sum_{n=0}^{\infty} a_n x^{n+r}$ is a solution $\Rightarrow r(r-1) + r p(0) + q(0) = 0 \longrightarrow$ which root to chose?

4) The radius of convergence of the series solution is the largest radius that avoids any other singular points

<u>Theorem</u>: Suppose the roots of the <u>indicial equation</u> $r(r-1) + r p(0) + q(0) = 0$ are r_1 and r_2. Then the fundamental set of solutions of

$$x^2 y'' + x p(x) y' + q(x) y = 0$$

on $(0, \infty)$ is given by

i) <u>$r_1 > r_2$ & $r_1 - r_2 \notin \mathbb{N}$</u>

a_n and b_n obey the same recurrence relation but a_0 and b_0 may be chosen arbitrarly

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

$\Rightarrow$ both roots give independent solutions

ii) $r_1 > r_2$ & $r_1 - r_2 \in \mathbb{N}$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \qquad y_2(x) = K y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

→ y_2 will have undetermined b_0 which then determines K and b_n up to but not including $b_{r_1-r_2}$, which can be set arbitrarily. This then determines the rest of b_n

→ note that K can be zero

essentially $K=0$ b_n depends on a_0 and b_0

iii) $r_1 = r_2 =: r$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r}$$

iv) $r_{1,2} = \lambda \pm i\mu$ → a_n and b_n obey the same recurrence relation but a_0 and b_0 may be chosen arbitrarily

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\lambda} \cos(\mu \ln x), \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\lambda} \sin(\mu \ln x)$$

↳ complex: $\sum_{n=0}^{\infty} a_n x^{n+\lambda+i\mu} = \sum_{n=0}^{\infty} a_n x^{n+\lambda} \underbrace{x^{i\mu}}_{= e^{i\mu \ln(x)}} = \sum_{n=0}^{\infty} a_n x^{n+\lambda} (\cos(\mu \ln x) + i \sin(\mu \ln x))$

Ex: $x^2 y'' - x y' + (1-x) y = 0$

Standard form: $y'' - \frac{1}{x} y' + \frac{1-x}{x^2} y = 0$

- $x = 0$ is a regular singular point
- $x \neq 0$ are ordinary points

Frobenius: $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$\begin{aligned}
x^2 y'' - x y' + (1-x) y &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} (1-x) \\
&= \sum_{n=0}^{\infty} a_n x^{n+r} \left((n+r)(n+r-1) - (n+r) + 1 \right) - \sum_{n=0}^{\infty} a_n x^{n+r+1} \\
&= \sum_{n=0}^{\infty} a_n x^{n+r} \left((n+r)(n+r-2) + 1 \right) - \sum_{n=1}^{\infty} a_{n-1} x^{n+r}
\end{aligned}$$

Indicial eq: x^r: $a_n (r(r-2)+1) = 0 \Rightarrow r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$

Recurrence: $n \geq 1$: x^{n+r}: $a_n((n+r)(n+r-2)+1) - a_{n-1} = 0$

→ $r = 1 \Rightarrow a_n(\overbrace{(n+1)(n-1)}^{n^2-1} + 1) = a_{n-1} \Rightarrow a_n = \frac{a_{n-1}}{n^2}$

↳ a_0, $a_1 = \frac{a_0}{1^2}$, $a_2 = \frac{a_0}{1^2 \cdot 2^2}$, $a_3 = \frac{a_0}{1^2 \cdot 2^2 \cdot 3^2}$ $\Rightarrow a_n = \frac{a_0}{(n!)^2}$

$$\Rightarrow y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_0 \frac{1}{(n!)^2} x^{n+1} = \underline{a_0 x \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}}, \quad a_0 \in \mathbb{R}$$

$\rightarrow$ we have repeated root $r=1$ and solution $y_1(x) = a_0 \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2}$

Second solution: $y_2(x) = \ln(x)\, y_1(x) + \sum_{m=0}^{\infty} b_m x^{m+1}$

$$y_2 = \ln x \sum_{m=0}^{\infty} a_m x^{m+1} + \sum_{m=0}^{\infty} b_m x^{m+1}$$

$$y_2' = \ln x \sum_{m=0}^{\infty} (m+1) a_m x^m + \sum_{m=0}^{\infty} a_m x^m + \sum_{m=0}^{\infty} b_m (m+1) x^m$$

$$y_2'' = \ln x \sum_{m=0}^{\infty} (m+1) m\, a_m x^{m-1} + \sum_{m=0}^{\infty} (m+1) a_m x^{m-1} + \sum_{m=0}^{\infty} m\, a_m x^{m-1} + \sum_{m=0}^{\infty} b_m (m+1) m\, x^{m-1}$$

Note: $x^2 y'' - x y' + (1-x) y = 0 \Rightarrow$ get coefficients

x^{m+1}: $(m+1) a_m + m a_m + b_m (m+1) m - \left(a_m + b_m (m+1)\right) + b_m - b_{m-1} = 0$

$$\Rightarrow a_m (m+1+m-1) + b_m \left((m+1)m - (m+1) + 1\right) = b_{m-1} \qquad \frac{a_{m-1}}{m^2}$$

$$\Rightarrow 2m a_m + m^2 b_m = b_{m-1} \Rightarrow b_m = \frac{b_{m-1}}{m^2} - 2\frac{a_m}{m} = \frac{b_{m-1}}{m^2} - 2\frac{a_{m-1}}{m^3}$$

We have derived $b_m = \frac{b_{m-1}}{m^2} - 2\frac{a_{m-1}}{m^3}$

$$b_1 = \frac{b_0}{1^2} - 2\frac{a_0}{1^3} \qquad \rightarrow a_1 = \frac{a_0}{1^2}$$

$$b_2 = \frac{b_0}{1^2 \cdot 2^2} - 2\frac{a_0}{1^3 2^2} - 2\frac{a_0}{1^2 \cdot 2^3} = \frac{b_2}{2!^2} - 2\frac{a_0}{2!^2}\left(\frac{1}{1} + \frac{1}{2}\right) \qquad \rightarrow a_2 = \frac{a_0}{2!^2}$$

$$b_3 = \frac{b_0}{2!^2 3^2} - 2\frac{a_0}{2!^2 3^2}\left(\frac{1}{1} + \frac{1}{2}\right) - 2\frac{a_0}{2!^2 \cdot 3^3} = \frac{b_0}{3!^2} - 2\frac{a_0}{3!^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)$$

$\Rightarrow$ in general $b_m = \frac{b_0}{m!^2} - 2\frac{a_0}{m!^2} H_m$

$$\Rightarrow y_2 = a_0 \ln x \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2} + \sum_{m=0}^{\infty} \left(\frac{b_0}{m!^2} - 2\frac{a_0}{m!^2} H_m\right) x^{m+1}$$

$$= (a_0 \ln x + b_0) \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2} - 2a_0 \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2} H_m$$

y_1 is in fact contained here in $b_0 \sum \frac{x^{m+1}}{m!^2}$

$a_0, b_0 \in \mathbb{R}$

The general solution of $x^2 y'' - x y' + (1-x) y = 0$ is

$\rightarrow b_1 = 0 \Rightarrow b_3 = 0 \Rightarrow \ldots\ b_{ODD} = 0$

$$b_2 = \frac{-4}{2 \cdot 1} b_0 \rightarrow b_4 = \frac{(-4)^2}{4 \cdot 3 \cdot 2 \cdot 1} b_0 \Rightarrow b_{2m} = \frac{(-4)^m}{(2m)!} b_0$$

$\rightarrow$ since $k=0$ we have $y_2(x) = x^{-3} \sum_{m=0}^{\infty} b_m x^m = x^{-3} \sum_{m=0}^{\infty} b_{2m} x^{2m} = \frac{b_0}{x^3} \sum_{m=0}^{\infty} \frac{(-4)^m}{(2m)!} x^{2m}$

The general solution of $x^2 y'' + 6xy' + (4x^2+6) y = 0$ is

$$y(x) = \frac{a_0}{x^2} \sum_{m=0}^{\infty} \frac{(-4)^m}{(2m+1)!} x^{2m} + \frac{b_0}{x^3} \sum_{m=0}^{\infty} \frac{(-4)^m}{(2m)!} x^{2m}, \quad a_0, b_0 \in \mathbb{R}$$

Ex: $x^2y'' + 6xy' + (4x^2+6)y = 0$ *

Standard form: $y'' + \frac{6}{x}y' + \frac{4x^2+6}{x^2}y = 0$

- $x=0$ is a regular singular point
- $x\neq 0$ are ordinary points

Frobenius: $y(x)=\sum_{m=0}^{\infty} a_m x^{m+r} \Rightarrow y'=\sum_{m=0}^{\infty} a_m(m+r)x^{m+r-1} \Rightarrow y''=\sum_{m=0}^{\infty} a_m(m+r)(m+r-1)x^{m+r-2}$

*: $\sum a_m(m+r)(m+r-1)x^{m+r} + 6\sum a_m(m+r)x^{m+r} + (4x^2+6)\sum a_m x^{m+r} = 0$

$$\sum_{m=0}^{\infty} a_m x^{m+r}\left[(m+r)(m+r-1)+6(m+r)+6\right] + 4\sum_{m=0}^{\infty} a_m x^{m+r+2}=0$$

$$\sum_{m=0}^{\infty} a_m x^{m+r}\left[(m+r)(m+r+5)+6\right] + 4\sum_{m=2}^{\infty} a_{m-2}x^{m+r} = 0$$

$m=0$: x^r: $a_0(r(r+5)+6)=0 \Rightarrow r^2+5r+6=(r+2)(r+3)=0 \to r_1=-2,\ r_2=-3$

$m=1$: x^{r+1}: $a_1((r+1)(r+6)+6) = a_1(r^2+7r+12)=a_1(r+3)(r+4)=0 \Rightarrow a_1=0 \vee r=-3 \vee r=-4$

↳ since $r_1=-2>-3=r_2$, we choose -2 as our root $\Rightarrow a_1=0$

$m\geq 2$: x^{m+r}: $a_m[(m+r)(m+r+5)+6]+4a_{m-2}=0$

$r=-2$: $a_m[(m-2)(m+3)+6]+4a_{m-2}=0$ ... $(m-2)(m+3)+6=m^2+m=m(m+1)$

$\Rightarrow 4a_{m-2}+m(m+1)a_m=0 \Rightarrow a_m=\frac{-4a_{m-2}}{m(m+1)}$, $m\geq 2$

Finding a_m: $a_1=0 \Rightarrow a_3=0 \Rightarrow \dots a_{ODD}=0$

$a_2=\frac{-4}{2\cdot3}a_0 \Rightarrow a_4=\frac{(-4)^2}{2\cdot3\cdot4\cdot5}a_0 \Rightarrow a_6=\frac{(-4)^3}{7!}a_0 \Rightarrow a_{2m}=\frac{(-4)^m}{(2m+1)!}a_0$

$\Rightarrow y_1(x)=\sum_{m=0}^{\infty} a_m x^{m-2}=\sum_{\ell=0}^{\infty} a_{2\ell}x^{2\ell-2}=\frac{a_0}{x^2}\sum_{m=0}^{\infty} \frac{(-4)^m}{(2m+1)!}x^{2m}$

→ we have real roots $r_1=-2$, $r_2=-3$ $\Rightarrow r_1-r_2=1\in\mathbb{N}$

$\Rightarrow y_2=K\ln(x)y_1+\sum_{m=0}^{\infty} b_m x^{m+r_2} = K\ln(x)\sum_{m=0}^{\infty} a_m x^{m-2}+\sum_{m=0}^{\infty} b_m x^{m-3}$

$y_2'=K\ln x\sum_{m=0}^{\infty} a_m(m-2)x^{m-3}+K\sum_{m=0}^{\infty} a_m x^{m-3}+\sum_{m=0}^{\infty} b_m(m-3)x^{m-4}$

$y_2''=K\ln x\sum a_m(m-2)(m-3)x^{m-4}+K\sum a_m(m-2)x^{m-4}+K\sum a_m(m-3)x^{m-4}+\sum b_m(m-3)(m-4)x^{m-5}$

Coefficients: $x^2y''+6xy'+(4x^2+6)y=0$

x^{m-2}: $Ka_m(m-2)+Ka_m(m-3)+b_{m+1}(m-2)(m-3)+6(Ka_m+b_{m+1}(m-2))+6b_{m+1}+4b_{m-1}=0$

$m\geq 1$ $\quad Ka_m(m-2+m-3+6)+b_{m+1}(m^2-5m+6+6m-12+6)+4b_{m-1}=0$

(*) $Ka_m(2m+1)+b_{m+1}(m^2+m)+4b_{m-1}=0$

x^{-3}: $b_0(-3)(-4)+6b_0(-3)+6b_0=0 \Rightarrow 12b_0-18b_0+6b_0=0 \Rightarrow b_0\in\mathbb{R}$

x^{-2}: as x^{m-2} but without $4x^2y$ $\Rightarrow$ without $4b_{m-1}$, $m=0$ (*)

$\Rightarrow Ka_0+b_1\cdot 0=0 \Rightarrow Ka_0=0$ and we know $a_0\neq 0 \Rightarrow K=0$

↳ b_1 can be anything $\Rightarrow$ pick $b_1=0$

Therefore (*) becomes $b_{m+1}(m+1)m+4b_{m-1}=0 \Rightarrow b_{m+1}=\frac{-4b_{m-1}}{m(m+1)} \Rightarrow b_m=\frac{-4b_{m-2}}{m(m-1)}$, $m\geq 2$

Ex: Solve $x^2y'' + 3xy' + y = 0$ as an Euler equation and using method of Frobenius

a) Euler equation: guess $y = x^r \Rightarrow y' = rx^{r-1} \Rightarrow y'' = r(r-1)x^{r-2}$

indicial eq: $r(r-1) + 3r + 1 = 0$

$r^2 + 2r + 1 = (r+1)^2 = 0 \Rightarrow$ repeated root $r = -1$

$$\Rightarrow \underline{\underline{y(x) = (c_1 + c_2 \ln x)\, x^{-1}}}$$

b) Frobenius method

$x = 0$ is a regular singular point and $x \neq 0$ are ordinary points

$y = \sum a_m x^{m+r}$, $y' = \sum (m+r) a_m x^{m+r-1}$, $y'' = \sum (m+r)(m+r-1) a_m x^{m+r-2}$

Coefficients: $x^2y'' + 3xy + y = 0$

$x^r \Rightarrow m = 0$: $(0+r)(0+r-1)a_0 + 3(0+r)a_0 + a_0 = 0$

$r(r-1) + 3r + 1 = r^2 + 2r + 1 = (r+1)^2 = 0 \Rightarrow$ repeated $r = -1$

x^{m+r}: $a_m(m+r)(m+r-1) + 3(m+r)a_m + a_m = 0$

$m \geq 1$ $\quad a_m[(m+r)(m+r+2) + 1] \overset{r=-1}{=\!=\!=} a_m[\underbrace{(m-1)(m+1)}_{m^2-1} + 1] = m^2 a_m = 0 \Rightarrow a_m = 0$

$$\Rightarrow y_1(x) = \sum_{m=0}^{\infty} a_m x^{m-1} = a_0 x^{-1}$$

Second solution: $y_2(x) = \ln(x)\, y_1(x) + \sum_{m=0}^{\infty} b_m x^{m-1}$

$y_2(x) = a_0 \ln(x)\, x^{-1} + \sum_{m=0}^{\infty} b_m x^{m-1}$

$y_2'(x) = a_0 \ln(x)(-1x^{-2}) + a_0 x^{-2} + \sum_{m=0}^{\infty} b_m (m-1) x^{m-2}$

$y_2''(x) = a_0 \ln(x) \cdot 2x^{-3} - a_0 x^{-3} - 2a_0 x^{-3} + \sum_{m=0}^{\infty} b_m (m-1)(m-2) x^{m-3}$

Coefficients: $x^2y'' + 3xy' + y = 0$

x^{-1}: $-a_0 - 2a_0 + b_0(-1)(-2) + 3(a_0 + b_0(-1)) + b_0 = 0$

$a_0(-3+3) + b_0(2-3+1) = 0 \Rightarrow a_0 \in \mathbb{R},\ b_0 \in \mathbb{R}$

x^{m-1}: $b_m(m-1)(m-2) + b_m(m-1) + b_m = 0$

$m \geq 1$ $\quad \Rightarrow b_m[(m-1)(m-1) + 1] = b_m[\underbrace{(m-1)^2}_{\geq 0} + \underbrace{1}_{>0}] = 0 \Rightarrow b_m = 0,\ m \geq 1$

$\Rightarrow y_2(x) = \ln(x)\, y_1(x) + b_0 x^{-1}$

$= a_0 \ln(x) x^{-1} + b_0 x^{-1} = \underline{\underline{(b_0 + a_0 \ln x)\, x^{-1}}}$, $a_0, b_0 \in \mathbb{R}$

↳ this also gives the general solution

Ex: Solve $y'' + y = 0$ using Frobenius method and otherwise

a) Normally: char eq: $r^2 + 1 = 0 \Rightarrow r = \pm i$

complex: $y = e^{it} = \cos t + i \sin t$

real: $y(x) = C_1 \cos x + C_2 \sin x$

b) Frobenius

$\forall x \in \mathbb{R}$ is an ordinary point of this equation $\Rightarrow$ don't need Frobenius

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$y'' + y = 0$: $\displaystyle\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

x^n: $(n+2)(n+1) a_{n+2} + a_n = 0 \Rightarrow a_{n+2} = -a_n \frac{1}{(n+1)(n+2)}$ $\quad n \geq 0$

Even: a_0, $a_2 = a_0 \frac{-1}{1\cdot 2}$, $a_4 = a_0 \frac{(-1)^2}{1\cdot2\cdot3\cdot4}$ $\Rightarrow a_{2n} = a_0 \frac{(-1)^n}{(2n)!}$

Odd: a_1, $a_3 = a_1 \frac{-1}{2\cdot 3}$; $a_5 = a_1 \frac{(-1)^2}{2\cdot3\cdot4\cdot5}$ $\Rightarrow a_{2n+1} = a_1 \frac{(-1)^n}{(2n+1)!}$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} =$$

$$= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1) \frac{x^{2n+1}}{(2n+1)!} = a_0 \cos x + a_1 \sin x, \quad a_0, a_1 \in \mathbb{R}$$

Ex: Solve $4xy'' + 2y' + y = 0$ using Frobenius method

Standard form: $y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$

- $x \neq 0$ are ordinary points
- $x = 0$ is a singular point $\to$ is it regular?

$x^2\left(\frac{1}{4x}\right) = \frac{1}{4}x$ and $x\left(\frac{1}{2x}\right) = \frac{1}{2}$ are both analytical $\Rightarrow$ it is regular

Frobenius: $y = \displaystyle\sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \displaystyle\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \displaystyle\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

Coefficients $4xy'' + 2y' + y = 0$

x^{r-1}: $4(a_0(r)(r-1)) + 2(a_0(r)) = 0 \Rightarrow 2a_0 r[2(r-1)+1] = 0$

$\Rightarrow$ indicial eq: $r(2r-1) = 0 \Rightarrow r_1 = \frac{1}{2} > r_2 = 0$ & $r_1 - r_2 = \frac{1}{2} \notin \mathbb{N}$

$n \geq 0$: x^{n+r}: $4[a_{n+1}(n+1+r)(n+r)] + 2[a_{n+1}(n+r+1)] + a_n = 0$

note: $4(n+r+1)(n+r) + 2(n+r+1) = 2(n+r+1)(2(n+r)+1) = \overbrace{2(n+r+1)(2n+2r+1)}^{*}$

$\Rightarrow a_{n+1} \cdot * + a_n = 0 \Rightarrow a_{n+1} = a_n \frac{-1}{*} = a_n \frac{-1}{2(n+r+1)(2n+2r+1)}$

$r_1 = \frac{1}{2}$: $a_0 \in \mathbb{R}$, $a_{n+1} = \frac{-1}{2(n+\frac{3}{2})(2n+2)} a_n = \frac{-1}{(2n+3)(2n+2)} a_n \Rightarrow a_n = \frac{-1}{2n(2n+1)} a_{n-1}$

$r_2 = 0$: $b_0 \in \mathbb{R}$, $b_{n+1} = \frac{-1}{2(n+1)(2n+1)} b_n = \frac{-1}{(2n+1)(2n+2)} b_n \Rightarrow b_n = \frac{-1}{2n(2n-1)} b_{n-1}$

$\to$ solving $4xy'' + 2y' + y = 0$, $\quad y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} = x^r \sum_{m=0}^{\infty} a_m x^m$

$r_1 = \frac{1}{2} \Rightarrow a_m = \frac{-1}{2m(2m+1)} a_{m-1}$ | $r_2 = 0 \Rightarrow b_m = \frac{-1}{2m(2m-1)} b_{m-1}$

- $a_0 \in R$, $a_1 = \frac{-1}{2\cdot 3} a_0$, $a_2 = \frac{(-1)^2}{2\cdot3\cdot4\cdot5} a_0$, $a_3 = \frac{(-1)^3}{2\cdot3\cdot4\cdot5\cdot6\cdot7} a_0 \Rightarrow a_m = \frac{(-1)^m}{(2m+1)!} a_0$
- $b_0 \in R$, $b_1 = \frac{-1}{2\cdot 1} b_0$, $b_2 = \frac{(-1)^2}{4\cdot3\cdot2\cdot1} b_0 \Rightarrow b_m = \frac{(-1)^m}{(2m)!} b_0$

$\Rightarrow$ General solution of $4xy'' + 2y' + y = 0$ is

$$y(x) = a_0 \sqrt{x} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^m + b_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^m = \underline{\underline{a_0 \sin(\sqrt{x}) + b_0 \cos(\sqrt{x})}}$$

👁 $\cos(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \Rightarrow \cos(\sqrt{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^m$

$\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \Rightarrow \sin(\sqrt{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{m+\frac{1}{2}}$

Ex: $\underline{y'' - 2xy' + 2y = 0}$

a, Find the general solution about $x_0 = 0$

b, Solve the initial value problem with $y(0) = 1$ and $y'(0) = 2$

- $\forall x \in R$ are ordinary points $\Rightarrow$ no singular points

$y = \sum_{m=0}^{\infty} a_m x^m \Rightarrow y' = \sum_{m=0}^{\infty} m a_m x^{m-1} \Rightarrow y'' = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}$ $\hookrightarrow 0$ for $m = 0,1$

$$\Rightarrow y'' - 2xy' + 2y = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2\sum_{m=0}^{\infty} m a_m x^m + 2\sum_{m=0}^{\infty} a_m x^m$$
$$= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - 2\sum_{m=0}^{\infty} a_m x^m (m-1)$$

coefficients: x^m: $(m+2)(m+1)a_{m+2} - 2a_m(m-1) = 0 \Rightarrow a_{m+2} = \frac{2a_m(m-1)}{(m+1)(m+2)}$

$a_0, a_1 \in R$ $\to$ ODD: $a_3 = a_1 \frac{2\cdot 0}{2\cdot 3} = 0 \Rightarrow a_5 = 0 \Rightarrow \dots a_{odd} = 0$

$\hookrightarrow$ EVEN: $a_2 = \frac{2a_0(-1)}{1\cdot 2} = -a_0 \Rightarrow a_4 = \frac{2a_2\cdot 1}{3\cdot 4} = -\frac{2}{3\cdot 4} a_0$

$a_6 = -\frac{2}{3\cdot4} a_0 \cdot \frac{2\cdot 3}{5\cdot 6} = -4a_0 \frac{1\cdot 3}{3\cdot4\cdot5\cdot6}$

$a_8 = -8a_0 \frac{1\cdot 3}{3\cdot4\cdot5\cdot6} \cdot \frac{5}{7\cdot 8} = -16a_0 \cdot \frac{1\cdot3\cdot5}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8} = -2^4 a_0 \cdot \frac{5!!}{8!}$

$a_{10} = -2^5 a_0 \frac{5!!}{8!} \cdot \frac{7}{9\cdot 10} = -2^5 a_0 \frac{7!!}{10!}$, $a_{12} = -2^6 a_0 \frac{9!!}{12!}$

$$\Rightarrow a_{2m} = -2^m a_0 \frac{(2m-3)!!}{(2m)!} = -2^m a_0 \frac{(2m-3)!!}{(2m)!!\,(2m-1)!!} = -a_0 \frac{2^m (2m-3)!!}{(2^m m!)(2m-1)(2m-3)!!} = -\frac{a_0}{(2m-1)m!}$$

$$\Rightarrow y(x) = \sum_{m=0}^{\infty} a_m x^m = a_1 x + \sum_{m=0}^{\infty} a_{2m} x^{2m} = \underline{\underline{a_1 x - a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m-1)m!}}}, \quad a_0, a_1 \in R$$

b) $y(0) = 1 \Rightarrow \left(\sum a_n x^n\right)(0) = 1 \Rightarrow a_0 + 0 = 1 \Rightarrow a_0 = 1$

$y'(0) = 2 \Rightarrow \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{=0} = 2 \Rightarrow 1 \cdot a_1 = 2 \Rightarrow a_1 = 2$

$$\Rightarrow \underline{\underline{y(x) = 2x - \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n-1)n!}}}$$

• <u>Solving ODEs using Fourier Transforms</u>

<u>Ex</u>: Solve $y' + 2y = h(x)$, where $h(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

<u>Recall</u>: $\frac{d^n}{dx^n} f(x) \xleftrightarrow{\mathcal{F}} (ik)^n \hat{f}(k)$

$\Rightarrow$ take the F.T. on both sides:

$$ik\hat{y} + 2\hat{y} = \hat{h} \Rightarrow \hat{y}(ik+2) = \hat{h} \Rightarrow \hat{y} = \frac{1}{ik+2}\hat{h}$$

$\Rightarrow$ we need to find $\hat{h}$

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(ik+1)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{ik+1} e^{-x(ik+1)} \Big|_0^{\infty} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ik+1}$$

$$\Rightarrow \hat{y}(k) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(ik+2)(ik+1)} = \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{ik+2} + \frac{1}{ik+1} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ik+1} - \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{2}}{i\frac{k}{2}+1} = \hat{h}(k) - \frac{1}{2}\hat{h}\left(\frac{k}{2}\right).$$

$$\Rightarrow \underline{\underline{y(x) = h(x) - h(2x) = \begin{cases} e^{-x} - e^{-2x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}}}$$

<u>General idea</u>:

1) F.T. the whole equation, utilizing the derivative rule to get rid of the derivatives
2) express the F.T. of the unknown function
3) inverse F.T. to get the unknown function

$\|\cdot\|: V \to \mathbb{R}$ is a norm $\equiv$

i) $\|v\| \geq 0$ & $\|v\| = 0 \iff v = 0$

ii) $\|\lambda v\| = |\lambda| \cdot \|v\|$

iii) $\|u + v\| \leq \|u\| + \|v\|$

$\langle \cdot | \cdot \rangle : V^2 \to \mathbb{R}$ is an inner product $\equiv$

i) $\langle u|u\rangle \geq 0$ & $\langle u|u\rangle = 0 \iff u = 0$

ii) $\langle u|v\rangle = \langle v|u\rangle$

iii) $\langle \lambda u|v\rangle = \lambda \langle u|v\rangle$

iv) $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$

Real Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{\ell}\right)$$

$\nearrow f$ ODD $\Rightarrow a_n = 0$

$\searrow f$ EVEN $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\ell}\int_{x_0}^{x_0+\ell} f(x)\,dx, \quad a_n = \frac{2}{\ell}\int_{x_0}^{x_0+\ell} f(x)\cos\left(\frac{2\pi n x}{\ell}\right)dx, \quad b_n = \frac{2}{\ell}\int_{x_0}^{x_0+\ell} f(x)\sin\left(\frac{2\pi n x}{\ell}\right)dx$$

Complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{\ell}\right), \quad c_n = \frac{1}{\ell}\int_{x_0}^{x_0+\ell} f(x)\exp\left(\frac{-2\pi i n x}{\ell}\right)dx \to f \text{ real} \Rightarrow \overline{c_n} = c_{-n}$$

Parsevals Theorem

modulus

$$\frac{1}{\ell}\int_{x_0}^{x_0+\ell} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2} \quad \to \langle \tilde{c}_1 | \tilde{c}_2 \rangle_{\mathbb{C}} = \frac{1}{\ell}\langle f|g\rangle$$

Fourier Transform

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx, \quad \mathcal{F}^{-1}[\hat{f}](x) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)\,e^{ikx}\,dx$$

$\to \mathcal{F}(\mathcal{F}(f))(x) = f(-x)$

$\to \mathcal{F}^{-1}(\mathcal{F}(f)) = f$

$\to \mathcal{F}^{-1}(\mathcal{F}^{-1}(\hat{f}))(k) = \hat{f}(-k)$

$\downarrow$

$\lim_{x \to \pm\infty} f(x) = 0$

Know: $\mathcal{F}\left[e^{-\frac{1}{2}x^2}\right] = e^{-\frac{1}{2}k^2}$, $\mathcal{F}[\mathrm{rect}(x)] = \sqrt{\frac{2}{\pi}}\,\mathrm{sinc}(k)$, $\mathcal{F}[\mathrm{sinc}(x)] = \sqrt{\frac{\pi}{2}}\,\mathrm{rect}(k)$

Rules: $\alpha f(x) + \beta g(x) \xleftrightarrow{\mathcal{F}} \alpha\hat{f}(k) + \beta\hat{g}(k)$

$f(\alpha x) \longleftrightarrow \frac{1}{|\alpha|}\hat{f}\left(\frac{k}{\alpha}\right)$

$f(x-a) \longleftrightarrow e^{-ika}\hat{f}(k)$

$e^{iax}f(x) \longleftrightarrow \hat{f}(k-a)$

f even $\iff \hat{f}$ even

f odd $\iff \hat{f}$ odd

$\frac{d^n}{dx^n}f(x) \longleftrightarrow (ik)^n\hat{f}(x)$ ... derivative rule

$x^n f(x) \longleftrightarrow (i)^n \frac{d^n}{dk^n}\hat{f}(k)$

Dirac Delta Function

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \quad \& \quad \int_{-\infty}^{\infty}\delta(x)\,dx = 1 \quad \Rightarrow \int_{-\infty}^{\infty}\delta(x-a)\,f(x)\,dx = f(a)$$

$$\delta(g(x)) = \begin{cases} \frac{\delta(0)}{|g'(x)|}, & g(x) = 0 \\ 0, & g(x) \neq 0 \end{cases} \quad \Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(g(x))\,dx = \sum_i \frac{f(x_i)}{|g'(x_i)|} \quad \text{where } g(x_i) = 0$$

$$\mathcal{F}[\delta] = \frac{1}{\sqrt{2\pi}}, \quad \mathcal{F}[1] = \sqrt{2\pi}\,\delta(k), \quad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx$$

Plancharel's Theorem

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|\hat{f}(k)|^2\,dk$$

Convolutions

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

Rules: $f*g = g*f$

$(f*g)*h = f*(g*h)$

$f*(g+h) = (f*g)+(f*h)$

$(\alpha f)*(\beta g) = \alpha\beta \cdot (f*g)$

$f(x+\alpha) * g(x+\beta) = (f*g)(x+\alpha+\beta)$

$f(\alpha x) * g(\alpha x) = \frac{1}{|\alpha|}(f*g)(\alpha x)$

$f*\delta = \delta * f = f$

$(f*g)(x) \xleftrightarrow{\mathcal{F}} \sqrt{2\pi}\, \hat{f}(k) g(k)$

$f(x)\, g(x) \xleftrightarrow{\mathcal{F}} \frac{1}{\sqrt{2\pi}} (\hat{f} * \hat{g})(k)$

The Discrete Fourier Transform

N^{th} primitive root of unity

$$\mathcal{F} : (f_0, \dots f_{N-1}) \mapsto (c_0, \dots c_{N-1}) \equiv \forall m : c_m = \frac{1}{N} \sum_{k=0}^{N-1} f_k\, \omega^{mk} = \frac{1}{N} p_f(\omega^m), \quad \omega = e^{-\frac{2\pi}{N} i}$$

$\hookrightarrow p_f(x) := \sum_{k=0}^{N-1} f_k x^k$ $\rightarrow$ evaluates the polynomial at $\{\omega^m \mid m \in [N-1]\}$

$\mathcal{F}$ is a linear map $\Rightarrow \vec{c} = \mathcal{F}(\vec{f}) \Leftrightarrow \vec{c} = \Omega \vec{f}$, $\Omega_{mk} = \frac{1}{N}\omega^{mk}$

$\Rightarrow$ inverse DFT: $\vec{f} = \Omega^{-1}\vec{c} = N\overline{\Omega}\,\vec{c}$ $\Rightarrow$ just use the complement $\overline{\omega}$ and $*N$

- $\hat{f}(x) = \sum_{m=0}^{N-1} c_m \exp(imx)$ ... $\forall k : \hat{f}\left(\frac{2\pi k}{N}\right) = f_k$

Parcevals Theorem: $\vec{c} = \mathcal{F}(\vec{f}) \Rightarrow \sum |c_k|^2 = \frac{1}{N} \sum |f_k|^2$

EXAM REVISION

① Show that $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto |x|+|y|$ is a norm on $\mathbb{R}^2$

1, positivity: $\|(x,y)\| \geq 0 \iff |x|+|y| \geq 0$ ... true for $\forall (x,y) \in \mathbb{R}^2$
$\|(x,y)\| = 0 \iff |x|+|y| = 0 \iff (x,y) = (0,0)$

2, scaling: $\|\lambda(x,y)\| = \|(\lambda x, \lambda y)\| = |\lambda x| + |\lambda y| = |\lambda|\cdot|x| + |\lambda|\cdot|y|$
$= |\lambda|(|x|+|y|) = |\lambda|\cdot\|(x,y)\|$

3, Δ-ineq: Want: $\|u+v\| \leq \|u\| + \|v\|$
$\|(x_1+x_2, y_1+y_2)\| = |x_1+x_2| + |y_1+y_2| \leq |x_1|+|x_2| + |y_1|+|y_2|$
$= (|x_1|+|y_1|) + (|x_2|+|y_2|) = \|(x_1,y_1)\| + \|(x_2,y_2)\|$

② Show that $\langle\cdot,\cdot\rangle : \mathbb{R}^{2\times2} \times \mathbb{R}^{2\times2} \to \mathbb{R}$, $(A,B) \mapsto \operatorname{tr}(AB^T)$ is an inner product on $\mathbb{R}^{2\times2}$.

↳ I will show this for any $\mathbb{R}^{m\times n}$, $n \in \mathbb{N}$

$A,B \in \mathbb{R}^{m\times n} \Rightarrow \operatorname{tr}(AB^T) = \sum_{k=1}^{m} (AB^T)_{kk} = \sum_{k=1}^{m} \sum_{j=1}^{n} A_{kj} (B^T)_{jk} = \sum_{k=1}^{m} \sum_{j=1}^{n} a_{kj} b_{kj}$
$= \sum_{i,j=1}^{m} a_{ij} b_{ij}$ ... element-wise multiplication + sum

1, positivity: $\langle A|A\rangle = \sum_{i,j=1}^{m} a_{ij}^2 \geq 0$ & $\langle A|A\rangle = 0 \iff A = 0^{m\times n}$ ✓

2, symmetry: $\langle A|B\rangle = \sum_{i,j=1}^{m} a_{ij} b_{ij} = \sum_{i,j=1}^{m} b_{ij} a_{ij} = \langle B|A\rangle$ ✓

3, scaling: $\langle \lambda A|B\rangle = \sum_{i,j=1}^{m} \lambda a_{ij} b_{ij} = \lambda \sum_{i,j} a_{ij} b_{ij} = \lambda \langle A|B\rangle$ ✓

4, linearity: $\langle A+B|C\rangle = \sum_{i,j} (a_{ij}+b_{ij}) c_{ij} = \sum_{i,j} a_{ij} c_{ij} + \sum_{i,j} b_{ij} c_{ij} = \langle A|C\rangle + \langle B|C\rangle$

③ Find all $x \in \mathbb{R}$ s.t. $A = \begin{bmatrix} 1 & -2 \\ 5 & x \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ x & x \end{bmatrix}$ are orthogonal with respect to ↑

$0 = \langle A|B\rangle = 1\cdot2 + (-2)(-2) + 5x + x^2 = x^2 + 5x + 7$

$x_{1,2} = \frac{-5 \pm \sqrt{25-28}}{2a} \notin \mathbb{R} \Rightarrow \nexists x \in \mathbb{R}$ s.t. $\langle A|B\rangle = 0$

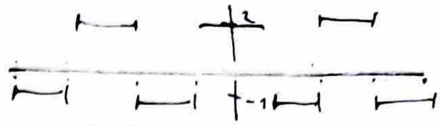
④ Evaluate $\sum_{i=2}^{4} \sum_{j=3}^{5} \delta_{ij} (i^2+j)^2$ ... $\delta_{ij} = 1$ if $i=j$ else 0

$i \in \{2,3,4\}$, $j \in \{3,4,5\} \to 3, 4$

$\Rightarrow (3^2+3)^2 + (4^2+4)^2 = 12^2 + 20^2 = 144 + 400 = \underline{\underline{544}}$

5) Find the real Fourier series of $f(x)=\begin{cases}2, & x\in(-1,1)\\ -1, & x\in(1,3)\end{cases}$, $f(x+4)=f(x) \to \ell=4$

$$f(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty} a_n\cos\left(\frac{2\pi n x}{\ell}\right)+\sum_{n=0}^{\infty} b_n \sin\left(\frac{2\pi n x}{\ell}\right)$$



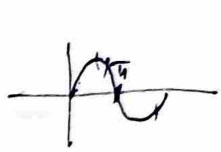
$$a_0=\frac{2}{\ell}\int_{x_0}^{x_0+\ell} f(x)\,dx, \quad a_n=\frac{2}{\ell}\int_{x_0}^{x_0+\ell} f(x)\cos\left(\frac{2\pi n x}{\ell}\right)dx, \quad b_n=\frac{2}{\ell}\int_{x_0}^{x_0+\ell} f(x)\sin\left(\frac{2\pi n x}{\ell}\right)dx$$

- $a_0=\frac{2}{\ell}\int_{-1}^{3} f(x)\,dx=\frac{2}{4}\int_{-1}^{1} 2\,dx+\frac{2}{4}\int_{1}^{3}(-1)\,dx=\frac{1}{2}\cdot 4+\frac{1}{2}(-2)=1$

- $a_n=\frac{2}{4}\int_{-1}^{3} f(x)\cos\left(\frac{\pi n x}{2}\right)dx=\frac{1}{2}\int_{-1}^{1} 2\cos\left(\frac{\pi n x}{2}\right)dx-\frac{1}{2}\int_{1}^{3}\cos\left(\frac{\pi n x}{2}\right)dx$

$$=\frac{2}{\pi n}\left[\sin\left(\frac{\pi n x}{2}\right)\right]_{-1}^{1}-\frac{1}{2}\cdot\frac{2}{\pi n}\left[\sin\left(\frac{\pi n x}{2}\right)\right]_{1}^{3}$$

$$=\frac{2}{\pi n}\left[\sin\left(\frac{\pi}{2}n\right)-\sin\left(-\frac{\pi}{2}n\right)\right]-\frac{1}{\pi n}\left[\sin\left(\frac{3}{2}\pi n\right)-\sin\left(\frac{\pi}{2}n\right)\right]$$



$$=\frac{5}{\pi n}\sin\left(\frac{\pi}{2}n\right)-\frac{1}{\pi n}\sin\left(\frac{\pi}{2}n+\pi n\right)$$

↳ 1 for 1, 5, 9 ...
−1 for 3, 7, 11

- n EVEN: sin = 0 $\Rightarrow a_n=0$
- n ODD: $\sin\left(\frac{\pi}{2}n+\pi n\right)=-\sin\left(\frac{\pi}{2}n\right)$

$$a_n=\begin{cases}0, & n \text{ even}\\ \frac{6}{\pi n}\sin\left(\frac{\pi}{2}n\right), & n \text{ odd}\end{cases}$$

↳ $n=2k+1 \Rightarrow a_{2k+1}=\frac{6}{\pi(2k+1)}\cdot(-1)^k$

- $b_n=0$ because f is even and sin is odd $\Rightarrow \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \text{EVEN}\cdot\text{ODD}=\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}\text{ODD}=0$

$$\Rightarrow f(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty} a_n\cos\left(\frac{\pi n x}{2}\right)=\frac{1}{2}+\frac{6}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\frac{\pi(2k+1)x}{2}\right)$$

6) Show that the coefficients of a $\mathbb{C}$ F-series of a real even function are real

$f:\mathbb{R}\to\mathbb{R}$, $f(-x)=f(x)$, $f(x)=\sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{\ell}\right)$, $c_n=\frac{1}{\ell}\int_{x_0}^{x_0+\ell} f(x)\exp\left(\frac{-2\pi i n x}{\ell}\right)dx$

$$\Rightarrow c_n=\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x)\exp\left(\frac{-2\pi i n x}{\ell}\right)dx=\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \underset{\text{EVEN}}{f(x)}\left(\underset{\text{EVEN}}{\cos\left(\frac{2\pi i n x}{\ell}\right)}-i\underset{\text{ODD}}{\sin\left(\frac{2\pi i n x}{\ell}\right)}\right)dx$$

$$=\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x)\cos(\dots)\,dx-\frac{i}{\ell}\underbrace{\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x)\sin(\dots)\,dx}_{\text{ODD}\ \Rightarrow 0}=\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(x)\cos(\dots)\,dx \in\mathbb{R}$$

7. Find the Fourier transform of

$$f(x)=\begin{cases} e^{3x}, & x<0 \\ 0, & x>0 \end{cases}, \quad g(x)=\begin{cases} 0, & x<0 \\ e^{-3x}, & x>0 \end{cases}, \quad h(x)=e^{-3|x|}$$

$$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{3x}\, e^{-ikx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{x(3-ik)}\,dx=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{3-ik}\left[e^{x(3-ik)}\right]_{-\infty}^{0}$$

$$=\frac{1}{\sqrt{2\pi}(3-ik)}\left[1-0\right]=\frac{1}{\sqrt{2\pi}(3-ik)}$$

$$\hat{g}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\, e^{-ikx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-3x}\, e^{-ikx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x(3+ik)}\,dx=\frac{1}{2\pi}\,\frac{-1}{(3+ik)}\left[e^{-x(3+ik)}\right]_{0}^{\infty}$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\frac{-1}{(3+ik)}\cdot\left[0-1\right]=\frac{1}{\sqrt{2\pi}(3+ik)}$$

$$h(x)=f(x)+g(x) \Rightarrow \hat{h}(k)=\hat{f}(k)+\hat{g}(k)=\frac{1}{\sqrt{2\pi}(3-ik)}+\frac{1}{\sqrt{2\pi}(3+ik)}=\underline{\underline{\frac{6}{\sqrt{2\pi}(9+k^2)}}}$$

8. $N=4$, $(f_0,f_1,f_2,f_3)=(2,6,10,8) \Rightarrow$ Find (c_0,c_1,c_2,c_3) using FFT

$$c_m=\frac{1}{N}\sum_{k=0}^{3} f_k\,\omega^{mk}, \quad \omega=e^{-\frac{2\pi}{4}i}=e^{-\frac{\pi}{2}i}=-i \Rightarrow \omega^2=-1,\ \omega^3=i,\ \omega^4=1$$

$$\vec{c}=\Omega\vec{f}=\frac{1}{N}\begin{bmatrix}1&1&1&1\\1&\omega&\omega^2&\omega^3\\1&\omega^2&\omega^4&\omega^6\\1&\omega^3&\omega^6&\omega^9\end{bmatrix}\cdot\vec{f}=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{bmatrix}\cdot\begin{bmatrix}2\\6\\10\\8\end{bmatrix}$$

$$E: \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}2\\10\end{bmatrix}=\begin{bmatrix}12\\-8\end{bmatrix}$$

$$O: \begin{bmatrix}1&1\\-i&i\end{bmatrix}\begin{bmatrix}6\\8\end{bmatrix}=\begin{bmatrix}14\\2i\end{bmatrix}$$

$$\vec{c}=\frac{1}{4}\begin{bmatrix}[E]+[O]\\ [E]-[O]\end{bmatrix}=\frac{1}{4}\begin{bmatrix}12+14\\-8+2i\\12-14\\-8-2i\end{bmatrix}=\frac{1}{4}\begin{bmatrix}26\\-8+2i\\-2\\-8-2i\end{bmatrix}=\begin{bmatrix}13/2\\-2+i/2\\-1/2\\-2-i/2\end{bmatrix}$$

9. Solve $xy'=y+xy$, $x\neq 0,\ y\neq 0$

$$x\frac{dy}{dx}=y(1+x) \Rightarrow \int\frac{dy}{y}=\int\frac{1+x}{x}\,dx=\int\frac{1}{x}+1\,dx$$

$$\Rightarrow \ln|y|=\ln|x|+x+C$$

$$|y|=\underbrace{|x|\cdot e^{x}\cdot A}_{>0},\ A\in\mathbb{R}^+ \Leftrightarrow \underline{y=K\cdot|x|e^{x}},\ K\in\mathbb{R}\setminus\{0\}$$

10. Solve $xy'-5y=x^7$, $x<0$

$$y'-\frac{5}{x}y=x^6 \Rightarrow \mu(x)=e^{\int-\frac{5}{x}dx}=e^{-5\ln|x|}=|x|^{-5}\overset{x<0}{=}(-x)^{-5}=-x^{-5}$$

$$y(x)=-x^5\int -x^{-5}\cdot x^6\,dx=x^5\int x\,dx=x^5\left(\frac{x^2}{2}+C\right)$$

$$=\underline{\frac{1}{2}x^7+C\cdot x^5},\quad C\in\mathbb{R}$$

(11) Solve the IVP $y'' - y' - 6y = \sin(2x)$, $y(0) = \frac{105}{52}$ $y'(0) = \frac{21}{26}$

char eq: $r^2 - r - 6 = (r+2)(r-3) = 0 \Rightarrow y_h = C_1 e^{-2x} + C_2 e^{3x}$

particular sol: $y_p(x) = A\cos(2x) + B\sin(2x)$

$y_p'(x) = -2A\sin(2x) + 2B\cos(2x)$

$y_p''(x) = -4A\cos(2x) - 4B\sin(2x)$

$\Rightarrow -4A\cdot C - 4B\cdot S + 2A\cdot S - 2B\cdot C - 6A\cdot C - 6B\cdot S = S$

$\sin: -4B + 2A - 6B = 1 \Rightarrow 2A - 10B = 1$ } $52A = 1 \Rightarrow A = \frac{1}{52}$

$\cos: -4A - 2B - 6A = 0 \Rightarrow 10A + 2B = 0$ } $\frac{10}{52} + 2B = 0 \Rightarrow B = \frac{-5}{52}$

$\Rightarrow y(x) = C_1 e^{-2x} + C_2 e^{3x} + \frac{1}{52}\cos(2x) - \frac{5}{52}\sin(2x)$

$y'(x) = -2C_1 e^{-2x} + 3C_2 e^{3x} - \frac{2}{52}\sin(2x) - \frac{10}{52}\cos(2x)$

IVP: $y(0) = \frac{105}{52} = C_1 + C_2 + \frac{1}{52} \Rightarrow C_1 + C_2 = \frac{104}{52} = 2$ } $5C_2 = 5 \Rightarrow C_2 = 1$

$y'(0) = \frac{42}{52} = -2C_1 + 3C_2 - \frac{10}{52}$ $\quad -2C_1 + 3C_2 = 1$ } $\Rightarrow C_1 = 1$

$\Rightarrow \underline{\underline{y(x) = e^{-2x} + e^{3x} + \frac{1}{52}\cos(2x) - \frac{5}{52}\sin(2x)}}$

(12) Evaluate $\int_{-\frac{\pi}{2}}^{2\pi} \cos\left(\frac{x}{3}\right)\delta(x^3 - \pi^2 x)\,dx$, δ... Dirac delta

$g(x) := x^3 - \pi^2 x = x(x^2 - \pi^2) = x(x-\pi)(x+\pi) = 0 \iff x \in \{0, \pi, -\pi\}$

$g'(x) = 3x^2 - \pi^2 \Rightarrow |g'(0)| = |-\pi^2| = \pi^2$, $|g'(\pm\pi)| = |3\pi^2 - \pi^2| = 2\pi^2$

$\Rightarrow \int_{-\frac{\pi}{2}}^{2\pi} \cos\left(\frac{x}{3}\right)\delta(g(x))\,dx = \frac{\cos\left(\frac{x}{3}\right)}{|g'(x)|}\Big|_0 + \frac{\cos\left(\frac{x}{3}\right)}{|g'(x)|}\Big|_\pi = \frac{1}{\pi^2} + \frac{\cos\left(\frac{\pi}{3}\right)}{2\pi^2} = \frac{1}{\pi^2}\left(1 + \frac{1}{2}\cdot\frac{1}{2}\right) = \underline{\underline{\frac{5}{4\pi^2}}}$

(13) Evaluate $\int_0^{2\pi} \cos(x)\delta(x^4 - \pi^4)\,dx$

$g(x) = x^4 - \pi^4 = (x^2 - \pi^2)(x^2 + \pi^2) = 0 \iff x = \pm\pi$

$g'(x) = 4x^3 \Rightarrow |g'(\pm\pi)| = |\pm 4\pi^3| = 4\pi^3$

$\Rightarrow \int_0^{2\pi} \cos(x)\delta(g(x))\,dx = \frac{\cos(\pi)}{4\pi^3} = \underline{\underline{-\frac{1}{4\pi^3}}}$

(14) $y'' - 2xy' + 2y = 0$

$-\ \forall x \in \mathbb{R}$ is an ordinal point

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} a_n \cdot n\, x^{n-1} \Rightarrow y'' = \sum_{n=0}^{\infty} a_n \cdot n(n-1)\, x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - 2\sum_{n=0}^{\infty} n \cdot a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n \Big((n+2)(n+1)\, a_{n+2} - 2n\, a_n + 2a_n \Big) = 0$$

$$\Rightarrow \forall n \geq 0: \quad a_{n+2} = \frac{a_n(2n-2)}{(n+2)(n+1)} = 2a_n \frac{(n-1)}{(n+2)(n+1)}$$

n-ODD: a_1, $a_3 = 2a_1 \cdot 0 = 0 \Rightarrow a_{ODD} = 0$

n-EVEN: a_0, $a_2 = 2a_0 \frac{-1}{1\cdot 2}$, $a_4 = 4a_0 \frac{-1\cdot 1}{4\cdot 3\cdot 2\cdot 1} = -4a_0 \frac{1}{4!}$

$$a_6 = -2^3 a_0 \frac{3}{6!}, \quad a_8 = -2^4 a_0 \frac{1\cdot 3\cdot 5}{8!}, \quad a_{10} = -2^5 a_0 \frac{1\cdot 3\cdot 5\cdot 7}{10!}$$

$$\Rightarrow a_{2n} = -2^n a_0 \frac{(2n-3)!!}{(2n)!} = -2^n a_0 \frac{(2n-3)!!}{(2n)!!\,(2n-1)!!} = -a_0 \frac{1}{(2n-1)n!}, \quad n \geq 0$$

$$\Rightarrow y(x) = a_1 x + \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_1 x - a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n-1)n!}, \quad a_0, a_1 \in \mathbb{R}$$

non-zero polynomial solution: $a_0 = 0$, $a_1 = 1$ gives $y(x) = x$