Metric Spaces

Def: A <u>metric space</u> (MS) is a pair (M,d) where $M \neq \emptyset$ is a set and $d: M \times M \to \mathbb{R}$ is a metric i.e.

1, $\forall x,y \in M: d(x,y) \geq 0$ & $d(x,y) = 0 \Leftrightarrow x = y$ ... positivity
2, $\forall x,y \in M: d(x,y) = d(y,x)$ ... symmetry
3, $\forall x,y,z \in M: d(x,z) \leq d(x,y) + d(y,z)$ ... Δ-ineq.

In fact, $d(x,y) \geq 0$ follows from the other axioms. $\Rightarrow$ we only need to show $d(x,y) = 0 \Leftrightarrow x = y$ when proving stuff
↳ assume $d(x,y) < 0$
$\Rightarrow 0 = d(x,x) = d(x,y) + d(y,x) = 2 \cdot d(x,y) < 0$ ↯

Def: The subset $X \subseteq M$ determines a <u>subspace</u> of (M,d) which we call (X,d)

Def: An <u>isometry</u> f of metric spaces (M, d_M) and (N, d_N) is a bijection $f: M \to N$ which preserves distances
$$\forall x,y \in M: \ d_M(x,y) = d_N(f(x), f(y))$$
→ if it exists we say that (M, d_M) and (N, d_N) are <u>isometric</u>.

<u>Idea</u>: Isometric spaces are practically indistinguishable

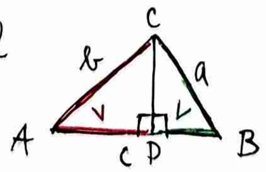
- <u>Euclidean Space</u>

Def: The <u>n-dim. Euclidean space</u> is $\mathbb{E}_n := (\mathbb{R}^n, e_n)$ where $e_n(\tilde{x}, \tilde{y}) = \sqrt{\sum_i (x_i - y_i)^2}$
↳ (X, e_n) where $X \subseteq \mathbb{R}^n$ are also called Euclidean

<u>Theorem</u>: $\mathbb{E}_n = (\mathbb{R}^n, e_n)$ is a metric space

PT: $c^2 = a^2 + b^2$
(*) $\Rightarrow c > a$
$c > b$

↳ clearly 1 and 2 hold
↳ for $n=1$ the Δ-ineq is trivial
↳ for $n=2$ consider any triangle



where WLOG: $c \geq b \geq a$

→ clearly $a \leq b + c$ and $b \leq a + c$, need to show $c \leq a + b$
→ draw a height from C to find the point $D \in AB$
↳ if $D \notin AB$ then $a > c$ or $b > c$
→ divide the triangle to 2 right angle triangles
(*): $c = \underline{d(A,D)} + \underline{d(D,B)} \leq \underline{b} + \underline{a}$

→ if $n>2$ we reduce it geometrically to $n=2$

↳ any three different non-collinear points in $\mathbb{R}^n$ determine a unique two dimensional plane $R \subset \mathbb{R}^n$ isometric to $\mathbb{R}^2$

→ the distances between the points are preserved → use $n=2$ ∎

Ex: The distance in a graph is a metric

Ex: (F,d) where F is the set of functions which have the Riemann integral on the interval $[a,b] \subseteq \mathbb{R}$ and

$$d(f,g) := \int_a^b |f(t)-g(t)|\,dt$$

→ ① and ② obviously hold

Δ ineq: $d(f,g) \leq d(f,h) + d(h,g)$

$$d(f,h)+d(h,g) = \int_a^b |f(t)-h(t)|\,dt + \int_a^b |h(t)-g(t)|\,dt$$
$$\geq \int_a^b |(f-h)+(h-g)|\,dt = \int_a^b |f-g|\,dt = d(f,g)$$ ∎

• The Spherical Metric

Def: The unit sphere is $S := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2+x_2^2+x_3^2 = 1\}$

Define $s: S\times S \to [0,\pi]$ as $s(\tilde{x},\tilde{y}) = \begin{cases} 0, & \tilde{x}=\tilde{y} \\ \varphi, & \tilde{x}\neq\tilde{y} \end{cases}$ ⟶

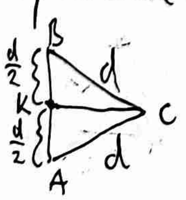
Fact: (S,s) is a metric space

Thm: The upper hemisphere $H = \{(x_1,x_2,x_3) \in S \mid x_3 \geq 0\}$ is not flat.

↳ (H,s) is not isometric to any Euclidean space $(X,e_m) \subseteq \mathbb{E}_m$.

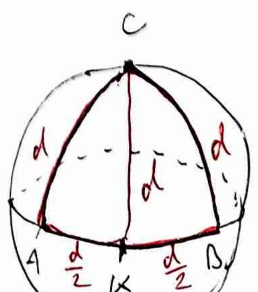
Pf: We find a property which holds for any quadruple of points in $\mathbb{E}_m$ but not in H

equilateral Δ



$e_m(K,C) = \sqrt{d^2-(\frac{1}{2}d)^2} = \frac{\sqrt{3}}{2}d < d$

→ need to find 4 points on H which violate it



$s(K,C) = d$

↯

∎

- **Ultrametric Spaces** (non-Archimedean)

Def: The MS (M,d) is called ultrametric $\equiv$ d satisfies the strong Δ-ineq

$$\forall x,y,z \in M: \ d(x,y) \le \max\{d(x,z), d(z,y)\}$$

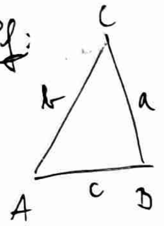
if $d(x,z) \neq d(z,y)$, then $d(x,y) = \max\{d(x,z), d(z,y)\}$

$\hookrightarrow$ let $d(x,z) < d(z,y)$ and $d(x,y) < \max\{d(x,z), d(z,y)\} = d(z,y)$

- strong Δ ineq: $d(z,y) \le \max\{d(z,x), d(x,y)\} < d(z,y)$ $\lightning$

Lemma: Every triangle in an UMS (M,d) is isosceles Δ.

Pf:



let $a \ge b, c$

Δ-ineq: $a \le \max\{b, c\}$ $\Rightarrow a = b$ or $a = c$ ∎

Def: An open ball with center a and radius $r > 0$ in (M,d) is $B(a,r) := \{x \in M \mid d(a,x) < r\}$

$B(a,r) \neq \emptyset$ because $a \in B(a,r)$

Lemma: For every ball in an UMS, every point is its center.

Pf: pick any $b \in B(a,r)$ ... $d(a,b) < r$

need: $d(a,x) < r \iff d(b,x) < r$

Δ-ineq: $d(b,x) \le \max\{d(b,a), d(a,x)\} < r$ ∎

- **p-adic Metric and Norm**

Def: Let p be a prime and $n \in \mathbb{Z}$. We define the p-adic order of n as

$$\mathrm{ord}_p(n) = \max\{m \in \mathbb{N}_0 \mid p^m \mid n\}, \quad \mathrm{ord}_p(0) := +\infty.$$

For $a \in \mathbb{Z}$, $b \in \mathbb{Z}\setminus\{0\}$ we define

$$\mathrm{ord}_p(a/b) := \mathrm{ord}_p(a) - \mathrm{ord}_p(b)$$

Ex: $\mathrm{ord}_5(297/100) = 0 - 2 = -2$, $\mathrm{ord}_3(297/100) = \mathrm{ord}_3(27 \cdot 11) - \mathrm{ord}_3(100) = 3 - 0$

$\frac{a}{b} = \frac{c}{d} \Rightarrow \mathrm{ord}_p(\frac{a}{b}) = \mathrm{ord}_p(\frac{c}{d})$ $\because \mathrm{ord}(\frac{q \cdot a}{q \cdot b}) = \mathrm{ord}(qa) - \mathrm{ord}(qb) = (\mathrm{ord}(q) + \mathrm{ord}(a)) - (\mathrm{ord}(q) + \mathrm{ord}(b))$

$\mathrm{ord}_p(\alpha \cdot \beta) = \mathrm{ord}_p(\alpha) + \mathrm{ord}_p(\beta)$, $\alpha, \beta \in \mathbb{Q}$ $\swarrow$ Fundamental Thm of Arithmetic used

$\hookrightarrow \mathrm{ord}(\frac{a}{b} \cdot \frac{c}{d}) = \mathrm{ord}(ac) - \mathrm{ord}(bd) = \mathrm{ord}(a) + \mathrm{ord}(c) - \mathrm{ord}(b) - \mathrm{ord}(d) = \mathrm{ord}(\frac{a}{b}) + \mathrm{ord}(\frac{c}{d})$

Def: The p-adic norm for $c \in (0,1)$ is the function

$|\cdot|_p : \mathbb{Q} \to \mathbb{R}^+$, $|\alpha|_p := c^{\mathrm{ord}_p(\alpha)}$, $|0|_p = c^{+\infty} := 0$

Ex: $c = \frac{1}{2} \to \left|\frac{1}{100}\right|_5 = \left(\frac{1}{2}\right)^{\mathrm{ord}_5(1/100)} = \left(\frac{1}{2}\right)^{0-2} = 4$

$|\alpha \cdot \beta|_p = |\alpha|_p \cdot |\beta|_p$

$\hookrightarrow |\alpha \cdot \beta|_p = c^{\mathrm{ord}(\alpha\beta)} = c^{\mathrm{ord}\,\alpha + \mathrm{ord}\,\beta} = c^{\mathrm{ord}\,\alpha} \cdot c^{\mathrm{ord}\,\beta} = |\alpha|_p \cdot |\beta|_p$

Intuition

$\mathrm{ord}_p \sim \log$

$|\cdot|_p \sim e^{\log} \sim x$

$\log(x y) = \log x + \log y$

$(x \cdot y) = (x) \cdot (y)$

• Normed Fields

Def: A normed field is a field $(F, 0_F, 1_F, \oplus, \cdot)$ equipped with the norm $|\cdot|_F : F \to \mathbb{R}^+$, s.t.

1) $\forall x \in F : |x|_F = 0 \iff x = 0_F$
2) $\forall x,y \in F : |x \cdot y|_F = |x|_F \cdot |y|_F$ ... multiplicativity
3) $\forall x,y \in F : |x \oplus y|_F \le |x|_F + |y|_F$ ... Δ-ineq

Lemma: $|1_F|_F = 1$, $\forall x \in F : |-x|_F = |x|_F$, $\forall x = 0_F : |x^{-1}|_F = 1/|x|_F$

Pf: • $\forall x \in F : x \cdot 1_F = x \Rightarrow |x \cdot 1_F| = |x| \cdot |1_F| = |x| \Rightarrow |1_F| = 1$

• $$\left.\begin{aligned} |(-x)\cdot(-x)| &= |-x|\cdot|-x| = |-x|^2 \\ &= |x \cdot x| = |x|\cdot|x| = |x|^2 \end{aligned}\right\} |x|^2 = |-x|^2 \;\&\; |\cdot| : F \to \mathbb{R}^+ \Rightarrow |x| = |-x|$$

• $|x \cdot x^{-1}| = |x| \cdot |x^{-1}| = |1_F| \Rightarrow |x|\cdot|x^{-1}| = 1 \Rightarrow |x^{-1}| = 1/|x|$

Lemma: For every normed field $(F, |\cdot|_F)$, the function $d(x,y) := |x-y|_F$ is a metric on F.
When $|\cdot|_F$ satisfies the strong Δ-ineq, then d is an ultrametric

Corollary: It is easy to construct metric spaces from normed fields.

Pf: ① $d(x,y) = 0 \iff |x-y| = 0 \iff x - y = 0_F \iff x = y$

② $d(x,y) = |x-y| = |y-x| = d(y-x)$

③ $d(x,z) + d(z,y) = |x-z| + |z-y| \ge |(x-z)+(z-y)| = |x-y| = d(x,y)$

④ $|x \oplus y| \le \max\{|x|,|y|\} \Rightarrow \max\{d(x,z), d(z,y)\} = \max\{|x-z|, |z-y|\} \ge |x-y| = d(x,y)$ ∎

Theorem: For every prime p and $c \in (0,1)$, $(\mathbb{Q}, |\cdot|_p)$ is a normed field.
The corresponding metric space $(\mathbb{Q}, d)$ is ultrametric.

Pf: We need to show that $|\cdot|_p$ is a norm with the strong Δ-ineq.

① $|x|_p = 0 \iff x = 0$ ✓ ... from the definition of $|\cdot|_p$

② $|x \cdot y|_p = |x|_p \cdot |y|_p$ ✓ ... proven earlier

③ stron Δ-ineq $\leftarrow$ want

$$\forall \alpha, \beta \in \mathbb{Q}: \ |\alpha+\beta|_p \le \max\{|\alpha|_p, |\beta|_p\}$$
$$c^{\operatorname{ord}(\alpha+\beta)} \le \max\{c^{\operatorname{ord}\alpha}, c^{\operatorname{ord}\beta}\} \quad , c \in (0,1)$$
$$\operatorname{ord}(\alpha+\beta) \ge \min\{\operatorname{ord}\alpha, \operatorname{ord}\beta\}$$

$\rightarrow$ let $\alpha = a/b$, $\beta = c/d$ ... when $\alpha = 0$ or $\beta = 0$ it is trivial

$\Rightarrow$ let $\operatorname{ord}_p(\alpha) = n$, $\operatorname{ord}_p(\beta) = m$

$\hookrightarrow \alpha = \frac{a}{b} = p^n \cdot \frac{a'}{b'}$, $\beta = \frac{c}{d} = p^m \cdot \frac{c'}{d'}$, a', b', c', d' not divisible by p

$\rightarrow$ WLOG let $n \le m$

$$\alpha + \beta = p^n \frac{a'}{b'} + p^m \frac{c'}{d'} = p^n \cdot \frac{a'd' + p^{m-n} c'b'}{b'd'} =: p^n \cdot \frac{e}{f}$$

$\rightarrow p \nmid f \Rightarrow \operatorname{ord}_p\left(\frac{e}{f}\right) \ge 0 \Rightarrow \operatorname{ord}(\alpha+\beta) \ge n = \min(\operatorname{ord}\alpha, \operatorname{ord}\beta)$ ∎

• Ostrowski's Theorem

Def: On any field $\mathbb{F}$ we have the trivial norm $\|x\| = \begin{cases} 0, & x = 0_F \\ 1, & x \ne 0_F \end{cases}$

Lemma: Let $\|\cdot\|: \mathbb{F} \to \mathbb{R}^+$ be a norm on $\mathbb{F}$. Then $f(\|\cdot\|): \mathbb{F} \to \mathbb{R}^+_0$, $a \to f(\|a\|)$ is a norm $\iff$

$$\left.\begin{array}{l} \text{i) } f(x) \ge 0 \ \& \ f(x) = 0 \iff x = 0 \\ \text{ii) } f(x \cdot y) = f(x) \cdot f(y) \\ \text{iii) } f(x+y) \le f(x) + f(y) \end{array}\right\} f: \mathbb{R}^+_0 \to \mathbb{R}^+_0 \text{ is a norm}$$

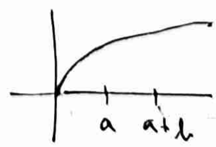
Pf: (i) shows ①, (ii) shows ②, (iii) shows ③

$\nearrow$ modified absolute value

Lemma: Denote $|\cdot|$ the standard norm on $\mathbb{C}$. Then $|\cdot|^c$ is a norm $\iff c \in (0,1]$

Pf: i) ✓ ii) $|a \cdot b|^c = |a|^c \cdot |b|^c$

iii) need: $(a+b)^c \le a^c + b^c$, $a, b \ge 0$



$\hookrightarrow$ true because $f(x) = x^c$ is increasing concave

Def: For $\alpha \in \mathbb{Q}$ and p prime, the canonical p-adic norm is

$$\|\alpha\|_p = p^{-\operatorname{ord}_p(\alpha)}$$ ...... we set $c = \frac{1}{p}$ in the general p-adic norm

OSTROVSKI THEOREM

Lemma 1: For $c>0$ is $|\cdot|^c$ a norm (on $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) $\iff c \le 1$ ... $c \in (0,1]$

Lemma 2: Let $\|\cdot\|$ be a non-trivial norm on $\mathbb{Q}$. Then $\exists n \in \mathbb{N}$ s.t. $n \ge 2$ & $\|n\| \ne 1$.

Lemma 3: For every coprime (gcd = 1) $a, b \in \mathbb{Z}$ exist $c, d \in \mathbb{Z}$ s.t. $ac + db = 1$.

Theorem (Ostrovski): Let $\|\cdot\|$ be a norm on $\mathbb{Q}$. Then one of the following holds:

1. $\|\cdot\|$ is a trivial norm
2. $\exists c \in (0,1]$ s.t. $\|x\| = |x|^c$ ... modified absolute value
3. $\exists c \in (0,1)$ s.t. $\|x\| = |x|_p = c^{\operatorname{ord}_p(x)}$ ... p-adic norm

Proof: Let $\|\cdot\|$ be non-trivial, then by lemma 2 $\exists n \in \mathbb{N}$, $n \ge 2$ and $\|n\| \ne 1$.

$\Rightarrow$ either (1): $\exists n \in \mathbb{N}$, $n \ge 2$ s.t. $\|n\| > 1$. (2) $\forall n \in \mathbb{N}: \|n\| \le 1$ & $\exists n \ge 2: \|n\| = 0$.

(1) $\exists n \in \mathbb{N}$, $n \ge 2$ s.t. $\|n\| > 1$ $\Rightarrow$ let $n_0 \ge 2$ be the smallest such n

⊛1 ☼ $m \in \mathbb{N}$ & $1 \le m < n_0 \Rightarrow \|m\| \le 1$ ...

⊛2 ☼ because $n_0 \ge 2$ & $\|n_0\| > 1$: $\exists$ unique $c > 0$ s.t. $\|n_0\| = n_0^c$

We can write any $n \in \mathbb{N}$ in base-n_0 as

$$n = a_0 + a_1 n_0 + a_2 n_0^2 + \dots + a_s n_0^s, \quad a_i \in [n_0 - 1],\ a_s \ne 0$$

$$\Rightarrow \|n\| = \left\| \sum a_i n_0^i \right\| \le \sum \|a_i \cdot n_0^i\| = \sum \|a_i\| \cdot \|n_0^i\| = \sum_{i=0}^{s} \|a_i\| \cdot \|n_0\|^i$$

$$\overset{⊛1}{\le} \sum_{i=0}^{s} \|n_0\|^i \overset{⊛2}{=} \sum_{i=0}^{s} n_0^{c \cdot i} = n_0^0 + n_0^c + n_0^{2c} + \dots + n_0^{sc} \quad ⊛2$$

$$= n_0^{sc} \left(\frac{1}{n_0^{sc}} + \frac{1}{n_0^{(s-1)c}} + \dots + \frac{1}{n_0^c} + \frac{1}{n_0^0} \right) = n_0^{sc} \sum_{i=0}^{s} \left(\frac{1}{n_0^c} \right)^i \le n_0^{sc} \sum_{i=0}^{\infty} \left(\frac{1}{\|n_0\|} \right)^i$$

$$\overset{n \ge n_0^s}{\le} n^c \cdot A, \quad A = \sum_{i=0}^{\infty} \left(\frac{1}{\|n_0\|} \right)^i = \frac{1}{1 - \frac{1}{\|n_0\|}} = \frac{\|n_0\|}{\|n_0\| - 1} \ge 1 \because \|n_0\| > 1$$

Goal: show $\|n\| = n^c$ for $n \in \mathbb{N}$

→ then from multiplicativity of norms we have $\frac{a}{b} \in \mathbb{Q}: \left\|\frac{a}{b}\right\| = \frac{\|a\|}{\|b\|} = \left(\frac{a}{b}\right)^c$

$\Rightarrow$ by lemma 1 then $c \in (0,1]$ and therefore $\|\cdot\|$ is modified absolute value

Claim $\|n\| \le n^c$: we know $\|n\| \le n^c \cdot A$, $A \ge 1$ for any $n \in \mathbb{N}$

$\Rightarrow$ let $m \in \mathbb{N}$: $\|n\|^m = \|n^m\| \le (n^m)^c \cdot A = (n^c)^m \cdot A$

→ taking m-th root: $\|n\| \le (n^c) \cdot \sqrt[m]{A}$

→ now we take the limit transition $m \to \infty$: $\sqrt[m]{A} \to 1 \Rightarrow \|n\| \le n^c$ ⊛3 claim: $\|m\| \geq m^c$

→ again consider expansion of m in base m_0

$$m = a_0 + a_1 m_0 + a_2 m_0^2 + \dots + a_s m_0^s \quad , \quad a_i \in [m_0 - 1]$$

$m_0^s \leq m < m_0^{s+1}$ ⊛4 ... $m_0 = 10$, $m = 1293$: $1000 \leq 1293 < 10000$

$\|m_0\|^{s+1} = \|m_0^{s+1}\| = \|m + (m_0^{s+1} - m)\| \leq \|m\| + \|m_0^{s+1} - m\|$

$\Rightarrow \|m\| \geq \|m_0\|^{s+1} - \|m_0^{s+1} - m\|$

⊛2: $\|m_0\| = m_0^c \Rightarrow \|m_0\|^{s+1} = m_0^{(s+1)c}$

⊛3: $\|m_0^{s+1} - m\| \leq (m_0^{s+1} - m)^c$

$$\geq m_0^{(s+1)c} - (m_0^{s+1} - m)^c$$

$$\overset{⊛4}{\geq} m_0^{(s+1)c} - (m_0^{s+1} - m_0^s)^c = m_0^{(s+1)c} - \left[m_0^{s+1}\left(1 - \frac{1}{m_0}\right)\right]^c$$

$$= m_0^{(s+1)c}\left[1 - \left(1 - \frac{1}{m_0}\right)^c\right] = m_0^{(s+1)c} \cdot B \overset{⊛4}{\geq} m^c \cdot B$$

$\Rightarrow$ we have: $\|m\| \geq m^c \cdot B$, $B = 1 - \left(1 - \frac{1}{m_0}\right)^c > 0$ ∵ $m_0 \geq 2 \Rightarrow 1 - \frac{1}{m_0} \in \left[\frac{1}{2}, 1\right)$

→ again we use the trick with the m-th roots to get

$\|m\| \geq m^c \cdot \sqrt[m]{B}$, $m \to \infty$ we get $\sqrt[m]{B} \to 1$

Therefore $\|m\| \leq m^c$ & $\|m\| \geq m^c \Rightarrow \|m\| = m^c$ and $\|\cdot\|$ is modified abs.-value..

(2) $\forall m \in \mathbb{N}: \|m\| \leq 1$ & $\exists m \in \mathbb{N}: m \geq 2$, $\|m\| < 1$ ⇒ let $m_0 \geq 2$ be min. such m

claim: $m_0 = p$ is a prime number and $\|\cdot\|$ is a p-adic norm

↳ if $m_0 = a \cdot b$, $1 < a, b < m_0$, then $1 > \|m_0\| = \|a \cdot b\| = \|a\| \cdot \|b\| = 1 \cdot 1 = 1$ ↯

know: $\|p\| =: C \in (0,1)$ ... $p \neq 0 \Rightarrow \|p\| \neq 0$ & $\|p\| < 1$

claim: $\|x\|$ is the p-adic norm $C^{\mathrm{ord}_p(x)}$

- $q \neq p$ prime: we need $\|q\| = C^{\mathrm{ord}_p(q)} = C^0 = 1$

  ↳ because $q \in \mathbb{N}$: $\|q\| \leq 1$ ... suppose $\|q\| < 1$

  → take large $m \in \mathbb{N}$ s.t. $\|p\|^m, \|q\|^m < \frac{1}{2}$

  → $a \in \mathbb{N}: \|a\| \leq 1$

  → $-a \in \mathbb{N}: \|a\| = \|-a\| \leq 1$

  Lemma 3: $\exists a, b \in \mathbb{Z}$ s.t. $a \cdot q^m + b \cdot p^m = 1$ ... Note: $a \in \mathbb{Z} \Rightarrow \|a\| \leq 1$

  $\Rightarrow 1 = \|1\| = \|a q^m + b p^m\| \overset{\Delta\text{-ineq}}{\leq} \|a\| \cdot \|q\|^m + \|b\| \cdot \|p\|^m < 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$ ↯

- $x \in \mathbb{Q}$ not prime: we need $\|x\| = C^{\mathrm{ord}_p(x)}$, $C = \|p\|$

$x = q_1 \cdot q_2 \cdots q_m$ ⟨ $q_i = p \Rightarrow \|q_i\| = C$; $q_i \neq p \Rightarrow \|q_i\| = 1$ ⟩ $\|x\| = \|q_1\| \cdot \|q_2\| \cdots \|q_m\| = \underline{C^{\mathrm{ord}_p(x)}}$ ↱ $\# q_i = p$

▨

- Metric Spaces Revision

Def: We say that $(a_n) \subset (X,d)$ has the limit $a \in X$ $\equiv$

$(\forall \varepsilon > 0)(\exists n_0): n \geq n_0 \Rightarrow d(a_n, a) < \varepsilon$

Notation: We write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ $\rightarrow B(a,\varepsilon) = \{x \in X \mid d(a,x) < \varepsilon\}$

Def: $M \subseteq (X,d)$ is open in $X \equiv (\forall a \in M)(\exists \varepsilon > 0): B(a,\varepsilon) \subseteq M$.

Def: $M \subseteq (X,d)$ is closed in $X \equiv \forall (a_n) \subset M : a_n \to a \Rightarrow a \in M$

$\emptyset$ and X are both open & closed in X

Proposition: M is open/closed in $X \Leftrightarrow X \setminus M$ is closed/open in X.

Def: $M \subseteq (X,d)$ is bounded in $X \equiv (\exists a \in M)(\exists r > 0): M \subseteq B(a,r)$

Def: $M \subseteq (X,d)$ is compact in $X \equiv$ every sequence of M has a convergent subsequence

$(\forall (a_n) \subset M): \exists (a_m) \subseteq (a_n) : a_m \to a \in M$

Theorem (Bolzano–Weierstrass): Closed bounded intervals $[a,b] \subseteq \mathbb{R}$ are compact.

In every metric space, every finite set is compact.

A, B compact $\Rightarrow A \cap B$ and $A \cup B$ compact

A, B open $\Rightarrow A \cap B$ and $A \cup B$ open

A, B closed $\Rightarrow A \cap B$ and $A \cup B$ closed

A, B bounded $\Rightarrow A \cap B$ and $A \cup B$ bounded

$M \subseteq (X,d)$ compact & $A \subseteq M$ closed $\Rightarrow A$ compact in X.

Theorem: $M \subseteq (X,d)$ compact $\Rightarrow M$ bounded & closed

Theorem: $M \subseteq \mathbb{E}_n$ compact $\Leftrightarrow M$ bounded & closed

Pf: $\Leftarrow$: Enclose M into a brick $[-a,a]^n$ ... M is bounded

$\hookrightarrow [-a,a]$ is compact & A, B compact $\Rightarrow A \times B$ compact $\Rightarrow [-a,a]^n$ compact

$\rightarrow M \subseteq [-a,a]^n$ is closed $\rightarrow$ compact ▨

<u>Proposition</u>: $M \subseteq (X,d)$, $A \subseteq M$ is open / closed in M

$\iff \exists \tilde{A} \subseteq X$ open / closed in X & $A = \tilde{A} \cap M$

↳ proof in excersizes wiko 3

<u>Proposition</u>: $M \subseteq (X,d)$, $A \subseteq M$ is closed in $X \Rightarrow A$ is closed in M

$A \subseteq M$ is compact in $X \iff A$ is compact in M

<u>Def</u>: For m.s. $(X_1, d_1), \dots (X_n, d_n)$ define their product

$$\prod_i (X_i, d_i) = (X_1, d_1) \times \dots \times (X_n, d_n) := \left(\prod_i X_i, d\right)$$

where $\prod X_i = X_1 \times \dots \times X_n$ and $d(\tilde{x}, \tilde{y}) := \max_i d_i(x_i, y_i)$.

<u>Def</u>: Metrics d_1, d_2 on X are strongly equivalent $\equiv$

$$\exists \alpha, \beta > 0: \ \alpha \cdot d_1(x,y) \le d_2(x,y) \le \beta \cdot d_1(x,y)$$

<u>Fact</u>: So called topological properties are preserved under changing equivalent metrics

- convergence of sequences
- continuity of functions
- open / closed / bounded / compact sets

etc.

<u>Theorem</u>: The following matrices are strongly equivalent.

- Euclidean: $e_n(\tilde{x}, \tilde{y}) := \sqrt{\sum_i |x_i - y_i|^2}$ ... L^2
- Manhattan: $d_s(\tilde{x}, \tilde{y}) := \sum_i |x_i - y_i|$ ... L^1
- Chebyshev: $d_m(\tilde{x}, \tilde{y}) := \max_i |x_i - y_i|$ ... L^∞
- general L^p metric: $L^p(\tilde{x}, \tilde{y}) := \left(\sum_i |x_i - y_i|^p\right)^{\frac{1}{p}}$, $p \ge 1$

<u>Pf</u>: We will show that Chebyshev is equivalent to L^p for any p

1. $\max_i |x_i - y_i| \le \left(\sum_i |x_i - y_i|^p\right)^{\frac{1}{p}} \le \beta \cdot \max_i |x_i - y_i| \quad \Rightarrow \alpha = 1, \ \beta = n^{\frac{1}{p}}$

$$\left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} d_m(\tilde{x}, \tilde{y})^p\right)^{\frac{1}{p}} = \left(n \cdot (d_m(\tilde{x}, \tilde{y}))^p\right)^{\frac{1}{p}} = n^{\frac{1}{p}} \cdot d_m(\tilde{x}, \tilde{y}) \quad \blacksquare$$

<u>Corollary</u>: In the definition of a product space, we could have used any L^p metric.

↳ Euclidean: $d(\tilde{x}, \tilde{y}) := \sqrt{\sum_i |d_i(x_i, y_i)|^2}$

$\mathbb{E}_n \times \mathbb{E}_m = \mathbb{E}_{n+m}$ ... $(\mathbb{R}^n, e_n) \times (\mathbb{R}^m, e_m) = (\mathbb{R}^{n+m}, e_{n+m})$

↳ equal up to a formality → vectors of $\mathbb{E}_{n+m}$ are $v \in \mathbb{R}^{n+m}$ $(x_1, \dots, x_{n+m})$

↳ vectors of $\mathbb{E}_n \times \mathbb{E}_m$ are $v \in \mathbb{R}^n \times \mathbb{R}^m$ $((x_1 \dots x_n)(x \dots x))$

Theorem: The product of compact spaces is compact. $(X, d_X) \times (Y, d_Y)$

Pf: We have $(a_n) \subset X \times Y$ and want a convergent subsequence

1. pick $(a_{n_k})_k \subset (a_n)_n$ s.t. the x-part $(x_{n_k})_k$ converges in X
2. pick $(a_{n_{k_j}})_j \subset (a_{n_k})_k$ s.t. the y-part $(y_{n_{k_j}})_j$ converges in Y ■

- Continuous functions

Def: The function $f: (X, d_X) \to (Y, d_Y)$ is continuous at $a \in X \equiv$

$$(\forall \varepsilon > 0)(\exists \delta > 0) : (\forall x \in X): d_X(a, x) < \delta \Rightarrow d_Y(f(a), f(x)) < \varepsilon$$

Def: f is continuous $\equiv$ it is continuous at $\forall a \in X$.

Def: For $f: X \to Y$ define

- image of $A \subseteq X$: $f[A] := \{f(x) \in Y \mid x \in A\}$
- preimage of $B \subseteq Y$: $f^{-1}[B] := \{x \in X \mid f(x) \in B\}$

$f: (X, d_X) \to (Y, d_Y)$ is continuous $\Leftrightarrow$

$$(\forall a \in X)(\forall \varepsilon > 0)(\exists \delta > 0): f[B_X(a, \delta)] \subseteq B_Y(f(a), \varepsilon)$$

Theorem (Heine): $f: (X, d_X) \to (Y, d_Y)$ is continuous at $a \in X$ $\Leftrightarrow$

$\forall (a_n) \subset X$, $a_n \to a \Rightarrow f(a_n) \subset Y$, $f(a_n) \to f(a)$ $\quad \lim_{n\to\infty} f(a_n) = f(\lim_{n \to \infty} a_n)$

Idea: The image of a convergent sequence converges to the image of the limit

Pf: $\Rightarrow$: claim: $(\forall \varepsilon > 0)(\exists n_0): (n \geq n_0 \Rightarrow d_Y(f(a_n), f(a)) < \varepsilon)$

↳ continuity of f for this ε: $(\exists \delta > 0)$: $d_X(a_n, a) < \delta \Rightarrow d_Y(f(a_n), f(a)) < \varepsilon$

↳ convergence of (a_n) for this δ: $(\exists n_0): (n \geq n_0 \Rightarrow d_X(a_n, a) < \delta)$

$\Leftarrow$: for contradiction: $\exists \varepsilon > 0$ s.t. $(\forall \delta)(\exists x \in X)$: $d_X(x, a) < \delta$ & $d_Y(f(x), f(a)) \geq \varepsilon$

↳ for every $\delta = \frac{1}{n}$ we have x_n $\Rightarrow$ create sequence

$x_n \to a$ but $f(x_n) \not\to f(a)$ ■

Lemma: Let $f:(X,d_X)\to(Y,d_Y)$ be continuous. Then

$$A\subseteq X \text{ compact} \Rightarrow f[A]\subseteq Y \text{ compact}$$

Pf: Let $(y_n)\subset f[A]$ we want a convergent subsequence with $\lim \in f[A]$

↳ define $(x_n)\subset A$ s.t. $f(x_n)=y_n$

↳ A compact $\Rightarrow$ we have $(x_{n_k})_k\subseteq(x_n)_n$ with limit $a\in A$

→ f continuous so using Heine: $f(x_{n_k})=(y_{n_k})_k$ converges to $f(a)\in f[A]$ ∎

Theorem: Let $f:(X,d)\to\mathbb{R}$ be continuous and $A\subseteq X$ compact.

Then f attains both a min. and max. value on A.

Pf: $f[A]\subseteq\mathbb{R}$ is compact $\Rightarrow$ bounded $\Rightarrow$ has a infimum $m\in\mathbb{R}$ and sup. $M\in\mathbb{R}$

↳ we need to show $m,M\in f[A]$ ... lets show $M\in f[A]$

→ $M=\sup f[A] \Rightarrow \exists(a_n)\subset A$ s.t. $\lim_{n\to\infty} f(a_n)=M$

→ A is compact $\Rightarrow \exists(a_{n_k})_k\subseteq(a_n)_n$ s.t. $(a_{n_k})_k\to a\in A$

→ f is continuous $\Rightarrow f(a_{n_k})\to f(a)\in f[A]$

→ but $f(a_n)_n$ converges to M & $f(a_{n_k})_k$ is a subsequence of $f(a_n)_n$

$\Rightarrow \lim_{k\to\infty} f(a_{n_k})=\lim_{n\to\infty} f(a_n) \Rightarrow M=f(a)\in f[A]$ ∎

Theorem: Equivalent definitions of continuity of $f:(X,d_X)\to(Y,d_Y)$

① $(\forall a\in X)(\forall\varepsilon>0)(\exists\delta>0): f[B(a,\delta)]\subseteq B(f(a),\varepsilon)$ ... $\varepsilon-\delta$ definition

② $\forall(a_n)\subset X,\ a_n\to a:\ f(a_n)\to f(a)$ ... Heine's definition

③ $\forall B\subseteq Y$ open: $f^{-1}[B]$ open ... topological definition

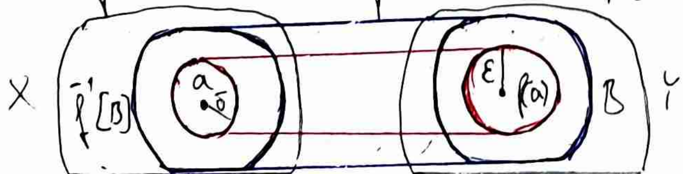
④ $\forall B\subseteq Y$ closed: $f^{-1}[B]$ closed

Pf: We know ① $\Leftrightarrow$ ②, lets show ① $\Leftrightarrow$ ③.

$\Rightarrow$: Let $B\subseteq Y$ open and $a\in f^{-1}[B]$. We need to find $\delta>0$ s.t. $B(a,\delta)\subseteq f^{-1}[B]$.

↳ B open & $f(a)\in B \Rightarrow \exists\varepsilon>0$ s.t. $B(f(a),\varepsilon)\subseteq B$

↳ f continuous $\Rightarrow$ for this $a\in X$, $\varepsilon>0$ $\exists\delta>0$ s.t. $f[B(a,\delta)]\subseteq B(f(a),\varepsilon)\subseteq B$



↳ $f[B(a,\delta)]\subseteq B$

$\Rightarrow B(a,\delta)\subseteq f^{-1}[B]$

$\Leftarrow$: Let $a \in X$, $\varepsilon > 0$, we need $\delta > 0$ s.t. $f[B(a,\delta)] \subseteq B(f(a), \varepsilon)$

$\to$ the open ball $B(f(a), \varepsilon)$ is open : (no shit Sherlock)

$\Rightarrow$ using ③ we have $f^{-1}[B(f(a), \varepsilon)] =: A$ open as well

$a \in A$ & A open $\Rightarrow \exists \delta > 0: B(a, \delta) \subseteq A$

$\Rightarrow$ therefore $f[B(a,\delta)] \subseteq f[A] = B(f(a), \varepsilon)$ ■

<u>Proposition</u>: Let (X, d_X) and (Y, d_Y) be m.s. and $M \subseteq X$. Then

$f: (M, d_X) \to (Y, d_Y)$ is <u>continuous</u> $\Leftrightarrow$

$(\forall B \subseteq Y \text{ open})(\exists A \subseteq X \text{ open}): f^{-1}[B] = M \cap A$

Pf: Recall $R \subseteq M \subseteq X$ open in $M \Leftrightarrow \exists \tilde{R} \subseteq X$ open in X s.t. $R = \tilde{R} \cap M$.

$\therefore$ $f^{-1}[B] \subseteq M \subseteq X$ open in $M \Leftrightarrow \exists A \subseteq X$ open in X s.t. $f^{-1}[B] = A \cap M$

$\Rightarrow$ we are essentially saying: $(\forall B \subseteq Y \text{ open}): (f^{-1}[B] \subseteq M \text{ open in } M)$

↳ topological definition of continuity ■

<u>Proposition</u>: Let (X, d_X) and (Y, d_Y) be m.s. and $M \subseteq X$ be <u>compact</u>.

And let $f: (M, d_X) \to (Y, d_Y)$ be <u>continuous</u> and <u>injective</u>.

Then the <u>inverse</u> $f^{-1}: (f[M], d_Y) \to (M, d_X)$ is also <u>continuous</u>.

Pf: We will use the topological definition of continuity ④.

↳ f^{-1} continuous $\Leftrightarrow$

$(\forall A \subseteq M \text{ closed in } M): (f^{-1})^{-1}[A] = f[A]$ is closed in $f[M]$

$\to$ M is compact & $A \subseteq M$ closed $\Rightarrow$ A is compact in M

$\to$ continuous image of a compact set is compact $\to$ $f[A]$ is compact in $f[M]$

$\to$ every compact set is closed $\Rightarrow$ $f[A]$ is closed in $f[M]$ ■

Def: $f: (X, d_X) \to (Y, d_Y)$ is a <u>homeomorphism</u> $\equiv$

① f is a bijection

② f and f^{-1} are continuous

Def: (X, d_X) and (Y, d_Y) are <u>homeomorphic</u> $\equiv \exists$ homeomorphism $f: X \to Y$.

Def: Metrics d_1, d_2 are <u>equivalent</u> on $X \equiv id: (X, d_1) \to (X, d_2)$, $x \mapsto x$ is a homomorphism.

<u>Proposition</u>: <u>Strongly equivalent</u> metrics are <u>equivalent</u>.

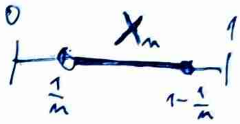
- <u>The Heine-Borel Theorem</u>

<u>Def</u>: Let $A \subseteq (M,d)$. We call $\{X_i\}_{i\in I}$ an <u>open cover</u> of $A \equiv A \subseteq \bigcup_{i\in I} X_i$ → <u>all X_i open</u>

<u>Def</u>: $A \subseteq (M,d)$ is <u>topologically compact</u> $\equiv$ <u>every open cover</u> $\{X_i\}_{i\in I}$ of A <u>has a finite subcover</u> $\{X_j\}_{j\in J}$, $J \subseteq I$ finite.

<u>Example</u>:

- $[0,1]$ is topologically compact ... we can always create a finite subcover
- $(0,1)$ is not topologically compact



⇒ consider sets $\{X_n\}_{n\in\mathbb{N}}$, $X_n = \left(\frac{1}{n}, 1-\frac{1}{n}\right)$

→ clearly $\bigcup_{n\in\mathbb{N}} X_n = (0,1)$, but there is no finite subcover

<u>Theorem (Heine-Borel)</u> Let $A \subseteq (M,d)$. Then: <u>A compact</u> $\iff$ <u>A top. compact</u>.

<u>Proof</u>: <u>WLOG assume $A = M$.</u> ⊛

<u>⇒</u>: let (M,d) be compact and let $M = \bigcup X_i$ be its open cover.

<u>goal</u>: find finite subcover

<u>Lemma</u>: For $\forall \delta > 0$ $\exists$ <u>finite cover</u> of M with open <u>balls of radius δ</u>

↳ $\exists$ finite $S_\delta \subseteq M$ s.t. $\bigcup_{a\in S_\delta} B(a,\delta) = M$

<u>Pf</u>: Suppose that this doesn't hold for some $\delta > 0$, that is

for $\forall$ finite $S \subseteq M$: $M - \left(\bigcup_{a\in S} B(a,\delta)\right) \neq \emptyset$

<u>Claim</u>: $\exists (a_n) \subset M$ s.t. $m < n \Rightarrow d(a_m, a_n) \geq \delta$

⇒ ↯ with compactness⊛ – there is no convergent subsequence

↳ choose $a_1 \in M$ arbitrarily. Now suppose we already have $a_1, \dots a_n$ and we will choose a_{n+1}

⇒ let $S := \{a_1, \dots a_n\}$ and choose $a_{n+1} \in M - \left(\bigcup_{a\in S} B(a,\delta)\right) \neq \emptyset$ ▨

<u>Lemma</u>: Suppose $\{X_i\}_{i\in I}$ doesn't have a finite subcover of M. Then for $\forall n \in \mathbb{N}$:

$\exists b_n \in S_{\frac{1}{n}}$ s.t. $\forall i \in I$: $B(b_n, \frac{1}{n}) \nsubseteq X_i$ → there is a ball which isn't completely covered by any X_i

<u>Pf</u>: Suppose that this doesn't hold for some $n_0 \in \mathbb{N}$ and that for $\forall b \in S_{\frac{1}{n_0}}$ $\exists i_b \in I$ s.t. $B(b, \frac{1}{n_0}) \subseteq X_{i_b}$. But then consider the finite ball cover

$M = \bigcup_{b\in S_{\frac{1}{n_0}}} B(b, \frac{1}{n_0})$ ⇝ $J := \{i_b \mid b \in S_{\frac{1}{n_0}}\}$ ⇒ $\{X_j\}_{j\in J}$ finite subcover ↯ ▨

Now to show "$\Longrightarrow$" suppose $\{X_i\}$ doesn't have a finite subcover, so by the lemma we have a sequence $(b_n) \subset M$ of ball centers.

$\to$ M compact $\to$ (b_n) has a convergent subsequence $(b_k) \subset (b_n)$

$\hookrightarrow$ denote the limit $b := \lim_{k\to\infty} b_k \in M$

$\to$ because $\{X_i\}$ cover M, $\exists j$ s.t. $b \in X_j$

$\to$ X_j open $\Rightarrow \exists\, r>0$ s.t. $B(b,r) \subseteq X_j$ ⊛ ... near a contradiction with the property of b_n ... lemma

$\Rightarrow$ we will take a large enough k s.t.

$\frac{1}{k} < \frac{r}{2}$ & $d(b, b_k) < \frac{r}{2}$

← by the definition of b_n

$\to$ to get the contradiction, consider the ball $B(b_k, \frac{1}{k}) \not\subseteq X_j$

$\hookrightarrow$ $x \in B(b_k, \frac{1}{k})$: $d(x,b) \le \underbrace{d(x,b_k)}_{\le \frac{1}{k} < \frac{r}{2}} + \underbrace{d(b_k,b)}_{< \frac{r}{2}} < \frac{r}{2} + \frac{r}{2} = r$

$\Rightarrow$ therefore $B(b_k, \frac{1}{k}) \subseteq B(b,r) \overset{⊛}{\subseteq} X_j$ $\lightning$

Meaning that there has to be a finite subcover of $\{X_i\}$

$\Longleftarrow$: let (M,d) be top. compact and let $(a_n) \subset M$. We want convergent $(a_k) \subset (a_n)$.

Lemma: Let $(a_n) \subset M$. Then $\exists\, b \in M$ s.t. $\forall r>0$: $A_r := (a_n) \cap B(b,r)$ is infinite.

Corollary: We can clearly select a subsequence $(a_{n_k})_k \subset (a_n)_n$ with limit $= b$.

$\to$ we will choose $\forall\, n_k$ s.t. $d(b, a_{n_k}) < \frac{1}{k}$

$\hookrightarrow$ suppose we already have $a_{n_1}, a_{n_2}, \dots a_{n_k}$ and now we want $a_{n_{k+1}}$

$\to$ pick any $a_i \in A_{\frac{1}{k+1}}$ s.t. $i > n_k$ ... we can $\because$ $A_{\frac{1}{k+1}}$ is infinite

$\Rightarrow$ M is compact

$\hookrightarrow$ Proof: Suppose $(\forall b \in M)(\exists\, r_b > 0)$ s.t. $A_b := (a_n) \cap B(b, r_b)$ is finite

$\to$ now consider the ball covering (we take every point of M as a center)

$M = \bigcup_{b\in M} B(b, r_b)$ and choose a finite subcovering (M top. compact)

given by the finite ball center set $N \subseteq M$

$\Rightarrow M = \bigcup_{b\in N} B(b, r_b) =: NB$

$\to$ consider what happens with A_b ... $\overbrace{\bigcup_{b\in N} A_b}^{\left(\bigcup_{b\in N} B(b, r_b)\right) \cap (a_n)}$ is finite $\because$ finite union of finite sets

$\Rightarrow$ but (a_n) is infinite $\Rightarrow \exists\, a_j \in (a_n)$ s.t. $a_j \notin NB = M$ $\lightning$

$\hookrightarrow$ and only finitely many a_i are covered by NB ■

Connected sets

Def: $M \subseteq (X,d)$ is <u>arc connected</u> $\equiv \forall a, b \in M$

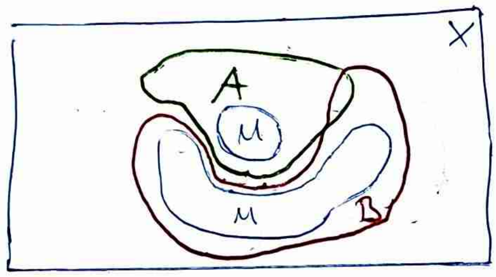
$$\exists \varphi : [\alpha, \beta] \to M \text{ continuous s.t. } \varphi(\alpha) = a, \ \varphi(\beta) = b.$$

Def: $M \subseteq (X,d)$ is <u>clopen</u> $\equiv$ it is closed and open

$\emptyset$ and X are clopen $\leftarrow$ <u>trivial clopen sets</u>

Def: $M \subseteq (X,d)$ is <u>connected</u> $\equiv$ it cannot be cut, meaning

$$\nexists A, B \subseteq X \text{ open s.t. } M \subseteq A \cup B \ \& \ \begin{matrix} M \cap A \neq \emptyset \\ M \cap B \neq \emptyset \end{matrix} \ \& \ M \cap A \cap B = \emptyset$$



$\to$ M is disconnected

$\to$ cut by A, B

A, B can intersect outside M

Theorem: (X,d) is <u>connected</u> $\iff$ <u>the only clopen subsets are $\emptyset$ and X</u>

$M \subseteq (X,d)$ is connected $\iff$ the only clopen subsets of (M,d) are $\emptyset$ and M.

Equivalently: (X,d) is <u>connected</u> $\iff \nexists$ nonempty clopen $A, B \subseteq X$ s.t.

$$X = A \dot{\cup} B \ \& \ A \cap B = \emptyset$$

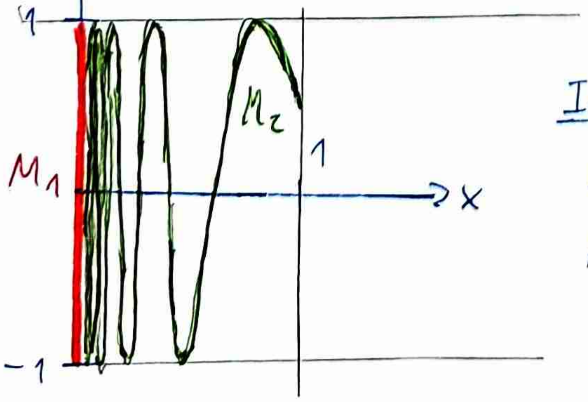
Example: $A \subseteq \mathbb{R}$ is <u>connected</u> $\iff$ it is a <u>closed interval</u> ... $A = [\alpha, \beta]$

Theorem: M <u>arc connected</u> $\Rightarrow$ M <u>connected</u>

M connected $\not\Rightarrow$ M arc connected

M <u>connected & open</u> $\Rightarrow$ M <u>arc connected</u>

Example: $M = \underbrace{\{0\} \times [-1,1]}_{M_1} \cup \underbrace{\{(t, \sin \tfrac{1}{t}) \mid 0 < t \leq 1\}}_{M_2} \subseteq \mathbb{R}^2$



Intuition:

M is connected

M is not arc connected

Theorem: Let $(X,d_1),(Y,d_2)$ be MS and $f: M\subseteq X \to Y$ be continuous.

M connected $\Rightarrow$ $f[M]\subseteq Y$ connected

Example: The complex unit circle is connected

$S:=\{z\in\mathbb{C} \mid |z|=1\}$ $\Rightarrow$ $f:[0,2\pi]\to\mathbb{C}$, $f(t)=\cos t + i\sin t$

f is continuous and $f\big[[0,2\pi]\big]=S$

$\Rightarrow$ because $[0,2\pi]$ is connected, so is $f\big[[0,2\pi]\big]=S$

The Fundamental Theorem of Algebra

Lemma 1: $(\forall x>0)(\forall n\in\mathbb{N})(\exists y>0): x=y^n$ $\to y=\sqrt[n]{x}$

we can deduce explicit formulas

Lemma 2: $(\forall z\in\mathbb{C})(\exists u\in\mathbb{C}): z=u^2$ $\to z=r\cdot e^{i\theta} \Rightarrow u_1=\sqrt{r}\cdot e^{i\frac{\theta}{2}}$

$z=r\,e^{i(\theta+2\pi)} \Rightarrow u_2=\sqrt{r}\cdot e^{i(\frac{\theta}{2}+\pi)}$

unit circle

Lemma 3: $(\forall z\in S)(\forall n \text{ odd})(\exists u\in S): z=u^n$

Pf: Let z be on the unit circle, we do not want to use sin and cos, because they are transcendental functions, so $z=re^{i\theta} \Rightarrow u=\sqrt[n]{r}\,e^{i\frac{\theta}{n}}$ is forbidden - using too complex tools.

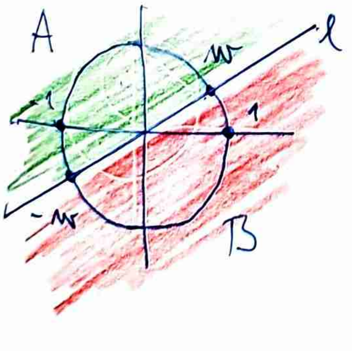
$\Rightarrow$ instead let $z\in S$, n be odd and we will show that the map

$f:S\to S$, $f(x)=x^n$, which is clearly continuous, is surjective

$\Rightarrow$ therefore $f^{-1}(z)$ is the u we are looking for

$\to$ for contradiction assume $\exists w\in S\setminus f[S]$ ... w has no n-th root

$\to$ because n is odd, we have $-w\in S\setminus f[S]$ ... if $-w=f(v)$, then $w=f(-v)$



$\Rightarrow$ consider the line $\ell\subseteq\mathbb{C}$ going through w and $-w$

ℓ divides $\mathbb{C}$ $\rightsquigarrow$ $\mathbb{C}=A\cup\ell\cup B$, A,B open halfplanes

claim: A and B cut $f[S]$ and make it disconnected

- $f[S]\subseteq A\cup B$ ... true $\because$ $f[S]\subseteq S$ & $S\cap(A\cup B)=S\setminus\{w,-w\}$
- $(\pm1)^n=(\pm1) \Rightarrow \pm1\in f[S]$ & ±1 lie in different halfplanes

  $\Rightarrow f[S]\cap A\neq\emptyset\neq f[S]\cap B$
- $f[S]\cap A\cap B=\emptyset$ $\because$ $A\cap B=\emptyset$

$\to$ but $f[S]$ is the continuous image of the connected set S, so $f[S]$ must also be connected $\lightning$

$\blacksquare$

Theorem: $\mathbb{C}$ contains all n-th roots

$$(\forall z \in \mathbb{C})(\forall n \in \mathbb{N})(\exists u \in \mathbb{C}): \ z = u^n$$

Proof: We will convert this to lemma 3. Note $z=0$ is trivial $\Rightarrow$ let $z \neq 0$

① WLOG n odd: if n even, then by lemma 2 exists $v \in \mathbb{C}$ s.t. $z = v^2$

$\Rightarrow$ we solve the problem for v and $\frac{n}{2}$ $\Rightarrow$ eventually odd n

② WLOG $z \in S$: if $z \notin S$, then $z = \alpha z'$, $z' \in S$, $\alpha \in \mathbb{R}^+$

$\left.\begin{array}{l} \rightarrow \text{we solve the problem for } z' = v^n \\ \rightarrow \text{by lemma 1 exists } \beta \in \mathbb{R}^+ \text{ s.t. } \alpha = \beta^n \end{array}\right\} z = \beta^n v^n = \underbrace{(\beta v)^n}_{u}$ ▨

Theorem (Fundamental Thm of Algebra): Every non-const. complex polynomial has a root.

Proof: Let $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$, $n \geq 1$, $a_n \neq 0$

The function $f(z) := |p(z)| : \mathbb{C} \to \mathbb{R}_0^+$ is continuous.

We will prove $f(u) = 0$ for some $u \in \mathbb{C}$, then also $p(u) = 0$ and u is a root of p

Plan: ① f attains a minimal value ② that minimum is 0

① let $K > 0$ be so large that

$$\frac{K^n |a_n|}{2} > |a_0| \quad \text{and} \quad \sum_{j=0}^{n-1} |a_j| K^{j-n} < \frac{|a_n|}{2}$$

Then for $\forall z \in \mathbb{C}$ we have the estimate that if $|z| > K$, then

$$f(z) = |p(z)| = |a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n| =$$

$$= |z^n| \cdot |a_n + a_0 \cdot z^{-n} + a_1 z^{1-n} + \dots + a_{n-1} z^{-1}| = |z^n| \cdot |a_n + \sum_{j=0}^{n-1} a_j z^{j-n}|$$

$|a_n + r| \geq |a_n| - |r|$

$$\geq |z|^n \left(|a_n| - \sum_{j=0}^{n-1} |a_j| \cdot |z|^{j-n}\right) > |z^n| \left(|a_n| - \frac{|a_n|}{2}\right) > |a_0|$$

Therefore $|z| > K$: $f(z) > |a_0| = |p(0)| = f(0)$

$\Rightarrow$ We define the rectangle $R = \{a + bi \mid -k \leq a, b \leq k\} \subseteq \mathbb{C}$.

- $z \in \mathbb{C} \setminus K \Rightarrow |z| > K \Rightarrow f(z) > f(0)$
- R is compact $\Rightarrow f$ attains a minima on $R \Rightarrow \exists u \in R$, $f(u)$ min. on R

↳ since $0 \in R$: $f(u) \leq f(0) \Rightarrow \forall z \in \mathbb{C}: f(u) \leq f(z) \Rightarrow f(u)$ min on $\mathbb{C}$

(2) <u>claim:</u> $f(u) = 0$ $\qquad\qquad\qquad\qquad$ $b_j \in \mathbb{C}$

→ we rewrite the polynomial $p(z)$ as $p(z) = \sum_{j=0}^{n} b_j (z-u)^j$, $b_n = a_n$

$f(u) = |p(u)| = |b_0|$ ... need $b_0 = 0$

→ suppose that $b \neq 0 \implies |b_0| > 0$

⟹ we find the first non-zero coefficient b_k in $p(z)$ ...

$$p(z) = b_0 + b_k (z-u)^k + \underbrace{b_{k+1}(z-u)^{k+1} + \dots + b_n (z-u)^n}_{q(z)}$$

$$b_0 \neq 0 \ , \ b_k \neq 0$$

We use the theorem about roots in $\mathbb{C}$ and take $\alpha \in \mathbb{C}$ s.t. $\alpha^k = -\frac{b_0}{b_k}$

$\lim_{z \to u} \frac{q(z)}{\frac{b_k}{2}(z-u)^k} = 0$ ... $q(z)$ has higher powers ⟹ goes to 0 quicker

Now we take $v = u + \delta \cdot \alpha$, $\delta > 0$ very small, so $v - u = \delta \cdot \alpha \to 0$

$$\implies |q(v)| < \frac{|b_k|}{2} |z - v|^k = \frac{|b_k|}{2} |\delta \alpha|^k = \frac{|b_k|}{2} \cdot \delta^k \cdot \left| -\frac{b_0}{b_k} \right| = \delta^k \cdot \frac{|b_0|}{2}$$

We get the contradiction that $f(v) < f(u)$:

$$f(v) = |p(v)| = |b_0 + b_k \alpha^k \delta^k + q(v)|$$

$\alpha^k b_k = -b_0$: $\quad = |b_0 - b_0 \delta^k + q(v)| = |b_0(1-\delta^k) + q(v)|$

Δ-ineq : $\quad \leq |b_0| \cdot (1 - \delta^k) + |q(v)|$

$$\leq |b_0| \cdot (1 - \delta^k) + \delta^k \frac{|b_0|}{2} = |b_0| \left(1 - \overbrace{\frac{\delta^k}{2}}^{\in (0,1)}\right) < |b_0|$$

⟹ we have $f(v) < |b_0| = f(u)$ ↯ $\qquad\qquad$ $\blacksquare$

Complete Spaces

Def: The sequence (a_n) is Cauchy $\equiv (\forall \varepsilon > 0)(\exists n_0): n, m \geq n_0 \Rightarrow d(a_n, a_m) < \varepsilon$

Every convergent sequence is Cauchy.

Not every Cauchy sequence is convergent

↳ $\mathbb{Q}$, $(a_n) = (1, 1.4, 1.41, 1.414, \dots) \rightsquigarrow \sqrt{2} \notin \mathbb{Q}$

Def: The metric (M,d) is complete $\equiv$ every Cauchy $(a_n) \subset M$ converges.

The set $X \subseteq M$ is complete $\equiv$ the subspace (X,d) is complete.

Properties:

1. $X \subseteq Y \subseteq (M,d)$: X complete in $(Y,d) \iff X$ complete in (M,d)
2. $(M_1, d_1), (M_2, d_2)$ complete $\Rightarrow (M_1, d_1) \times (M_2, d_2)$ complete
3. (M,d) complete & $X \subseteq M$ closed $\Rightarrow X$ complete

   ↳ $(a_n) \subset X$ Cauchy $\Rightarrow$ converges in M, but X is closed so converges to $\lim a_n = a \in X \Rightarrow X$ complete

4. (M,d) compact $\Rightarrow (M,d)$ complete

   ↳ proof in the back - ukol 4

5. (M,d) complete $\not\Rightarrow (M,d)$ compact ... take $\mathbb{R}$

   $\Rightarrow$ closed ... $\forall$ convergent (a_n) is Cauchy

   $\not\Rightarrow$ bounded ... $\mathbb{R}$

6. $\mathbb{R}$ is complete $\iff$ axiom of suprema
7. (M,d) finite $\Rightarrow (M,d)$ complete ... finite $\Rightarrow$ compact $\Rightarrow$ complete
8. M_1, M_2 complete $\Rightarrow M_1 \cap M_2$, $M_1 \cup M_2$ complete
   - countable union doesn't have to be complete
   - infinite intersection is always complete

Dense and Sparse Sets

Def: $X \subseteq (M,d)$ is <u>dense</u> in $M \equiv (\forall a \in M)(\forall r>0): B(a,r) \cap X \neq \emptyset$

$\rightarrow$ every ball in M intersects with X ... $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$

Def: $X \subseteq (M,d)$ is <u>sparse</u> in $M \equiv (\forall a \in M)(\forall R>0)$:

$(\exists b \in B(a,R))(\exists r>0): B(b,r) \subseteq B(a,R)$ & $B(b,r) \cap X = \emptyset$

$\rightarrow$ every ball in M contains a subball disjoined from X ... $\mathbb{Z}$ in $\mathbb{R}$

Properties: ! X not sparse $\not\Rightarrow$ X dense !

① $X \subseteq (M,d)$ <u>dense</u> $\iff$ $(\forall a \in M): \exists (x_n) \subset X$ s.t. $x_n \rightarrow a$

② $X \subseteq Y$ dense in Y & $Y \subseteq M$ dense in M $\Rightarrow$ X dense in M

③ $A, B \subseteq (M,d)$ sparse $\Rightarrow$ $A \cap B$, $A \cup B$ sparse

- infinite intersection is sparse ... the set doesn't get bigger
- infinite union doesn't have to be sparse

④ $A, B \subseteq (M,d)$ dense $\Rightarrow$ $A \cup B$ dense ... obviously

$\not\Rightarrow$ $A \cap B$ dense

$\rightarrow$ & A open $\rightarrow$ $A \cap B$ dense ... at least one needs to be open

Closure of a Set

Def: The <u>closure</u> of $M \subseteq (X,d)$ can be equivalently defined as

$\overline{M} := \bigcap \{A \subseteq X \mid A \text{ closed } \& \ M \subseteq A\}$ $\quad$ $M := \{x \in X \mid \exists (x_n) \subset M : (x_n) \rightarrow x\}$

$\overline{M} := \{x \in X \mid d(x,M) = 0\}$ $\quad$ $M := \{x \in X \mid \forall \varepsilon>0 : B(x,\varepsilon) \cap M \neq \emptyset\}$

Def: $d(x,M) := \inf\{d(x,a) \mid a \in M\}$, $d(A,B) := \inf\{d(a,b) \mid a \in A, b \in B\}$

Properties:

① A closed, B compact $\xRightarrow{\text{we need compactness ?}}$ $(d(A,B) = 0 \iff A \cap B \neq \emptyset)$

② $\overline{M}$ is closed ③ $M = \overline{M} \iff M$ closed ④ $\overline{\emptyset} = \emptyset$, $\overline{X} = X$

⑤ $\overline{\overline{M}} = \overline{M}$, $M \subseteq \overline{M}$, $\overline{A} \cap \overline{B} = \overline{A \cap B}$, $\overline{A} \cup \overline{B} = \overline{A \cup B}$

Baire's Theorem

Notation: $\overline{B}(a,r) := \overline{B(a,r)}$ ... closed ball

$r < R \Rightarrow \overline{B}(a,r) \subsetneq B(a,R)$

Theorem: Let (X, d_X), (Y, d_Y) be MSs and $M \subseteq X$ be dense.
Let $f, g: X \to Y$ be continuous and let $f|M = g|M$. Then $f = g$.

Intuition: The global behavior of a continuous function is fully determined by its behavior on any dense subset

$\Rightarrow$ if we know f on $\mathbb{Q}$, we know f on $\mathbb{R}$

Proof: Let $x \in X$ be arbitrary ... we want $f(x) = g(x)$.

M dense $\Rightarrow \exists (a_n) \subset M$ s.t. $\lim a_n = x$

Heine's def. of continuity: $f(x) = \lim f(a_n) = \lim g(a_n) = g(x)$ ∎

Recall: A countable union of sparse sets can be non-sparse ($\neq$ dense)

$\Rightarrow$ it is possible to escape sparsity

Baire: but it is impossible to reach completeness

Theorem (Baire): No complete metric space is a countable union of sparse sets.

(M,d) complete & $M = \bigcup_{i=1}^{\infty} X_i \Rightarrow \exists X_j$ not sparse

Proof: For contradiction assume that all X_i are sparse in M.

→ we will construct a sequence of nested closed balls $(\overline{B}_n)$ with centers $(c_n) \subset M$ converging to $a \in M$ s.t. $\forall i: a \notin X_i \Rightarrow a \notin \bigcup X_i$ ↯

→ let $B(b,1)$ be arbitrary

→ X_1 is sparse $\Rightarrow \exists$ subball $B(c_1, s_1) \subseteq B(b,1)$ disjoint from X_1

$\Rightarrow$ set $\overline{B}_1 := \overline{B}(c_1, r_1)$: where $r_1 = \min\left(\frac{s_1}{2}, \frac{1}{2}\right) \leq \frac{1}{2}$

$\overline{B}_1 \subsetneq B(c_1, s_1) \Rightarrow \overline{B}_1 \cap X_1 = \emptyset$

→ suppose that we have already defined the closed balls $\overline{B}_1 \supsetneq \overline{B}_2 \supsetneq \cdots \supsetneq \overline{B}_n$ s.t. for $\forall i$: $\overline{B}_i \cap X_i = \emptyset$ & $r_i \leq \frac{1}{2^i}$

→ since X_{n+1} is sparse $\Rightarrow \exists$ subball $B(c_{n+1}, s_{n+1}) \subseteq B(c_n, r_n)$ disjoint from X_{n+1}

$\Rightarrow$ define $\overline{B}_{n+1} := \overline{B}(c_{n+1}, r_{n+1})$ where $r_{n+1} := \min\left(\frac{s_{n+1}}{2}, \frac{1}{2^{n+1}}\right) \le \frac{1}{2^{n+1}}$

$\overline{B}_{n+1} \subsetneq B(c_{n+1}, s_{n+1}) \Rightarrow \overline{B}_{n+1} \cap X_{n+1} = \emptyset$ &

$\Rightarrow \overline{B}_{n+1} \subsetneq \overline{B}_n$

the sequence of the centers $(c_n) \subset M$ is Cauchy ← arbitrarily small

$m \ge n \Rightarrow \overline{B}_m \subsetneq \overline{B}_n \Rightarrow d(c_m, c_n) \le r_n \le \frac{1}{2^n}$

→ (M, d) is complete $\Rightarrow$ (c_n) converges to some $a \in M$

→ since all the balls are nested: $\overline{B}_1 \supsetneq \overline{B}_2 \supsetneq \cdots \supsetneq \overline{B}_n \supsetneq \cdots$

the limit of the centers $a \in \overline{B}_i$ for $\forall i$ ... because the balls are closed sets

$\Rightarrow$ for $\forall i$: $a \in \overline{B}_i \Rightarrow a \notin X_i$ $\Rightarrow$ always contain the limit

$\Rightarrow$ a lies in none of the sets X_i ↯ ∎

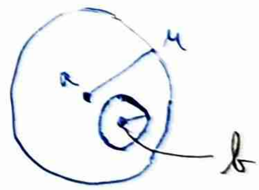
Isolated Points

Def: $a \in (M, d)$ is isolated $\equiv \exists r > 0$ s.t. $B(a, r) = \{a\}$

Lemma: $a \in M$ is not isolated $\Leftrightarrow$ $\{a\}$ is sparse in M.

Pf: $\Rightarrow$: Suppose $\forall r > 0$: $B(a, r) \supseteq \{a, b\}$, $b \in M$

claim: every ball in M has a subball without a



$\Leftarrow$: let $r > 0$, then $B(a, r)$ has a subball without a

$\Rightarrow$ the center of the subball $\neq a$ & is in $B(a, r)$ ∎

Theorem: Every complete metric space without isolated points is uncountable.

Proof: Suppose that is (M) was countable, then

$$M = \bigcup_{a \in M} \{a\} \text{ is a countable union}$$

Since each $a \in M$ is not isolate $\Rightarrow$ all $\{a\}$ are sparse

$\Rightarrow$ contradiction with Baire's theorem ↯ ∎

Non-differentiable Continuous Functions

Def: Let $I = [\alpha, \beta]$. Denote $C[I] := \{ f : I \to \mathbb{R} \mid f \text{ continuous} \}$

Def (Punctured neighborhood): $x \in \mathbb{R}$, $\delta > 0$: $P(x, \delta) := (x-\delta, x+\delta) \setminus \{x\}$

Def: $f : I \to \mathbb{R}$ is differentiable at $x \in I$ $\equiv$ it has a finite derivative $f'(x) \in \mathbb{R}$

Theorem (Wild functions exist): From now on let $I = [0,1]$.

There is a function $f \in C[I]$ s.t. for $(\forall x \in I)(\forall \delta > 0)$:

$$\sup \left\{ \left| \frac{f(y) - f(x)}{y - x} \right| \;\middle|\; y \in P(x, \delta) \cap I \right\} = +\infty$$

f is continuous on I but it is not differentiable at any $x \in I$.

Proof Idea:

1. Show that $C := (C[I], \underbrace{\|f-g\|_\infty}_{\text{maxim metric}})$ is a complete metric space
2. Define function sets $A_n \subseteq C[I]$ s.t.
$$A_n = \left\{ f \in C[I] \;\middle|\; (\exists x \in I)(\forall y \in I \setminus \{x\}) : \left| \frac{f(y) - f(x)}{y - x} \right| \leq n \right\}$$
3. Show that $\forall A_n$ is sparse in C
4. Apply Baire's theorem: C is complete & A_n sparse $\Rightarrow$
$$\exists f \in C[I] \setminus \bigcup_{n=1}^{\infty} A_n$$
5. Show that if f has the property that for $\forall x \in I$:
$$\sup \left\{ \left| \frac{f(y) - f(x)}{y - x} \right| \;\middle|\; y \in I \setminus \{x\} \right\} = +\infty$$
Then f has the property described in the theorem and we are done ■

Def: The function maxim metric on $C[I]$ is defined as

$$\|f - g\|_\infty := \sup \{ |f(x) - g(x)| \mid x \in I \}$$

This is a metric ... we use the Δ-ineq for the absolute value

$$\Rightarrow |a - b| \geq |a| - |b| \quad \because \; \underbrace{|a|}_{= |b + a - b|} \leq |b| + |a - b|$$

Theorem: The metric space $(C[I], \|f-g\|_\infty)$ is complete

Proof: Let $(f_n) \subset C[I]$ be a Cauchy sequence, that is

$$(\forall \varepsilon>0)(\exists n_0) : n, m \geq n_0 \Rightarrow \|f_n - f_m\| < \varepsilon \quad (*)$$

Therefor for $\forall x \in I$ is the sequence $(f_n(x)) \subset \mathbb{R}$ also Cauchy

$\rightarrow$ because $\mathbb{R}$ is complete, it is also convergent and we can define a new function

$$f: I \to \mathbb{R} \quad , \quad f(x) := \lim_{n\to\infty} f_n(x)$$

f has the property that point-wise $f_n \to f$ $(**)$

claim: $f_n \to f$ uniformly, that is $\|f - f_n\| \to 0$

$\Rightarrow$ let $\varepsilon>0$ be given. claim: $\exists N$ s.t. for $\forall x \in I$: $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

- take n_0 s.t. $(*)$ holds with $\frac{\varepsilon}{2}$
- take $m \geq n_0$ s.t. $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$ $(**)$ $\leftarrow$ specific $x \in I$

$\rightarrow$ now for $\forall n \geq n_0$ we have

$$\underbrace{|f_n(x) - f(x)|}_{a\,-\,b} \leq \underbrace{|f_n(x) - f_m(x)|}_{a\,-\,z} + \underbrace{|f_m(x) - f(x)|}_{+\;z\,-\,b} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\rightarrow$ we can set $N := n_0$ and this works for $\forall x \in I$

$\rightarrow$ we have shown that (f_n) converges to some function f, but for completeness we need $f \in C[I]$ ... f is continuous

claim: f is continuous on I

$\Rightarrow$ let $x_0 \in I$ and $\varepsilon > 0$ be given.

$\rightarrow$ we need: $\exists \delta>0$ s.t. $\forall x \in I$: $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

- take n_0 s.t. $n \geq n_0 \Rightarrow \|f - f_n\| \leq \frac{\varepsilon}{3}$ $\boxtimes$ ... $\|f - f_n\| \to 0$
- f_{n_0} is continuous $\Rightarrow$ take $\delta > 0$ s.t.

$$\forall x \in I : \quad |x - x_0| < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3} \quad \boxplus$$

$\Rightarrow$ we have:

$$\underbrace{|f(x) - f(x_0)|}_{a\,-\,b} \leq \underbrace{|f(x) - f_{n_0}(x)|}_{a\,-\,z_1} + \underbrace{|f_{n_0}(x) - f_{n_0}(x_0)|}_{z_1\,-\,z_2} + \underbrace{|f_{n_0}(x_0) - f(x_0)|}_{z_2\,-\,b} < \varepsilon$$

$\boxtimes < \varepsilon/3 \qquad \boxplus < \varepsilon/3 \qquad \boxtimes < \varepsilon/3$

Series Recap

Def: A series $\sum a_n = \sum_{n=1}^{\infty} a_n$ is a sequence $(a_n) \subset \mathbb{R}$ to which we assign the partial sums $(s_n) \subset \mathbb{R}$, $s_n = a_1 + \dots + a_n$.

The sum of a series $\sum a_n$ is $\lim_{n\to\infty} s_n$, if this limit exists

- a series is called
  - convergent $\equiv$ $\lim s_n$ exists and is finite
  - divergent $\equiv$ $\lim s_n$ doesnt exist or is infinite

Theorem (necessary condition of convergence): If $\sum a_n$ converges, then $(a_n) \to 0$

Pf: Otherwise s_n never stabilises

Examples

- $\sum \frac{1}{n} = +\infty$, $\sum \frac{1}{n(n+1)} = 1$, $\sum \frac{1}{n^2} = ?$ ... $\frac{\pi^2}{6}$
- $q \in (-1,1) \Rightarrow \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$

Riemann Rearrangement Theorem

Def: $\sum a_n$ is called a Riemann series $\equiv$

1. $(a_n) \to 0$ ... necessary condition
2. $\sum a_n^+ = +\infty$, $(a_n^+) \subset (a_n)$ is subsequence of positive terms
3. $\sum a_n^- = -\infty$, $(a_n^-) \subset (a_n)$ is subsequence of negative terms

Theorem (Rieman): Let $\sum a_n$ be a Riemann series. Then for $\forall S \in \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ there is a way to rearrange the sequence (a_n)

$\rightarrow$ formally, there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ s.t.

$$\sum_{n=1}^{\infty} a_{\pi(n)} = S$$ ... by reordering the RS we can get any sum

Example: Alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$ is Riemann.

Def: $\sum a_n$ is absolutely convergent $\equiv$ $\sum |a_n|$ is convergent

Theorem: Every convergent series is either

1. absolutely convergent
2. a Riemann series $\leftarrow$ conditionally convergent

Fourier Series - for details see notes from Fourier Analysis

Def: $f:[a,b]\to\mathbb{R}$ is piece-wise smooth (p-w. C^1) $\equiv$

① it has only finitely many discontinuities

② on each interval of continuity it is C^1

③ at discontinuities, it has finite onesided limits and derivatives

Def: For a l-periodic $\underbrace{\text{function}}_{\text{Riemann integrable}}$ f we define its Fourier series as the function

$$F_f(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right)$$

Where (for arbitrary x_0)

$$a_0 = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\,dx, \quad a_n = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\cos\left(\frac{2\pi n x}{l}\right)dx, \quad b_n = \frac{2}{l}\int_{x_0}^{x_0+l} f(x)\sin\left(\frac{2\pi n x}{l}\right)dx$$

Theorem (Dirichlet): Let $f:\mathbb{R}\to\mathbb{R}$ be l-periodic.

If f is piece-wise smooth on one period, then for $\forall a\in\mathbb{R}$

- f continuous at $a \Rightarrow F_f(a) = f(a)$
- f has discontinuity at $a \Rightarrow F_f(a) = \frac{1}{2}\left(f(a+0) + f(a-0)\right)$

one sided limits

Theorem (Bessel's inequality): If f has a Fourier series, then

$$\frac{1}{l}\int_{x_0}^{x_0+l} |f(x)|^2\,dx = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{a_n^2+b_n^2}{2}$$

Note:

f odd $\Rightarrow a_n = 0$

f even $\Rightarrow b_n = 0$

Theorem (Basel problem): $\zeta(2) = \sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof: Consider $f(x) = x^2$, $x\in[-\pi,\pi]$, $f(x+2\pi) = f(x)$

$\rightsquigarrow f$ even $\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$ $\quad \frac{2\pi n x}{l} = nx$ $\quad l = 2\pi$

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2$
- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx) = \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}\cdot\cos(\pi n)\right] = \frac{4}{n^2}\cdot(-1)^n$

D	I
$+\ x^2$	$\cos(nx)$
$-\ 2x$	$\frac{1}{n}\sin(nx)$
$+\ 2$	$-\frac{1}{n^2}\cos(nx)$
$-\ 0$	$-\frac{1}{n^3}\sin(nx)$

$\rightsquigarrow f(\pi) = \pi^2 = F_f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \underbrace{\cos(\pi n)}_{(-1)^n} \Rightarrow \frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2} = 4\zeta(2) \Rightarrow \underline{\zeta(2) = \frac{\pi^2}{6}}$ ▨

Random Walks in $\mathbb{Z}^d$

Definitions:

$\rightarrow N(v) := \{u \in V \mid uv \in E\}$

- a graph is (V, E), $E \subseteq \binom{V}{2}$
- G is k-regular $\equiv$ $\forall$ vertex has k neighbors
- G is locally finite $\equiv$ $\forall$ vertex has finitely many neighbors
- a walk w in G with start v_0 is any
  - $w = (v_0, v_1, \dots v_n)$ $\rightsquigarrow$ $|w| = n$
  - $w = (v_0, v_1, \dots)$ s.t. $\forall v_i v_{i+1} \in E$
- a recurrent walk revisits the start ... $\exists i \neq 0 : v_i = v_0$

Def:

$d_n(v_0, G) := \#$ walks w with start v_0 and $|w| = n$

$a_n(v_0, G) := \#$ recurrent walks w with start v_0 and $|w| = n$

If G is k-regular, then for $\forall v_0$: $d_n(v_0, G) = k^n$

Def: An automorphism of the graph G is a bijection $f: V \to V$ s.t.
$(\forall u, v \in V)$: $uv \in E \iff f(u) f(v) \in E$

Def: G is vertex transitive $\equiv (\forall u, v \in V)(\exists$ automorphism $f)$: $f(u) = v$.

Intuition: All verticies behave identically w.r.t. the edge relation

Proposition: The number of walks in a vertex transitive graph doesn't depend on the start (all vertices behave the same).

$\Rightarrow$ if G is transitive and locally finite, then $\forall u, v$: $\begin{aligned} d_n(u, G) &= d_n(v, G) \\ a_n(u, G) &= a_n(v, G) \end{aligned}$

Notation: In transitive (locally finite) graphs we can denote the number of walks and number of recursive walks of lenghts n from any start simply as

$$a_n(G) := a_n(v_0, G), \quad d_n(G) := d_n(v_0, G)$$

The infinite path $P := (\mathbb{Z}, \{\{n, n+1\} \mid n \in \mathbb{Z}\})$ is a transitive graph

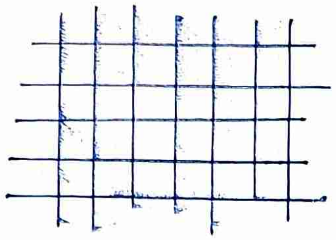
Every transitive locally finite graph is k-regular for some $k \in \mathbb{N}$

Def: We can generalize the path graph into an infinite square mesh as

$$\mathbb{Z}^k := \left(\mathbb{Z}^k, \left\{\{\tilde{u},\tilde{v}\} \,\middle|\, \sum_{i=1}^{k} |u_i - v_i| = 1\right\}\right) \quad , \quad \tilde{u} = (u_1, \dots u_n)$$

↳ $\tilde{u}\tilde{v} \in E$ ≡ they have all coordinates the same except for one, in which they differ by 1

$k = 2$:



↳ k coordinates can differ in 2 directions

👁 $\mathbb{Z}^k$ is vertex transitive and $2k$-regular $\Rightarrow d_n(\mathbb{Z}^k) = (2k)^n$

Theorem (G. Pólya 1921): For $k = 1$ and $k = 2$ it holds that

$$\lim_{n\to\infty} \frac{a_n(\mathbb{Z}^k)}{d_n(\mathbb{Z}^k)} = 1$$ ... a long random walk will almost surely revisit the start at some point

But for $k \geq 3$ it holds

$$\lim_{n\to\infty} \frac{a_n(\mathbb{Z}^k)}{d_n(\mathbb{Z}^k)} < 1$$ ... a long random walk never returns to the start with nonzero probability

↳ $\mathbb{P}[\text{escape}] \sim 1 - \frac{1}{2k}$, $k \to \infty$

Theorem (weak Abel's thm): If a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{R}[\![x]\!]$$

has a radius of convergence $\geq \rho \in \mathbb{R}^+$ ... it converges for $\forall x \in [0, \rho)$ and all $a_n \geq 0$, then we can define $A(\rho)$ as

$$A(\rho) := \lim_{x\to\rho^-} A(x) = \sum_{n=0}^{\infty} a_n \rho^n \in [0,\infty) \cup \{+\infty\}$$

↳ equality holds no matter if $A(\rho)$ is finite or $+\infty$

Proof: for $\forall N \in \mathbb{N}$ we have

$$\sum_{n=0}^{N} a_n \rho^n = \lim_{x\to\rho^-} \sum_{n=0}^{N} a_n x^n \leq \lim_{x\to\rho^-} A(x) = \lim_{x\to\rho^-} \sum_{n=0}^{\infty} a_n x^n \leq \sum_{n=0}^{\infty} a_n \rho^n$$

The limit transition $N \to +\infty$ gives

$$\sum_{n=0}^{\infty} a_n \rho^n \leq \lim_{x\to\rho^-} A(x) \leq \sum_{n=0}^{\infty} a_n \rho^n \quad \Longrightarrow \text{they are equal}$$ ∎

Proof of Pólya's theorem for $k=2$ – we will not show the general case

→ we will WLOG consider walks starting in the origin σ

⇒ let $k=2$ and $w=(v_0, v_1, \dots v_n)$, $v_0=\sigma$, $|w|=n$

- $d_n = \#$ walks w , $d_0=1$
- $a_n = \#$ walks w s.t. $\exists j\neq 0 : v_j=\sigma$, $a_0=0$
- $b_n := \#$ walks w s.t. $v_n=\sigma$, $b_0=1$
- $c_n := \#$ walks w s.t. $v_n=\sigma$ & $\forall 0<i<n: v_i\neq\sigma$, $c_0:=0$

$c_n \le b_n \le d_n$; $a_n \le d_n$, $d_n = 4^n$

→ we now split recurrent walks (a_n) into groups based on the first $v_j=\sigma$

$$a_n = \sum_{j=0}^{n} \overset{c_0=0}{c_j}\, d_{n-j} = \sum_{j=0}^{n} c_j\, 4^{n-j} = 4^n \sum_{j=0}^{n} \frac{c_j}{4^j} = d_n \cdot \sum_{j=0}^{n}\frac{c_j}{4^j}$$

$$\Rightarrow \sum_{j=0}^{n} \frac{c_j}{4^j} = \frac{a_n}{d_n} \le 1 \text{ , we want } \frac{a_n}{d_n} \to 1 \text{ as } n\to\infty$$

⇒ it is sufficient to prove that $\sum_{j=0}^{\infty}\frac{c_j}{4^j} = 1$

→ we will use generating functions and formal power series

$$B(x) := \sum_{n=0}^{\infty} b_n x^n \quad , \quad C(x) := \sum_{n=0}^{\infty} c_n \cdot x^n$$

→ when we look at the structural equation of $\mathcal{B}$, we can see that

$\mathcal{C}$ = walks that revisit σ only once – at the end
$\mathcal{C}^k$ = walks that revisit σ exactly k times, one of which is at the end
$\mathcal{B}$ = walks that visit σ at least once, one of the visits — " —

⇒ $\mathcal{B} = \{\emptyset\} + \mathcal{C} + \mathcal{C}^2 + \mathcal{C}^3 + \dots$ <u>Note:</u> $\{\emptyset\}$ counts $w=(v_0)\in\mathcal{B}$, $\notin \mathcal{C}^k$

$$\Rightarrow B(x) = 1 + C(x) + C(x)^2 + \dots = \frac{1}{1-C(x)}$$

→ plug in $x/4$: $C\left(\frac{x}{4}\right) = \sum_{n=0}^{\infty}\frac{c_n}{4^n}x^n$, $B\left(\frac{x}{4}\right) = \sum_{n=0}^{\infty}\frac{b_n}{4^n}x^n$

→ because $c_n \le b_n \le d_n = 4^n$, both of these have $R_{conv} \ge 1$

⇒ $C\left(\frac{x}{4}\right)$, $B\left(\frac{x}{4}\right)$ converge at least for $\forall x \in [0,1)$

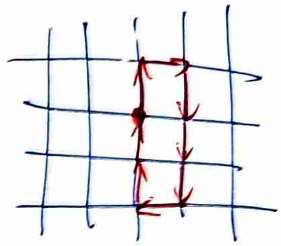
→ we want to show ⊛: $\sum_{j=0}^{\infty} \frac{c_n}{4^n} = C\left(\frac{1}{4}\right) = 1$

↳ Abels theorem says that $C\left(\frac{1}{4}\right) = \lim_{x\to 1^-} C\left(\frac{x}{4}\right)$ $\qquad \Uparrow \lim_{x\to 1^-} C\left(\frac{x}{4}\right) = 1$

⇒ it suffices to show that $\lim_{x\to 1^-} B\left(\frac{x}{4}\right) = \lim_{x\to 1^-} \frac{1}{1-C\left(\frac{x}{4}\right)} = +\infty$

→ Abels theorem once again says that

$$\lim_{x\to 1^-} B\left(\frac{x}{4}\right) = B\left(\frac{1}{4}\right) = \sum_{n=0}^{\infty} \frac{b_n}{4^n}$$



→ in order to show $B\left(\frac{1}{4}\right) = +\infty$, we will determine b_n

n odd $\Rightarrow b_n = 0$

→ <u>even walk lenghts</u>: $|w| = 2n$

k steps right ⇒ k steps left

⇒ $n-k$ steps up & $n-k$ steps down

⇒ # walks b_n with exactly k steps right =

$$= \binom{2n}{k, k, n-k, n-k} = \frac{(2n)!}{k!\,k!\,(n-k)!\,(n-k)!} = \frac{(2n)!}{n!\,n!} \cdot \left(\frac{n!}{k!\,(n-k)!}\right)^2 = \binom{2n}{n} \cdot \binom{n}{k}^2$$

$$\Rightarrow b_{2n} = \sum_{k=0}^{n} \binom{2n}{n} \cdot \binom{n}{k}^2 = \binom{2n}{n} \cdot \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \cdot \binom{2n}{n} = \binom{2n}{n}^2$$

→ we can now use the well known formula $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \Rightarrow b_{2n} \sim \frac{4^{2n}}{\pi n}$

$$\Rightarrow B\left(\frac{1}{4}\right) = \sum_{n=0}^{\infty} \frac{b_n}{4^n} = \sum_{n=0}^{\infty} \frac{b_{2n}}{4^{2n}} = \sum_{n=0}^{\infty} \frac{1}{\pi n} = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{n} = +\infty$$

↳ harmonic series

→ <u>summary</u>:

- we have shown $B\left(\frac{1}{4}\right) = +\infty$
- because $B(x) = \frac{1}{1-C(x)}$, this implies $C\left(\frac{1}{4}\right) = 1$
- so $\sum_{k=0}^{n} \frac{c_k}{4^k} = \frac{a_n}{d_n} \to 1$

Complex Analysis

- normed field $\mathbb{C}_{OF} = (\mathbb{C}, 0, 1, +, \cdot, |\cdot|)$, $|z| = |a+bi| = \sqrt{a^2+b^2}$
- metric space $(\mathbb{C}, |z_1 - z_2|)$ is isometric to $\mathbb{R}^2$ and therefore complete

Notation:

- $\mathcal{U}, \mathcal{U}_0, \mathcal{U}_1, \dots$ nonempty open subsets of $\mathbb{C}$
- $\mathcal{N}_{<\varepsilon}(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$, $\mathcal{N}^*_{<\varepsilon}(z_0) = \mathcal{N}_{<\varepsilon}(z_0) \setminus \{z_0\}$

Def: The derivative of $f: \mathcal{U} \to \mathbb{C}$ at $z_0 \in \mathbb{C}$ is

$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$. If $f'(z_0)$ exists, f is complex-differentiable at z_0

where $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L \equiv (\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \in \mathcal{U})$:

$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - L \right| \leq \varepsilon$$

Def: $f: \mathcal{U} \to \mathbb{C}$ is holomorphic on $\mathcal{U}$, $f \in \mathcal{H}[\mathcal{U}] \equiv \forall z \in \mathcal{U}: f'(z) \in \mathbb{C}$ exists

Def: $f: \mathbb{C} \to \mathbb{C}$ is entire (úplná) $\equiv f \in \mathcal{H}[\mathbb{C}]$

Properties of derivatives

→ let $f, g: \mathcal{U} \to \mathbb{C}$, $h: \mathcal{U}_0 \to \mathbb{C}$ be holomorphic and $\alpha, \beta \in \mathbb{C}$

1. $\alpha f + \beta g \in \mathcal{H}(\mathcal{U})$ $\qquad (\alpha f + \beta g)' = \alpha f' + \beta g'$
2. $f \cdot g \in \mathcal{H}(\mathcal{U})$ $\qquad (fg)' = f'g + fg'$
3. if $g \neq 0$ on $\mathcal{U}$, then $f/g \in \mathcal{H}(\mathcal{U})$ $\qquad (f/g)' = \frac{f'g - fg'}{g^2}$
4. if $h[\mathcal{U}_0] \subseteq \mathcal{U}$, then $f(h) \in \mathcal{H}(\mathcal{U}_0)$ $\qquad (f(h))' = f'(h) \cdot h'$

Examples:

- $(z^n)' = n \cdot z^{n-1}$, $(\alpha)' = 0$

Def: $f: U \to \mathbb{C}$ is <u>analytic</u> on $U \equiv (\forall z_0 \in U)(\exists \mathcal{N}_{<\varepsilon}(z_0) \subseteq U)$ we have

$$\forall z \in \mathcal{N}_{<\varepsilon}(z_0): \; f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad , \text{ for some } (a_n) \subset \mathbb{C}$$

$\Rightarrow$ the function <u>locally behaves as a power series</u>

Theorem: $f: U \to \mathbb{C}$ <u>analytic</u> $\Rightarrow$ <u>$f \in C^{\infty}[U]$</u> ... continuous $f^{(n)}(x)$ for $\forall n$

Pf: We can for $\forall z_0 \in U$ take some open disc $B \subseteq U$ containing z_0, differentiate the power series and use: f differentiable $\Rightarrow$ f continuous ∎

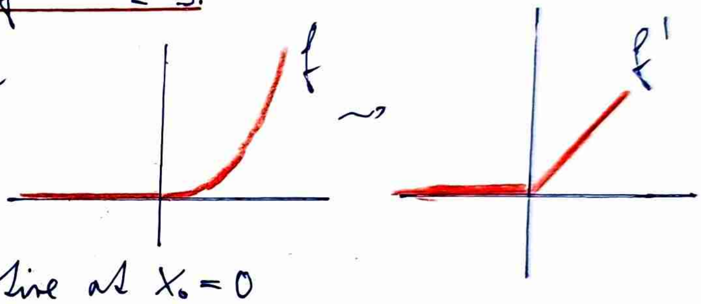
Theorem: $f: U \to \mathbb{C}$ <u>holomorphic</u> $\Rightarrow$ <u>analytic</u>.

↳ this is the main result of complex analysis

Corollary: $f: U \to \mathbb{C}$ <u>holomorphic</u> $\Rightarrow$ <u>$f \in C^{\infty}[U]$</u>. → continuous derivatives of all orders

Note: This doesn't hold in $\mathbb{R}$, consider

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



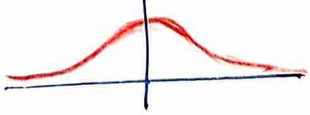
→ f doesn't have the second derivative at $x_0 = 0$

→ but all analytic functions have derivatives of all orders

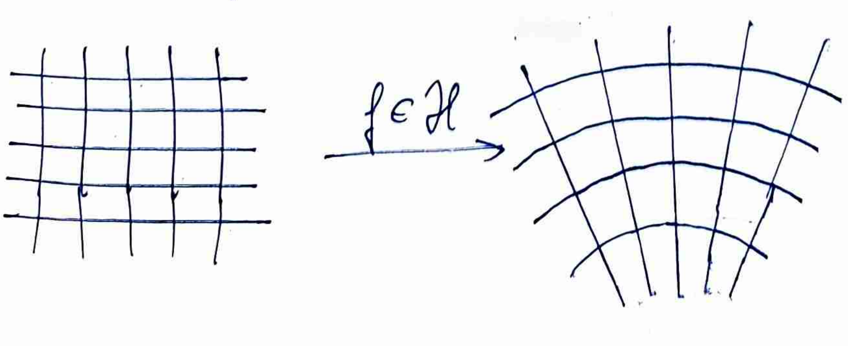
Def: $f: U \to \mathbb{C}$ is <u>bounded</u> $\equiv \exists m \geq 0$ s.t. $\forall z \in U: |f(z)| \leq m$

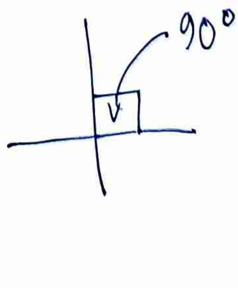
Theorem (Liouville): $f: \mathbb{C} \to \mathbb{C}$ <u>entire & bounded</u> $\Rightarrow$ f is <u>constant</u>.

Note: This is again not true for real functions

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = e^{-x^2}$  is entire & bounded

Intuition: Holomorphic, nonconstant functions <u>preserve angles</u>





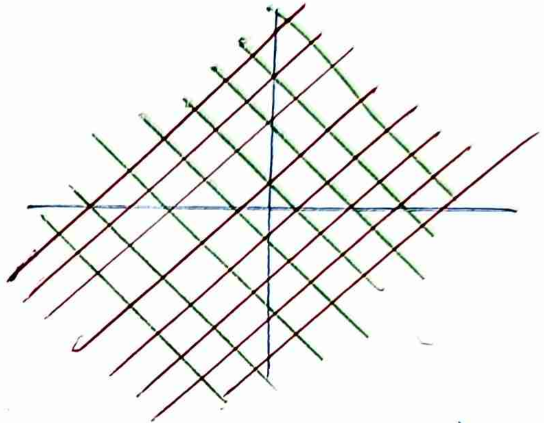
Ex: Načrtni množiny $\{Re(f) = C\}$, $\{Im(f) = C\}$

(0) $f(z) = e^{i\frac{\pi}{4}} z = \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right) \cdot z = \frac{\sqrt{2}}{2}(1+i)\overbrace{(x+iy)}^{z} = \frac{\sqrt{2}}{2}(x + iy + ix - y)$

$= \underbrace{\frac{\sqrt{2}}{2}(x-y)}_{Re} + i \underbrace{\frac{\sqrt{2}}{2}(x+y)}_{Im}$

$Re = \text{const} \iff x - y = \text{const}$

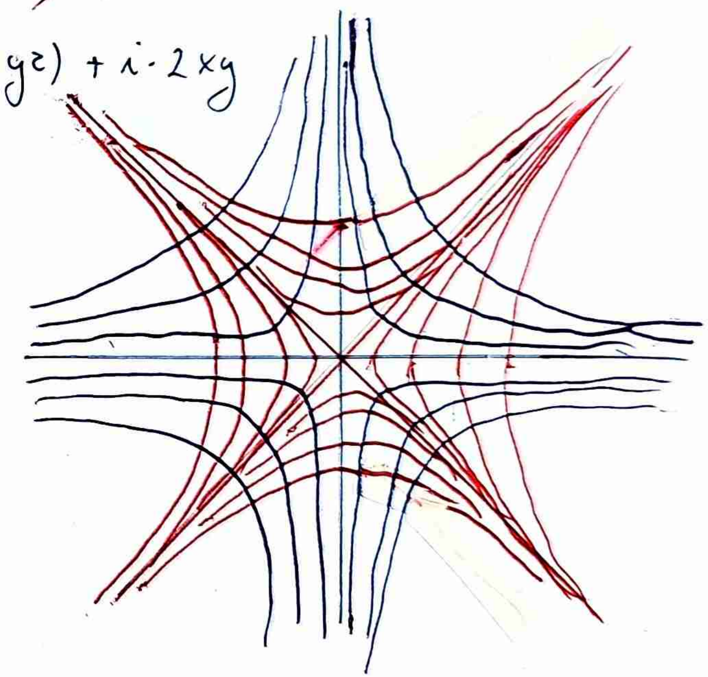
$Im = \text{const} \iff x + y = \text{const}$



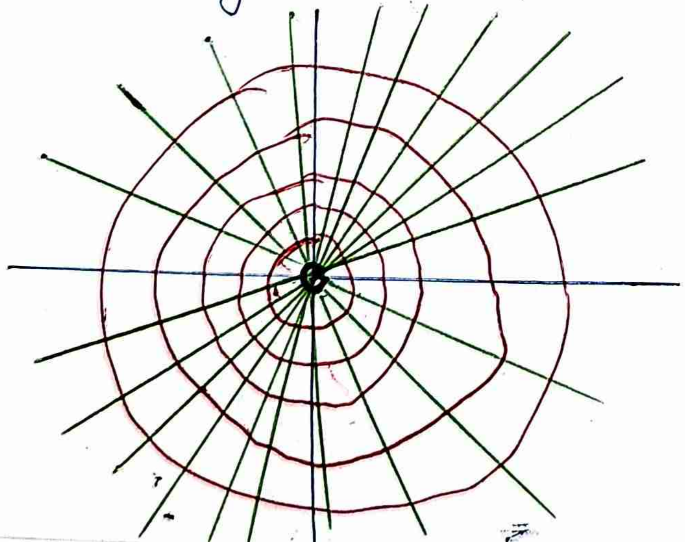
(1) $f(z) = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i \cdot 2xy$

$Re = \text{const} \iff y^2 = x^2 + C \iff |y| = \sqrt{x^2 + C}$

$Im = \text{const} \iff 2xy = C \iff y = \frac{C}{x}$



(2) $f(z) = Log(z) = \ln|z| + i\varphi$



Re ... $\ln|z| = C$ ... z na kruhu

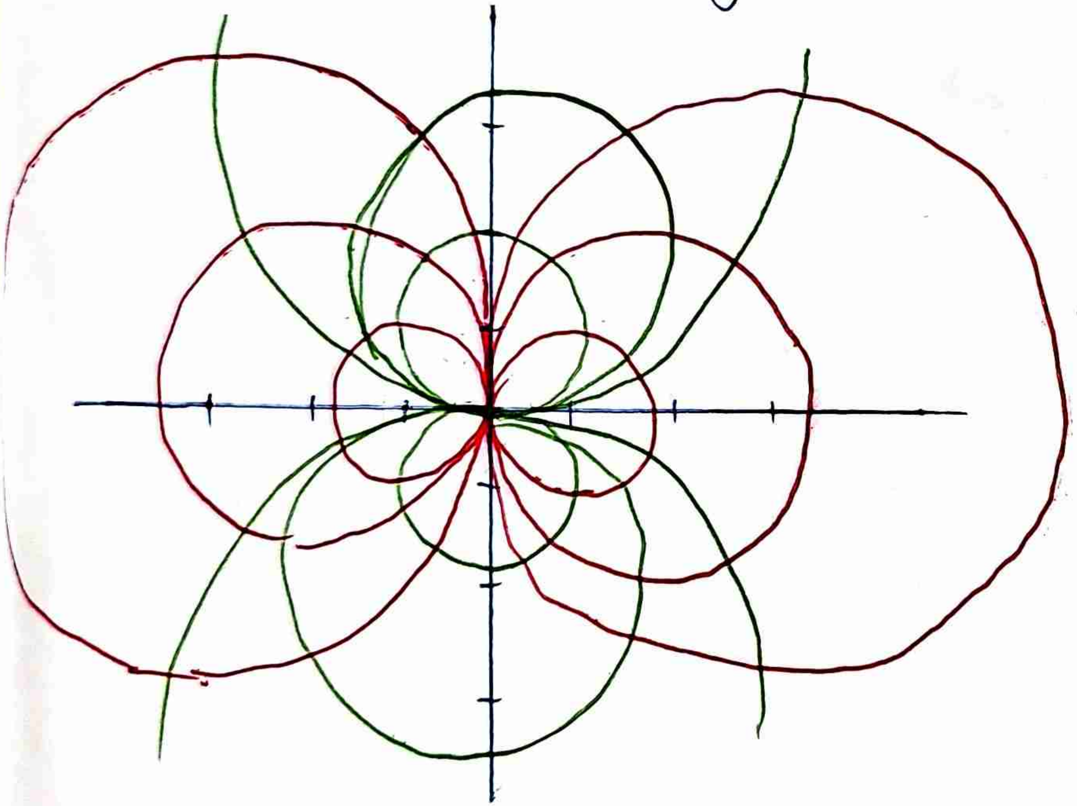
Im ... $\varphi = C$ ... stejný argument

1) $f(z) = \frac{1}{z}, \quad z \neq 0$

$$f(z) = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

$Re = const \Leftrightarrow x^2+y^2 = c\cdot x \Leftrightarrow (x^2+2cx+c^2)-c^2+y^2=0 \Leftrightarrow (x+c)^2+y^2=c^2$

$Im = const \Leftrightarrow x^2+y^2 = cy \Leftrightarrow \dots\ x^2+(y+c)^2=c^2$



Conjecture: Když se protne — s —, tak to je vždy kolmé.

Fundamental Theorem of Algebra Revisited

Theorem: (Liouville $\Rightarrow$ FTAlg): $P(z)$ polynomial with $\deg \geq 1 \Rightarrow \exists z_0 \in \mathbb{C}: P(z_0) = 0$.

Proof: For contradiction assume $\forall z_0 \in \mathbb{C}: P(z_0) \neq 0$

$\Rightarrow f(z) := \dfrac{1}{P(z)}$ is $\mathcal{H}(\mathbb{C})$ ... entire

claim: $f(z)$ is bounded

corollary: Liouville: $f(z)$ is constant $\Rightarrow P(z)$ is constant $\Rightarrow$ has $\deg = 1$ ↯

$\hookrightarrow |P(z)| \sim z^n$ for $|z| \to \infty$, where $n = \deg(P) \geq 1$

$\Rightarrow |P(z)| \to +\infty$ and $|f(z)| \to 0$ for $|z| \to \infty$

$\Rightarrow \exists R > 0$ s.t. $|f(z)| < 1$ for $\forall |z| > R$

but also $\exists m > 0$ s.t. $|f(z)| \leq m$ for $\forall |z| \leq R$

$\hookrightarrow B = \{z \in \mathbb{C} \mid |z| \leq R\}$ is compact & $|f(z)| : \mathbb{C} \to \mathbb{R}$ is continuous

$\Rightarrow$ it attains maxima on B ∎

Theorem (max. modulus principle): $f \in \mathcal{H}(U) \Rightarrow |f|$ doesn't have strict local maxima

$(\forall z_0 \in U)(\forall \delta > 0): \exists z \in \mathcal{N}_{<\delta}(z_0)$ s.t. $|f(z)| \geq |f(z_0)|$

Idea why this might hold:

Lemma: Let f be analytic at z_0, then one of the following occurs:

① f is locally constant around z_0 ... $\exists \varepsilon > 0: \forall z \in \mathcal{N}_{<\varepsilon}(z_0): f(z) = f(z_0)$

② f is locally strictly different from z_0 ... $\exists \varepsilon > 0: \forall z \in \mathcal{N}_{<\varepsilon}(z_0): f(z) \neq f(z_0)$

Proof: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ near z_0

a) $\forall n \geq 1: a_n = 0 \Rightarrow f(z) = a_0 = f(z_0) \Rightarrow$ ①

b) $\exists n \geq 1: a_n \neq 0 \Rightarrow$ let k be smallest such n

$$\Rightarrow f(z) = a_0 + \overset{\neq 0}{a_k}(z-z_0)^k + \sum_{n=k+1}^{\infty} a_n (z-z_0)^n = \underset{f(z_0)}{\underset{\parallel}{a_0}} + \underset{\text{for } z \neq z_0}{\underset{\neq 0}{(z-z_0)^k}} \underbrace{\left[a_k + \sum_{n=1}^{\infty} a_{k+n}(z-z_0)^n\right]}_{g(z)}$$

$\hookrightarrow g(z_0) = a_k \neq 0$

$\to g(z)$ is analytic at $z_0 \Rightarrow$ continuous around $z_0 \Rightarrow$ non-zero on some $\mathcal{N}_{<\varepsilon}(z_0)$ ∎

Curves in $\mathbb{C}$

→ final goal: prove ← Liouville's theorem
$f \in \mathcal{H}[\mathbb{C}] \Rightarrow f$ analytic in $\mathbb{C}$

→ less general than $f \in \mathcal{H}(\mathcal{U}) \Rightarrow$ analytic in $\mathcal{U}$

Def: A continuous $\varphi : [\alpha, \beta] \to \mathbb{C}$ is a curve with

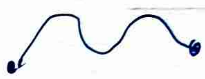
- start $\varphi(\alpha) \in \mathbb{C}$, end $\varphi(\beta) \in \mathbb{C}$
- image $\langle \varphi \rangle := \varphi[[\alpha,\beta]] = \{\varphi(t) \mid t \in [\alpha,\beta]\}$
- lenght $|\varphi| = \int_\alpha^\beta |\varphi'(t)|\, dt$

standard formula for lenght of a parametric curve

Note: $\varphi(t) = x(t) + i y(t) \Rightarrow \int_\alpha^\beta |\varphi'(t)|\,dt = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt$

Def: The curve $\varphi : [\alpha, \beta] \to \mathbb{C}$ is

- simple $\equiv \varphi$ is injective ... $t_1 \neq t_2 \Rightarrow \varphi(t_1) \neq \varphi(t_2)$



- closed $\equiv \varphi(\alpha) = \varphi(\beta)$

- simple closed $\equiv$ closed & $\varphi|_{[\alpha,\beta)}$ is injective

↳ also Jordan curve

Intuition: $\varphi'(t) \sim$ velocity vector of the curve parametrization
↳ the same $\langle \varphi \rangle$ can have different parametrizations with different velocities of how they move through the curve

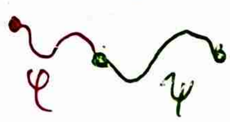
Def: The curve $\varphi : [\alpha, \beta] \to \mathbb{C}$ is

- regular $\equiv \varphi$ is differentiable & $\forall t \in (\alpha, \beta) : \varphi'(t) \neq 0$ → it doesn't stop
- smooth $\equiv \varphi \in C^1$ ... continuous derivative
- contour $\equiv$ piece-wise smooth ... up to finitely many discontinuities

Def (opposite curve): For the curve $\varphi : [\alpha, \beta] \to \mathbb{C}$ we define
$-\varphi : [\alpha, \beta] \to \mathbb{C}$ as $-\varphi(t) := \varphi(\alpha + \beta - t) \Rightarrow -\varphi : \begin{matrix} \alpha \mapsto \varphi(\beta) \\ \beta \mapsto \varphi(\alpha) \end{matrix}$

Def (curve composition): For curves $\varphi : [\alpha, \beta] \to \mathbb{C}$, $\psi : [\gamma, \delta] \to \mathbb{C}$ with $\varphi(\beta) = \psi(\gamma)$ define the curve $\varphi \oplus \psi$ by concatenating them together.

$\varphi \oplus \psi : [\alpha, \beta + (\delta - \gamma)] \to \mathbb{C}$, $t \mapsto \begin{cases} \varphi(t) , & t \in [\alpha, \beta) \\ \psi(t - \beta + \gamma), & t \in [\beta, \beta + (\delta - \gamma)] \end{cases}$

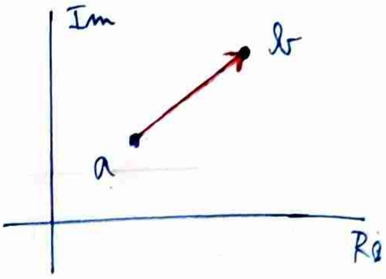


Segments and rectangles

Def: The curve φ is a segment $\equiv$ $\varphi(t)$ is a linear function

Def: For $a, b \in \mathbb{C}$ we define the segment $\overrightarrow{ab}$ as

$$\varphi : [0,1] \to \mathbb{C} \quad , \quad \varphi(t) := a + t(b-a)$$



Properties

① for the opposite segment it holds $-\overrightarrow{ab} = \overrightarrow{ba}$

② lenght of the segment is $|\overrightarrow{ab}| = |b-a|$

$$\hookrightarrow |\overrightarrow{ab}| = \int_0^1 |\varphi'(t)|\,dt = \int_0^1 |b-a|\,dt = \big[|b-a|t\big]_0^1 = |b-a|$$

Notation: When talking about segments we often mean the image

$\Rightarrow \overrightarrow{ab}$ ... curve , $\langle ab \rangle$ image

Def: A partition of the segment $\varphi : [\alpha, \beta] \to \mathbb{C}$ is any $p = (\underset{\varphi(\alpha)}{a_0}, a_1, \dots \underset{\varphi(\beta)}{a_k}) \subset \langle \varphi \rangle$ s.t.

$$\exists\, \alpha = t_0 < t_1 < \dots < t_k = \beta \quad \text{s.t.} \quad \forall i : \varphi(t_i) = a_i$$

Note: For segment $\overrightarrow{ab}$ we have partision $p = (a_0, a_1, \dots a_k)$ with $a_0 = a$, $a_k = b$

It defines k subsegments $\overrightarrow{a_{i-1} a_i}$ for $i \in [k]$

Def: The norm of the partition is the max. sub-segment lenght

$$\|p\| := \max_{i \in [k]} |\overrightarrow{a_{i-1} a_i}|$$

We have $\sum_{i=1}^{k} |\overrightarrow{a_{i-1} a_i}| = |\overrightarrow{ab}|$

Def (Cauchy sums): Let $\vec{u}$ be a segment and $p = (a_0, a_1, \dots a_k)$ its partition.

For $f : \langle u \rangle \to \mathbb{C}$ we define

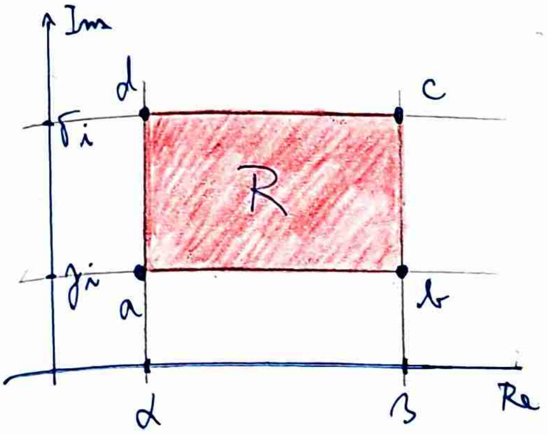
- $C(f, p) := \sum_{i=1}^{k} (a_i - a_{i-1}) f(a_i)$ ... Cauchy sum $\in \mathbb{C}$

- $C'(f, p) := \sum_{i=1}^{k} (a_i - a_{i-1}) f(a_{i-1})$ ... modified Cauchy sum $\in \mathbb{C}$

$$|C(f,p)| \,,\, |C'(f,p)| \le |\vec{u}| \cdot \sup_{z \in \langle u \rangle} |f(z)|$$

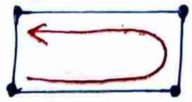
$\hookrightarrow$ replace $f(a_i)$ and $f(a_{i-1})$ by the max possible value $= \sup |f(z)|$

Def: The real numbers $\alpha < \beta$, $\gamma < \delta$ determine the rectangle



- $R = \{ z \in \mathbb{C} \mid \alpha \leq Re(z) \leq \beta \ \& \ \gamma \leq Im(z) \leq \delta \}$
- R is a square $\equiv \beta - \alpha = \delta - \gamma$
- canonical vertices $= (a, b, c, d) \in \mathbb{C}^4$

$d = \alpha + \delta i$ $\quad$ $c = \beta + \delta i$



$a = \alpha + \gamma i$ $\quad$ $b = \beta + \gamma i$

- The boundary of R is $\partial R := \langle ab \rangle \cup \langle bc \rangle \cup \langle cd \rangle \cup \langle da \rangle$
- The interior of R is $Int(R) := R \setminus \partial R$
- The perimeter of R is $Per(R) := |ab| + |bc| + |cd| + |da|$

Integrals and their existence

Def: For a segment $\vec{u}$ and a (usually continuous) $f : \langle u \rangle \to \mathbb{C}$ we define $\int_u f$.

If for every sequence (p_n) of partitions p_n of $\vec{u}$ with $\lim_{n \to \infty} \|p_n\| = 0$ the limit of the corresponding Cauchy sums

$$L = \lim_{n \to \infty} C(f, p_n) \in \mathbb{C} \text{ exists, we say that}$$

f has the integral L over $\vec{u}$ and write $\int_u f = L$.

Note: We could have also used a Riemann integral style definition using

$$R(f, p, q) = \sum_{i=1}^{k} (a_i - a_{i-1}) f(b_i) \quad \text{where } p = (a_0, \dots a_k)$$
$$q = (b_1, \dots b_k), \quad b_i \in \langle a_{i-1} a_i \rangle$$

For continuous functions f are the definitions using $C(f, p)$ and $R(f, p, q)$ equivalent, but for discontinuous its better to use the Riemann definition

Def: For a rectangle R with vertices (a, b, c, d) and $f : \partial R \to \mathbb{C}$ we define

$$\int_{\partial R} f := \int_{\vec{ab}} f + \int_{\vec{bc}} f + \int_{\vec{cd}} f + \int_{\vec{da}} f$$

Theorem: Suppose that $\vec{u}$ is a segment and R is a rectangle

① $f : \langle u \rangle \to \mathbb{C}$ continuous $\Rightarrow$ $\int_u f$ exists

② $f : \partial R \to \mathbb{C}$ continuous $\Rightarrow$ $\int_{\partial R} f$ exists

Theorem: For f, g continuous it holds that

continuous on compact $\Rightarrow$ sup = max

① $\int_u (\alpha f + \beta g) = \alpha \int_u f + \beta \int_u g$

② $\left| \int_u f \right| \le |\vec{u}| \cdot \max_{z \in \langle u \rangle} |f(z)|$, $\left| \int_{\partial R} f \right| \le \mathrm{Per}(R) \cdot \max_{z \in \partial R} |f(z)|$

③ $\int_{ba} f = - \int_{ab} f$

④ $\int_{ab} f = \int_{ac} f + \int_{cb} f$, $c \in \langle ab \rangle \setminus \{a, b\}$ $\leftarrow$ interior point

Contour Integrals

Def: Let $\varphi : [\alpha, \beta] \to \mathbb{C}$ be a contour (piece-wise C^1), $U \subseteq \mathbb{C}$ be open and containing φ ... $\langle \varphi \rangle \subseteq U$.

For $f : U \to \mathbb{C}$ define the integral of f along the curve φ as

$$\int_\varphi f := \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t) \, dt \quad \to \quad \int f(z) dz = \left| \begin{matrix} z = \varphi(t) \\ dz = \varphi'(t) dt \end{matrix} \right| = \int f(\varphi(t)) \varphi'(t) dt$$

If φ is closed, we write $\oint_\varphi f$ instead

The Cauchy sum integrals for segments we have defined previously evaluate to exactly this for continuous f ... just plug in φ and use Cauchy $\int$ = Riemann $\int$ (for continuous f)

Lemma: If $f : \mathbb{C} \to \mathbb{C}$ has primitive function $F : \mathbb{C} \to \mathbb{C}$, then

$$\int_{ab} f = F(b) - F(a)$$

Proof: Since $F' = f$, F is holomorphic $\Rightarrow F' = f$ is continuous

$\Rightarrow$ we can use the contour integral formula with $\varphi : [0,1] \to \mathbb{C}$, $\varphi(t) = a + (b-a)t$ ∎

Theorem (ML bound): f continuous $\Rightarrow$ $\int_\varphi f \le |\varphi| \cdot \max_{z \in \langle \varphi \rangle} |f(z)|$

Proof: $\langle \varphi \rangle$ compact $\Rightarrow$ f attains a maxima on $\langle \varphi \rangle$, call it M

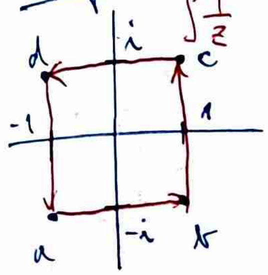
$$\int_\varphi f = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) dt \le \int_\alpha^\beta |f(\varphi(t))| \cdot |\varphi'(t)| dt \le M \cdot \int_\alpha^\beta |\varphi'(t)| dt = M \cdot |\varphi|$$ ∎

Not all $\oint$ are $= 0$

Theorem: Let S be the unit square. Then $\rho := \oint_{\partial S} \frac{1}{z} \neq 0$

$\to$ vertices $\pm 1, \pm i$

Proof: We will show this from the Cauchy sum definition and then using $\oint$



$\to$ canonical vertices a, b, c, d

$\to$ consider the n-equipartition (equal subsegments) of $\vec{ab}$

$p_{ab} = (a_0, a_1, \dots, a_n)$, $a_j = a + (b-a)\cdot \frac{j}{n}$

$\to$ equal lenghts $\Rightarrow a_j - a_{j-1} = \frac{b-a}{n}$

👁 multiplying $z \in \mathbb{C}$ by i rotates it counterclockwise by 90 degrees

$\Rightarrow p_{bc} = i \cdot p_{ab}, \quad p_{cd} = i \cdot p_{bc}, \quad p_{da} = i \cdot p_{cd}$

👁 $C(f, p_{ab}) = C(f, p_{bc}) = C(f, p_{cd}) = C(f, p_{da})$

$\hookrightarrow C(f, p_{ab}) = \sum_{j=1}^{n} (a_j - a_{j-1}) \cdot f(a_j) = \sum_{j=1}^{n} \frac{b-a}{n} \cdot \frac{1}{a + (b-a)\, j/n} = \quad \Big| \cdot \frac{i}{i} = 1$

$= \sum_{j=1}^{n} \frac{(ib - ia)}{n} \cdot \frac{1}{ia + (ib - ia)\, j/n} = \sum_{j=1}^{n} \frac{c-b}{n} \cdot \frac{1}{b + (c-a)\, j/n} = C(f, p_{bc})$

claim: $\operatorname{Im}(C(f, p_{ab})) \geq 1$

corollary: We let $n \to \infty$ and get $\operatorname{Im}(\rho) = 4 \cdot \operatorname{Im}(C(f, p_{ab})) \geq 4 \Rightarrow \rho \neq 0$

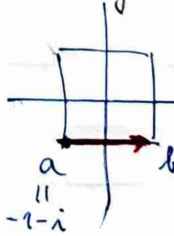
$\hookrightarrow$ plug in $a = -1-i$, $b = 1-i \Rightarrow b - a = 2 \Rightarrow a_j = \frac{2j}{n} - 1 - i$

$\Rightarrow C(f, p_{ab}) = \sum_{j=1}^{n} \frac{2}{n} \cdot \frac{1}{2j/n - 1 - i} \cdot \frac{2j/n - 1 + i}{2j/n - 1 + i} = \frac{2}{n} \sum_{j=1}^{n} \frac{2j/n - 1 + i}{(2j/n - 1)^2 + 1}$

$\Rightarrow \operatorname{Im}(C) = \frac{2}{n} \sum_{j=1}^{n} \frac{1}{(2j/n - 1)^2 + 1} \geq \frac{2}{n} \sum_{j=1}^{n} \frac{1}{2} = 1$ ▨

$\to$ in fact, S can be any simple closed contour containing the origin

Theorem $\oint_{\partial S} \frac{1}{z} = 2\pi i$, where S = unit square



$\varphi : [0,1] \to \mathbb{C}$

$\varphi(t) = a + (b-a)t$

$= 2t - 1 - i$

Pf: As observed in the previous proof, $\oint_{\partial S} \frac{1}{z} = 4 \cdot \int_{ab} \frac{1}{z}$ where

$4 \int_{ab} \frac{1}{z} = 4 \int_0^1 \frac{\varphi'(t)}{\varphi(t)}\, dt = 4 \int_0^1 \frac{2\, dt}{2t - 1 - i} = \Big| \begin{matrix} u = 2t - 1 \\ du = 2dt \end{matrix} \Big| = 4 \int_{u=-1}^{u=1} \frac{du}{u - i} \cdot \frac{u+i}{u+i}$

$= 4 \int_{-1}^{1} \frac{u + i}{u^2 + 1}\, du = 2 \int_{-1}^{1} \frac{2u}{u^2+1}\, du + 4i \int_{-1}^{1} \frac{1}{u^2+1}\, du$

$\to$ arctg is odd

$= 2 \Big[\ln|u^2 + 1| \Big]_{-1}^{1} + 4i \Big[\operatorname{arctg}(u) \Big]_{-1}^{1} = 2 \cdot \underbrace{(\ln(2) - \ln(2))}_{0} + 4i\, (\operatorname{arctg}(1) - \operatorname{arctg}(-1))$

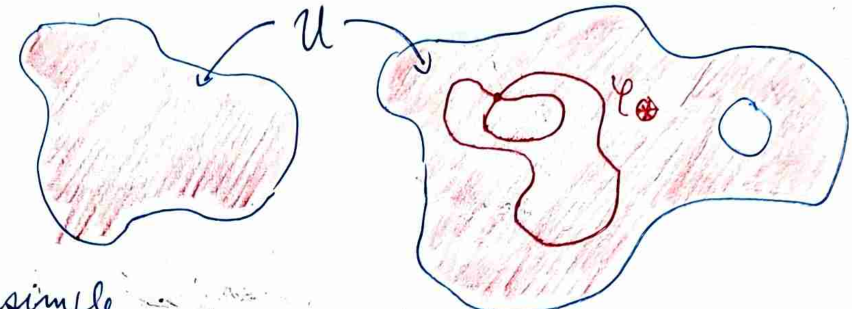
$= 4i \cdot 2 \operatorname{arctg}(1) = 8i \cdot \frac{\pi}{4} = 2\pi i$ ▨

For nice functions $\oint = 0$

Theorem (Cauchy-Goursat): Let $f: \mathcal{U} \to \mathbb{C}$ be holomorphic on $\mathcal{U}$.

If φ is a Jordan contour (piece-wise C^1) s.t. $\langle\varphi\rangle \cup \mathrm{Int}(\varphi) \subseteq \mathcal{U}$. Then

$$\oint_\varphi f = 0$$



Note: Since $\int_{\varphi \oplus \psi} f = \int_\varphi f + \int_\psi f$

⊛ we dont need φ to be simple

History:

→ we need this theorem to prove the Cauchy integral formula

→ and we need the Cauchy integral formula to show: $f \in \mathcal{H}[\mathcal{U}] \Rightarrow f \in C^\infty[\mathcal{U}]$

⇒ so we cannot assume: f' exists ⇒ f' continuous

↳ it holds, but it would be a cyclic proof

→ Cauchy originaly proved this theorem with the assumption: f' is continuous

→ Goursat later showed that its enough to assume that f' exists.

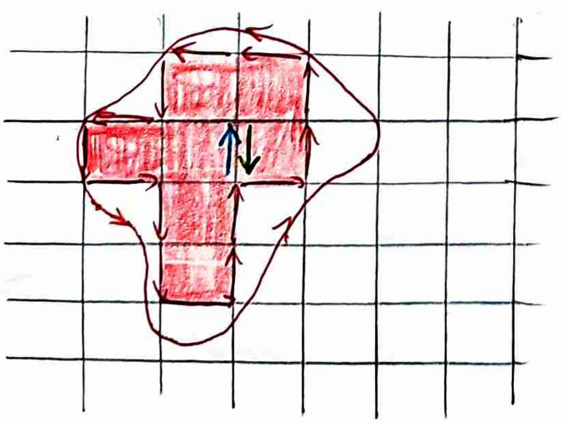
Proof idea:

① show that C-G holds when φ is a rectangle - we will show this

② approximate φ with a very fine rectangular mesh

③ show that $\oint_R f$ doesnt depend on the position of the rectangle R

④ we approximate $\oint_\varphi f$ using the rectangles inside φ



→ WLOG assume φ is simple ⊛

$M :=$ Rectangles inside $\mathrm{Int}(\varphi)$

$$\int_\varphi f \approx \sum_{R \in M} \int_{\partial R} f$$

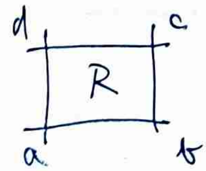
→ the touching boundaries cancel and only the outside boundary is left

↳ $\int_{ab} f = -\int_{ba} f$

Theorem (C-G. for R): Let $f \in \mathcal{H}[U]$ and $R \subseteq U$ be a rectangle. Then $\oint_{\partial R} f = 0$.

Proof: This will be long

Lemma 1: It is easy when f has a primitive function $F' = f$.



Pf: Recall: $\int_{ab} f = F(b) - F(a)$

$$\Rightarrow \oint_{\partial R} f = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da} = F(b) - F(a) + F(c) - F(b) + F(d) - F(c) + F(a) - F(d) = 0 \ \blacksquare$$

Def: The diameter of the set $X \subseteq \mathbb{C}$ is $Diam(X) := \sup\{|x-y| \mid x, y \in X\} \in \mathbb{R}_0^+ \cup \{+\infty\}$.

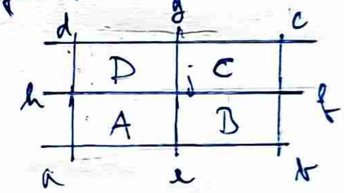
Lemma 2: Let $A_n \neq \emptyset$ be closed sets s.t. $\mathbb{C} \supseteq A_1 \supseteq A_2 \supseteq \cdots$ and $\lim_{n \to \infty} Diam(A_n) = 0$.

Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Proof: Because $Diam(A_n) \to 0$, it is easy to see that there exists a sequence $(a_n) \subset \mathbb{C}$ s.t. $\forall a_i \in A_i$ and $(a_n) \to a \in \mathbb{C}$

$\to$ because all A_n are closed, this $a \in A_n$ for $\forall n$. $\blacksquare$

Def: For a rectangle R we define its quarters $\to$



for each quarter Q we have

- $Per(Q) = \frac{1}{2} Per(R)$
- $Diam(Q) = \frac{1}{2} Diam(R) = \max\{|\beta - \alpha|, |\delta - \gamma|\}$

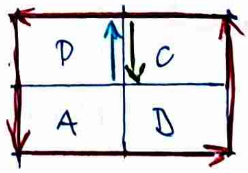
$\to$ rectangle defined by $\alpha < \beta$, $\gamma < \delta$

$R = \{z \mid Re(z) \in [\alpha, \beta],\ Im(z) \in [\gamma, \delta]\}$

Proof of C-G.: We first define a series of nested rectangles $R = R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots$ s.t. $\forall n$: R_{n+1} is a quarter of R_n and $|\oint_{\partial R_{n+1}} f| \geq \frac{1}{4} \cdot |\oint_{\partial R} f|$ (*)

$\to$ suppose that $R_0, R_1, \ldots R_n$ have already been defined, we want R_{n+1}

$\to$ let A, B, C, D be the quarters of R_n



the touching sides cancel

$$\oint_{\partial R_n} f = \oint_{\partial A} f + \oint_{\partial B} f + \oint_{\partial C} f + \oint_{\partial D} f$$

$\to$ we will pick a quarter such that (*) holds

$$|\oint_{\partial R_n} f| \leq |\oint_{\partial A} f| + |\oint_{\partial B} f| + |\oint_{\partial C} f| + |\oint_{\partial D} f|$$

$\Rightarrow \exists$ quarter Q s.t. $|\oint_{\partial Q} f| \geq \frac{1}{4} |\oint_{\partial R_n} f|$ $\Rightarrow$ choose $R_{n+1} := Q$

→ so now we have $U \supseteq R \supseteq R_1 \supseteq R_2 \supseteq \cdots$ s.t. $\left|\oint_{\partial R_{n+1}} f\right| \geq \frac{1}{4}\left|\oint_{\partial R_n} f\right|$ ⊛

→ because all R_n are closed and nonempty, by lemma 2:

$$\exists z_0 \in U: \quad z_0 \in \bigcap_{n=0}^{\infty} R_n$$

→ we will show that for $\forall \varepsilon > 0$: $\left|\oint_{\partial R} f\right| \leq \varepsilon \cdot \mathrm{Per}(R)^2 \xrightarrow{\varepsilon \to 0} 0.$

→ let $\varepsilon > 0$ be given.

Recall: f' exists ⇒ f continuous. We use continuity at $f(z_0)$

⇒ $\exists \delta > 0$ s.t. $\eta := \eta_{<\delta}(z_0) \subseteq U$ and on η we have

$|f(z) - f(z_0)|$ arbitrarily small (based on ε)

⇒ we can do a linear approximation of f on η as

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{g(z)} + \underbrace{\mu(z)(z - z_0)}_{h(z)} \quad , \mu : \eta \to \mathbb{C}$$

$|\mu(z)| \leq \varepsilon$

point is that we can shrink η so that $\mu(z)$ is arbitrarily bounded

👁 $g(z)$ is continuous and linear

👁 $h(z)$ is continuous on η ∵ $h(z) = f(z) - g(z)$ continuous

→ now let n be so large that $R_n \subseteq \eta$ ... here we are using $\mathrm{Diam}(R_n) \to 0$

$$\Rightarrow \oint_{\partial R_n} f = \underbrace{\oint_{\partial R_n} g}_{0} + \oint_{\partial R_n} h = \int_{\partial R} h$$

0 using lemma 1 ... g is linear ⇒ ~~has primitive function~~ ⇒ $\oint = 0$

→ now we estimate

lenght of bondary → max value

$$\left|\oint_{\partial R_n} f\right| = \left|\oint_{\partial R_n} h\right| \leq \mathrm{Per}(R_n) \cdot \max_{z \in \partial R_n} |h(z)|$$

$$= \mathrm{Per}(R_n) \cdot \max_{z \in \partial R_n} |\mu(z) \cdot (z - z_0)| \quad \ldots \ |\mu(z)| \leq \varepsilon \text{ on } \eta \text{ and } R_n \subseteq \eta$$

$$\leq \mathrm{Per}(R_n) \cdot \varepsilon \cdot \max_{z \in \partial R_n} |z - z_0| \leq \mathrm{Per}(R_n) \cdot \varepsilon \cdot \mathrm{Diam}(R_n)$$

R_j are nested quadrons

$$= \varepsilon \cdot \frac{\mathrm{Per}(R)}{2^n} \cdot \frac{\mathrm{Diam}(R)}{2^n} \leq \varepsilon \cdot \frac{1}{4^n} \cdot \mathrm{Per}(R)^2 \quad \leftarrow \mathrm{Diam}(R) \leq \mathrm{Per}(R)$$

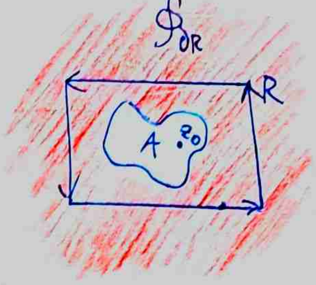
→ now we use ⊛

$$\Rightarrow \left|\oint_{\partial R_n} f\right| \geq \frac{1}{4^n}\left|\oint_{\partial R} f\right| \Rightarrow \left|\oint_{\partial R} f\right| \leq \varepsilon \cdot \mathrm{Per}(R)^2 \xrightarrow{\varepsilon \to 0} 0$$

▨

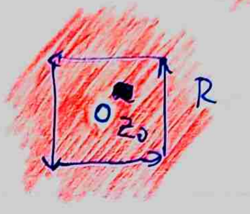
Cauchy integral formula

Lemma: Let $R \subseteq \mathbb{C}$ be a rectangle and $A \subseteq \mathrm{Int}(R)$ be compact.
Let $f, g : \mathbb{C} \setminus A \to \mathbb{C}$ be holomorphic. Then



① $\oint_{\partial R} (\alpha f + \beta g) = \alpha \oint_{\partial R} f + \beta \oint_{\partial R} g$

② if $A = \{z_0\}$ and f is bounded on some $\mathcal{U}^{\circ}_{<\varepsilon}(z_0)$, then
$$\oint_{\partial R} f = 0$$



$\oint_{\partial R} = 0$

③ if $z_0 \in \mathrm{Int}(R)$, then $\oint_{\partial R} \frac{1}{z - z_0} = 2\pi i$

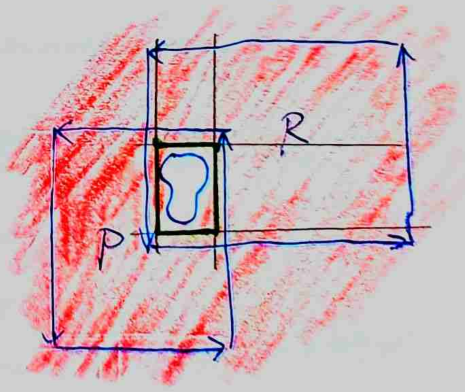
Proof: ① linearity of $\int_{\varphi} f$ holds in general

② let (R_n) be rectangles with $z_0 \in R_n$ and $\mathrm{Diam}(R_n) \to 0$
$$\oint_{\partial R_n} f \leq \mathrm{Per}(R_n) \cdot \max_{z \in \partial R_n} |f(z)| \xrightarrow{\mathrm{Per} \to 0} 0 \overset{⊛}{\Longrightarrow} \oint_R f = 0$$

③ let S be the unit square and $a+S$ a shifted copy with center at a
$$\oint_{\partial R} \frac{1}{z-a} \overset{⊛}{=} \oint_{\partial(a+S)} \frac{1}{z-a} = \oint_{\partial S} \frac{1}{z} = 2\pi i$$ ... we have shown before

Lemma ⊛ (Independence of $\oint_{\partial R} f$ on R): Let $R, P \subseteq \mathbb{C}$ be rectangles and let $A \subseteq \mathrm{Int}(R) \cap \mathrm{Int}(P)$ be compact. Let $f : \mathbb{C} \setminus A \to \mathbb{C}$ be holomorphic. Then
$$\oint_{\partial R} f = \oint_{\partial P} f$$

Proof:



There $\exists$ rectangle $T \subseteq \mathrm{Int}(R) \cap \mathrm{Int}(P)$ containing A

$\Rightarrow$ we extend the edges of T to divide R into 9 rectangles $R_1, R_2, \ldots R_8, T$

$$\oint_{\partial R} f = \oint_{\partial T} f + \underbrace{\sum_{j=1}^{8} \oint_{\partial R_j} f}_{0} = \oint_{\partial T} f$$

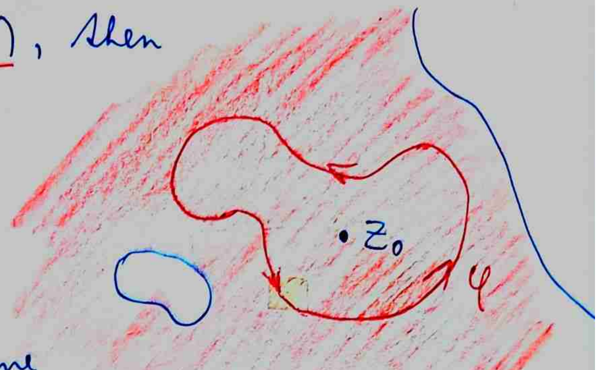
0 because R_j do not contain any holes $\to$ C.-G.: $\oint = 0$

$\rightsquigarrow$ similarly $\oint_{\partial P} f = \oint_T f$ ∎

Fact: In fact $\oint$ is completely independent on the curve φ as long as we transform it in a continuous manner – such curves are called homotopic

Theorem (Cauchy $\oint$ formula): Let $f \in \mathcal{H}[U]$ and $\mathcal{C}$ be a simple closed contour. If $\langle \mathcal{C} \rangle \cup \mathrm{Int}(\mathcal{C}) \subseteq U$ and $z_0 \in \mathrm{Int}(\mathcal{C})$, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}$$



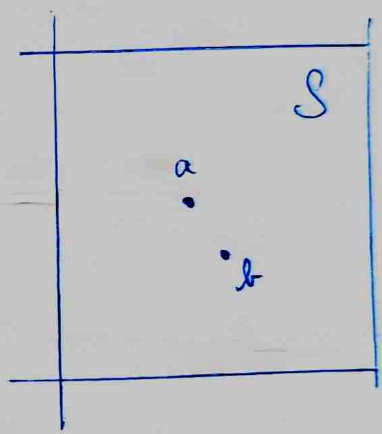
Proof: For $U = \mathbb{C}$ and $\mathcal{C} = R$ rectangle

→ since $f'(z_0)$ exists, $\frac{f(z) - f(z_0)}{z - z_0}$ is bounded on some punctured neighborhood $\mathcal{N}^{\circ}_{<\varepsilon}(z_0)$ and by the lemma on the previous page

$$0 = \oint_{\partial R} \frac{f(z) - f(z_0)}{z - z_0} = \oint_{\partial R} \frac{f(z)}{z - z_0} - f(z_0) \underbrace{\oint_{\partial R} \frac{1}{z - z_0}}_{2\pi i} \Rightarrow f(z_0) = \frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z - z_0} \quad \blacksquare$$

Proof of Liouville's theorem

Theorem: $f: \mathbb{C} \to \mathbb{C}$ entire & bounded $\Rightarrow$ f constant



Proof: Suppose $|f(z)| \leq M$. Let $a, b \in \mathbb{C}$, we will make a sufficiently large square $S \subseteq \mathbb{C}$ with side lengths s s.t. $a, b \in \mathrm{Int}(S)$ & for $\forall z \in \partial S$:

$$|z - a|, |z - b| \geq \frac{1}{3} s = \frac{1}{12} \mathrm{Per}(S)$$

By the Cauchy formula:

$$f(a) - f(b) = \frac{1}{2\pi i} \oint_{\partial S} \frac{f(z)}{z - a} - \frac{1}{2\pi i} \oint_{\partial S} \frac{f(z)}{z - b} = \frac{a - b}{2\pi i} \oint_{\partial S} \frac{f(z)}{(z - a)(z - b)}$$

→ we again use the maximum-length estimate

$$\left| \oint \right| \leq \mathrm{Per}(S) \cdot \max_{z \in \partial S} \frac{f(z)}{(z - a)(z - b)} \leq \mathrm{Per}(S) \cdot \frac{M}{\mathrm{Per}(S)^2 / 144} = \frac{144 M}{4 s} = \frac{36 M}{s}$$

→ we now let $s \to \infty$ and get $f(a) - f(b) \to 0$ $\blacksquare$

Proof of entire $\Rightarrow$ analytic — we will not show holomorphic $\Rightarrow$ analytic

→ let $f: \mathbb{C} \to \mathbb{C}$ be entire, let $z_0 \in \mathbb{C}$ and create a sufficiently large rectangle R containing both 0 and z_0 s.t. for $\forall z \in \partial R$

$$\left| \frac{z_0}{z} \right| = \frac{|z_0|}{|z|} \leq \frac{1}{2} \quad \text{and} \quad |z - z_0| \geq 1 \quad (\dagger)$$

→ for any $m \in \mathbb{N}$ we have using the Cauchy formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z - z_0} = \frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z} \cdot \frac{1}{1 - \frac{z_0}{z}} \, dz$$

→ now we use the formula $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^m + \frac{x^{m+1}}{1-x}$ for $x = \frac{z_0}{z_0}$

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z} \cdot \left(\sum_{k=0}^{m} \left(\frac{z_0}{z}\right)^k + \frac{(z_0/z)^{m+1}}{1 - z_0/z} \right) dz$$

$$= \sum_{k=0}^{m} \underbrace{\left(\frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z^{k+1}} dz \right)}_{a_m} z_0^k + \underbrace{\frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z - z_0} \cdot \left(\frac{z_0}{z}\right)^{m+1} dz}_{I_{m+1}}$$

→ now we again use the max-length estimate for I_{m+1}: for $m \to \infty$

$$|I_{m+1}| \leq Per(R) \cdot \max_{z \in \partial R} |f(z)| \cdot \underbrace{\frac{1}{|z - z_0|}}_{\leq 1} \cdot \underbrace{\left|\frac{z_0}{z}\right|^{m+1}}_{\leq \frac{1}{2}} \overset{*}{\leq} Per(R) \cdot \frac{1}{2^{m+1}} \cdot \max_{z \in \partial R} |f(z)|$$

→ because f is holomorphic, it is continuous and because ∂R is compact, f attains a maximum value on ∂R $\Rightarrow$ $\max_{z \in \partial R} |f(z)|$ is bounded

$\Rightarrow$ for $m \to \infty$ we get $|I_{m+1}| \to 0$ and

$$f(z_0) = \sum_{k=0}^{\infty} a_m z_0^k, \text{ where } a_m = \frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z^{k+1}} = \frac{1}{2\pi i} \oint_{\partial S} \frac{f(z)}{z^{k+1}}, \text{ where}$$

S is any rectangle containing the origin ■

Theorem (General Cauchy $\oint$ formula): Let $f \in \mathcal{H}[U]$ and φ be a simple closed contour. If $\langle \varphi \rangle \cup Int(\varphi) \subseteq U$ and $z_0 \in Int(\varphi)$, then

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \oint_{\varphi} \frac{f(z)}{(z - z_0)^{m+1}}$$

Proof: Now we know that holomorphic $\Rightarrow$ analytic, so near z_0 we have

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n \quad \Rightarrow \quad a_m = \frac{f^{(m)}(z_0)}{m!}, \text{ we claim: } a_m = \frac{1}{2\pi i} \oint_{\varphi} \frac{f(z)}{(z - z_0)^{m+1}}$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\varphi} \frac{f(z)}{(z - z_0)^{m+1}} = \frac{1}{2\pi i} \oint_{\varphi} \frac{\sum a_k (z - z_0)^k}{(z - z_0)^{m+1}} = \frac{1}{2\pi i} \oint_{\varphi} \sum_{k=0}^{\infty} a_k (z - z_0)^{k-m-1}$$

$$= \frac{1}{2\pi i} \left(\sum_{k=0}^{m-1} a_k \underbrace{\oint_{\varphi} (z - z_0)^{k-m-1}}_{* = 0} + a_m \underbrace{\oint_{\varphi} \frac{1}{z - z_0}}_{** = 2\pi i} + \sum_{k=m+1}^{\infty} a_k \underbrace{\oint_{\varphi} (z - z_0)^{k-m-1}}_{C.-G.:\ \oint = 0} \right) = a_m$$

→ WLOG assume φ is a circle centered at z_0 with small radius r: $\varphi: [0, 2\pi] \mapsto z_0 + r \cdot e^{it}$

$$\oint_{\varphi} \frac{1}{(z - z_0)^k} = \int_0^{2\pi} \frac{i r e^{it}}{(r e^{it})^k} dt = \frac{i}{r^{k-1}} \int_0^{2\pi} e^{it(1-k)} dt \quad \begin{cases} k = 1: \ i \int 1 = 2\pi i \ ** \\ k > 1: \ \int = \frac{1}{1-k} [e^{i 2\pi} - e^{i \cdot 0}] = 0 \ * \end{cases}$$

■

When is $\int_\gamma f = \int_\varphi f$?

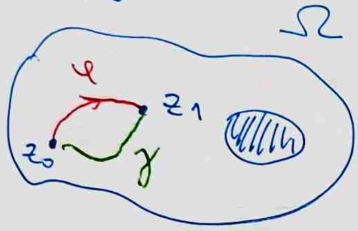
Suppose $\Omega \subseteq \mathbb{C}$ is a domain (open, connected), $f \in \mathcal{H}[\Omega]$, and we have contours $\gamma, \varphi \subseteq \Omega$, both $z_0 \to z_1$. When? $\int_\gamma f = \int_\varphi f$ (*)

(1) **Ω simply connected**

→ then f has an antiderivative F and $\int_\gamma f = \int_\varphi f = F(z_1) - F(z_0)$

⇒ (*) holds

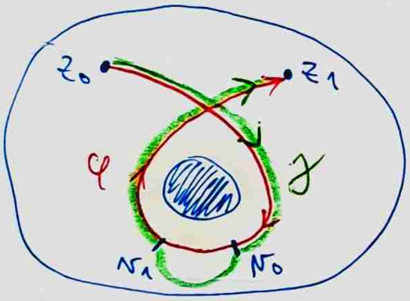
(2) **$\exists$ simply connected $\Omega' \subseteq \Omega$ containing φ, γ**



→ this is clearly convertible to (1)

⇒ (*) holds

(3) **γ, φ almost same**



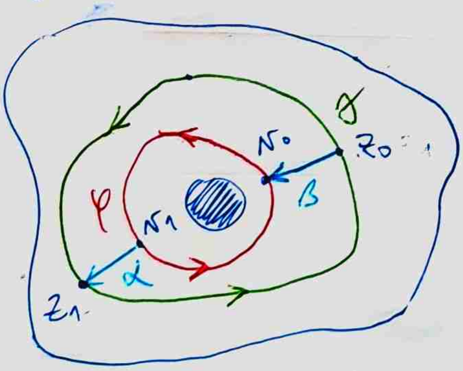
$\varphi = \varphi_1(z_0, v_0) \oplus \varphi_2(v_0, v_1) \oplus \varphi_3(v_1, z_1)$

$\gamma = \varphi_1(z_0, v_0) \oplus \gamma_2(v_0, v_1) \oplus \varphi_3(v_1, z_1)$

using (2): $\int_{\varphi_2} f = \int_{\gamma_2} f$

because $\int_\varphi + \int_\gamma = \int_{\varphi \oplus \gamma}$ ⇒ (*) holds

(4) **γ, φ jordan curves sharing interior holes** – not even touching



$\gamma = \gamma_1(z_0, z_1) \oplus \gamma_2(z_1, z_0)$

$\varphi = \varphi_1(v_0, v_1) \oplus \varphi_2(v_1, v_0)$

$\int_{\gamma_1} = \int_{\beta \oplus \varphi_1 \oplus \alpha}$ because (2)

$\int_{\gamma_2} = \int_{(-\alpha) \oplus \varphi_2 \oplus (-\beta)}$

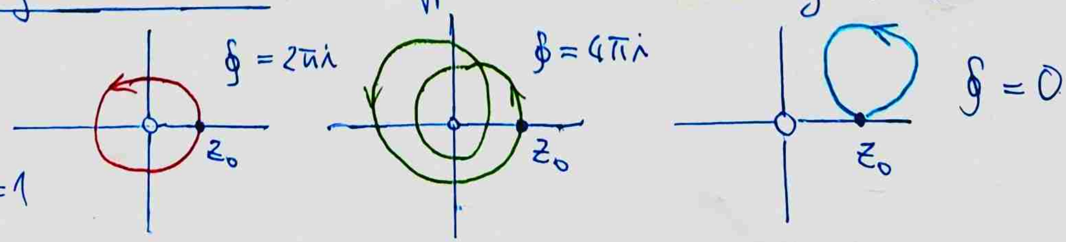
⇒ (*) holds

$$\Rightarrow \int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} = \int_\beta + \int_{\varphi_1} + \int_\alpha - \int_\alpha + \int_{\varphi_2} - \int_\beta = \int_{\varphi_1} + \int_{\varphi_2} = \int_\varphi$$

(5) **γ, φ touching closed curves**, but different holes or winding number ⇒ (*) breaks

$\Omega = \mathbb{C} \setminus \{0\}$

$f(z) = \frac{1}{z}$, $z_0 = 1$



Homotopic curves

Def: Let Ω be a domain (→ open & connected) and $\gamma_0, \gamma_1 \subseteq \Omega$ closed curves in Ω.
We say that γ_0 and γ_1 are (loop) homotopic $\equiv$

$\exists$ map $H(t,u)$, $H : [0,1] \times [0,1] \to \Omega$ s.t.

① for $\forall u \in [0,1]$, is the map $\varphi_u : [0,1] \to \Omega$, $\varphi_u(t) = H(t,u)$
a closed curve in Ω and $\underbrace{\langle \varphi_0 \rangle = \langle \gamma_0 \rangle}$, $\underbrace{\langle \varphi_1 \rangle = \langle \gamma_1 \rangle}$

② $H(t,u)$ is continuous in u & same orientation

Def: Let Ω be a domain and $\gamma_0, \gamma_1 \subseteq \Omega$ curves with the same endpoints.
We say that γ_0 and γ_1 are (end-to-end) homotopic $\equiv$

$\exists$ map $H(t,u)$, $H : [0,1] \times [0,1] \to \Omega$ s.t.

① for $\forall u \in [0,1]$, is the map $\varphi_u : [0,1] \to \Omega$, $\varphi_u(t) = H(t,u)$
a curve in Ω with the same endpoints and orientation as γ_0 and γ_1
and $\langle \varphi_0 \rangle = \langle \gamma_0 \rangle$ and $\langle \varphi_1 \rangle = \langle \gamma_1 \rangle$

② $H(t,u)$ is continuous in u

Intuition: Two curves are homotopic in Ω $\equiv$ $\exists$ continuous deformation between them that stays entirely within Ω.

Fact: Let Ω be a domain, γ, φ homotopic in Ω, $f \in \mathcal{H}[\Omega]$. Then

$\int_\gamma f = \int_\varphi f$ ... or $\oint_\gamma f = \oint_\varphi f$ if they are closed.

The Complex Logarithm

→ lets first try the square root – define as an inverse of $z \mapsto z^2$

⇒ consider $f: z \mapsto z^2$, $f^{-1}: z \mapsto \sqrt{z}$

→ look what happens in the unit circle $\Big<$ f^{-1}: halving angles; f: doubling angles

Theorem: If $f \in \mathcal{H}[\mathcal{U}(z_0)]$ and $f'(z_0) = 0$, then $f^{-1} \notin \mathcal{H}[\mathcal{U}(z_0)]$.

Proof: Let f be holomorphic on some $\mathcal{U}(z_0)$ ⇒ it is analytic at z_0

⇒ near z_0: $f(z) = \sum a_n (z-z_0)^n$

Well known result of formal power series:

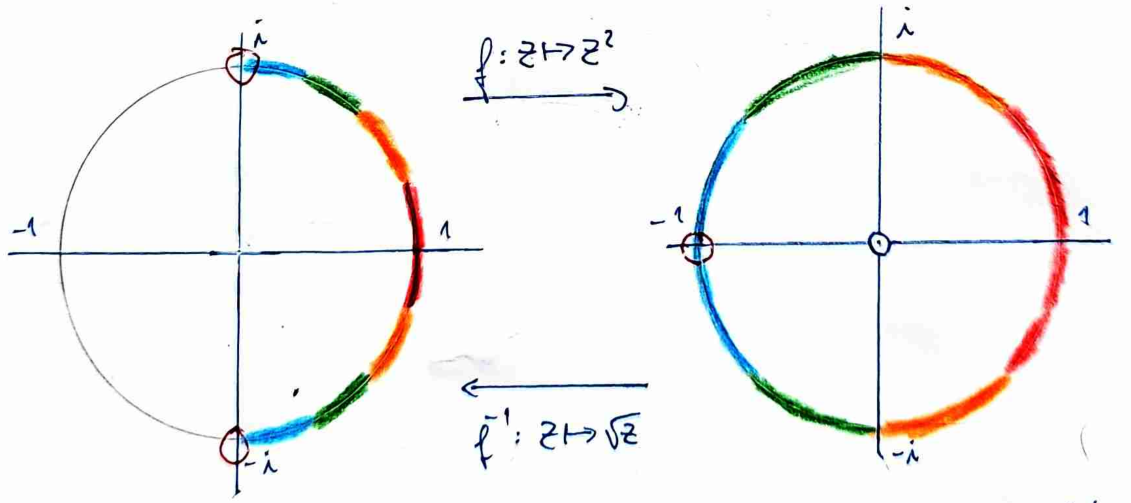
- A necessary condition for a FPS $A(x) = \sum a_n x^n$ to have a composition inverse is $a_1 \neq 0$

👁 $f'(z_0) = a_1$ ⇒ $f'(z_0) \neq 0$ is necessary for f^{-1} to be analytic at z_0 and we know analytic ⇔ holomorphic ∎

Consequence: We want $\sqrt{z}$ to be holomorphic ... an analytic continuation of $\sqrt{x}: \mathbb{R} \to \mathbb{R}$

⇒ necessary condition: $f'(z) = (z^2)' = 2z \neq 0$ ⇒ $z \neq 0$

⇒ we need to exclude $z_0 = 0$ from the definition domain of $\sqrt{z}$



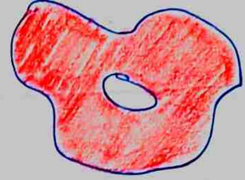
Problem: Clearly $f(i) = f(-i) = -1$, but what should $f^{-1}(-1) = \sqrt{-1}$ be?

Fact: $x \mapsto \sqrt{x}$ has no analytic continuation to any domain that contains any Jordan curve around the origin

⇒ we need to choose a "simple" domain – for example $\Omega = \mathbb{C} \setminus (-\infty, 0]$

and define: $z = re^{i\theta}$ ⇒ $\sqrt{z} := \sqrt{r} \cdot e^{i\frac{\theta}{2}}$

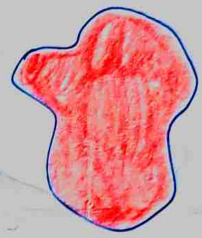
What is happening here?



Def: $\Omega \subseteq \mathbb{C}$ is a domain $\equiv$ it is open and connected

Def: $\Omega \subseteq \mathbb{C}$ is a simply connected domain $\equiv$ it is a domain and

- informally : doesn't contain any holes
- formally: $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ is connected
- differently: for $\forall$ Jordan curve $\varphi \subseteq \Omega$: $\mathrm{Int}(\varphi) \subseteq \Omega$



Def: We can define the complex logarithm on any SCD Ω s.t. $0 \notin \Omega$ as

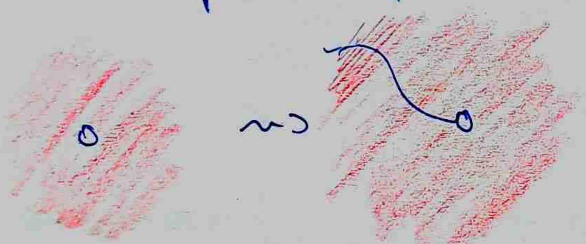
$$L(z_0) := \begin{cases} 0, & z_0 = 1 \\ \int_\gamma \frac{1}{z}, & \end{cases} \quad \text{where } \gamma \subseteq \Omega \text{ is any contour going } 1 \to z_0$$

Note: Usually $\Omega = \mathbb{C} \setminus (-\infty, 0]$

Why does this work?

→ $\frac{1}{z}$ not defined at 0 so we need to exclude 0 from Ω

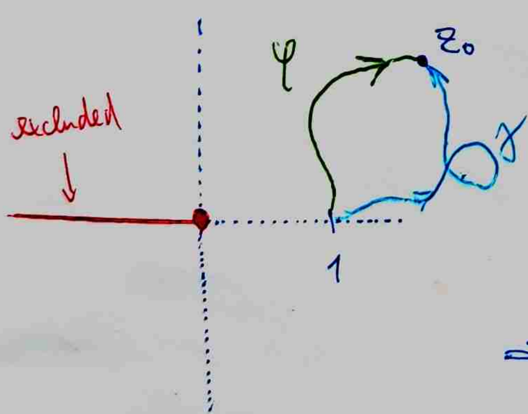
→ in order for the definition to be valid, we need Ω to be simply connected



⇒ we need to exclude some path going from 0 to infinity ⇐ branch cut

claim: When $\Omega \not\ni 0$ is a SCD, then the definition is correct

→ let $\varphi, \gamma \subseteq \Omega$ be two different $1 \to z_0$ curves : need $\int_\varphi \frac{1}{z} = \int_\gamma \frac{1}{z}$



⇒ define $\psi := \varphi \oplus (-\gamma)$ ... closed contour

Cauchy-Goursat: because $\frac{1}{z} \in \mathcal{H}[\Omega]$ and

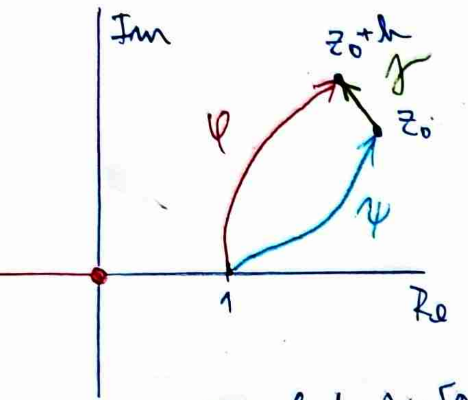
⊛ $\mathrm{Int}(\psi) \subseteq \Omega$ we have $\oint_\psi \frac{1}{z} = 0$

$$\Rightarrow \oint_\psi \frac{1}{z} = \int_\varphi \frac{1}{z} + \int_{-\gamma} \frac{1}{z} = \int_\varphi \frac{1}{z} - \int_\gamma \frac{1}{z} = 0 \quad \Rightarrow \int_\varphi \frac{1}{z} = \int_\gamma \frac{1}{z}$$

⊛ ψ isn't guaranteed to be a Jordan curve, but any closed curve can be composed of Jordan curves

Correctness of $L(z)$: If $\Omega \not\ni 0$ is S(D), then $L(z) \in \mathcal{H}[\Omega]$ and $L'(z) = \frac{1}{z}$.

$$L'(z_0) = \lim_{h \to 0} \frac{L(z_0+h) - L(z_0)}{h}$$



$$L(z_0+h) - L(z_0) = \int_\varphi - \int_\psi = \int_\gamma$$

$$\hookrightarrow \gamma = \underbrace{\varphi \oplus (-\psi) \oplus (-\gamma)}_{\text{Cauchy: } \oint = 0} \oplus \underbrace{\gamma}_{\int_\gamma}$$

$\Rightarrow$ let $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = z_0 + th$ $\Rightarrow \int_\gamma \frac{1}{z} = \int_0^1 \frac{h}{z_0+th}\, dt$

$$\Rightarrow \frac{L(z_0+h) - L(z_0)}{h} = \frac{1}{h}\int_0^1 \frac{h}{z_0+th}\, dt = \int_0^1 \frac{dt}{z_0+th} \xrightarrow{h \to 0} \int_0^1 \frac{dt}{z_0} = \frac{1}{z_0} \quad \blacksquare$$

Theorem: Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and Ω is a simply connected domain. Then f has a primitive function on Ω ... $\exists F(z) \in \mathcal{H}[\Omega]$ s.t. $F' = f$.

Proof: This is a direct generalization of what we just did.

① pick any $s \in \Omega$

② define for $z_0 \in \Omega$: $F(z_0) := \begin{cases} 0, & z_0 = s \\ \int_\gamma f, & z_0 \neq s \end{cases}$, γ = any $s \to z_0$ contour contained in Ω

① repeat what we did for the logarithm

$\hookrightarrow F = L$, $f = \frac{1}{z}$ $\blacksquare$

Correctness of $L(z)$: $\exp(L(z)) = z$

$\to$ define $y(z) := \exp(L(z))$, we will show $y(z) = z$

$$y'(z) = \exp(L(z)) \cdot L'(z) = y(z) \cdot \frac{1}{z}$$

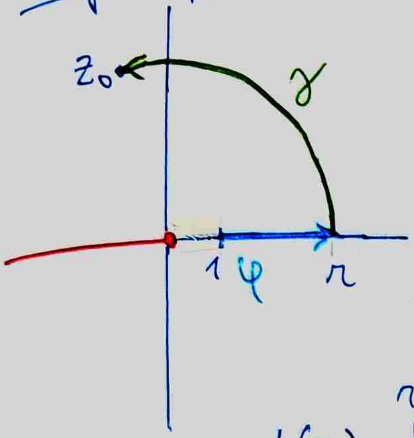
separable differential equation and $y(z) = z$ is a solution ... $1 = z \cdot \frac{1}{z}$ ✓

Innitial condition: We defined $L(1) = 0$ ... $f(1) = \exp(L(1)) = \exp(0) = 1$ ✓

Theorem (Log principle value): When $\Omega = \mathbb{C} \setminus (-\infty, 0]$ and $z \in \Omega$, then

$$L(z) = \ln|z| + \arg(z) \quad , \text{ where } \arg(z) \in (-\pi, \pi)$$

Proof: Suppose $z_0 = r \cdot e^{i\theta}$, $\theta \in (-\pi, \pi)$ ... this covers every $z \in \Omega$



$\varphi : [0,1] \to \mathbb{C}$, $\varphi(t) = 1 + (r-1)t$

$\gamma : [0, \theta] \to \mathbb{C}$, $\gamma(t) = r e^{it}$

$u = (r-1)t$
$du = (r-1)\, dt$

$$L(z_0) = \int_{\varphi \oplus \gamma} \frac{1}{z} = \int_\varphi \frac{1}{z} + \int_\gamma \frac{1}{z} = \int_0^1 \frac{(r-1)dt}{1+(r-1)t} + \int_0^\theta \frac{i r e^{it}}{r e^{it}} dt$$

$$\Rightarrow L(z_0) = \int_0^{r-1} \frac{du}{1+u} + \int_0^\theta i \, dt = \Big[\ln|1+u|\Big]_0^{r-1} + i\theta = \ln|r| + i\theta$$

$$\downarrow$$
$$r = |z_0| > 0 \qquad \blacksquare$$

Matrix functions

① matrix logarithm

→ let $\Omega = \mathbb{C} \setminus (-\infty, 0]$, $a \in \Omega$ and $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = 1 + t(a-1)$

$$L(a) = \int_\gamma \frac{1}{z} = \int_0^1 \frac{a-1}{1+t(a-1)} dt$$

→ $A \in \mathbb{C}^{n \times n}$ $\Rightarrow$ $\text{Log}(A) := \int_0^1 (A - I_n)\big(I_n + t(A - I_n)\big)^{-1} dt$

for this to be valid we need $I_n + t(A - I_n)$ to be invertible for $\forall t \in [0,1]$

↳ Fact: this holds $\iff \sigma(A) \cap (-\infty, 0] = \emptyset$ ... $\sigma(A)$ = eigenvalues of A

Remark:

⊛ $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot z^n$

$$\Rightarrow \ln(I_n + A) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot A^n$$

problem: ⊛ has radius of convergence $= 1$

⇒ this works only for A with spectral radius $\rho(A) < 1$

↳ $\max\{|\lambda| \mid \lambda \in \sigma(A)\}$

② complex exponential - using Cauchy ∮ formula

Cauchy: $\Omega \subseteq \mathbb{C}$ simply connected domain, $f(z) \in \mathcal{H}[\Omega]$,
$\varphi \subseteq \Omega$ jordan curve and $\mathrm{Int}(\varphi) \subseteq \Omega$. If $a \in \mathrm{Int}(\varphi)$, then

$$f(a) = \frac{1}{2\pi i} \oint_\varphi \frac{f(z)}{z-a}$$

exponential:

$$e^a = \frac{1}{2\pi i} \oint_\varphi \frac{e^z}{z-a} \quad , \quad a \in \mathrm{Int}(\varphi) \subseteq \Omega$$

matrix exp:

$$\exp(A) := \frac{1}{2\pi i} \oint_\varphi e^z \cdot (z \cdot I_n - A)^{-1} dz, \quad \sigma(A) \subseteq \mathrm{Int}(\varphi) \subseteq \Omega$$

general:

$$f(A) := \frac{1}{2\pi i} \oint_\varphi f(z) (z \cdot I_n - A)^{-1} dz, \quad \sigma(A) \subseteq \mathrm{Int}(\varphi) \subseteq \Omega$$

Singularities and poles

Def: Let $f \in \mathcal{H}[\mathcal{U}]$. Then f has an isolated singularity at $z_0 \in \mathbb{C} \equiv$

$z_0 \notin \mathcal{U}$ & $\exists \varepsilon > 0$ s.t. $\eta^*_{<\varepsilon}(z_0) \subseteq \mathcal{U}$.

↳ pierced neighborhood

Example:

$f(z) = \frac{1}{1-\exp(z)}$, $A := \{z \in \mathbb{C},\ e^z = 1\} = \{2\pi i \cdot k \mid k \in \mathbb{Z}\}$

↳ $f(z) \in \mathcal{H}[\mathbb{C} \setminus A]$ and points in A are isolated singularities of f

Def: Let z_0 be an isolated singularity of $f \in \mathcal{H}[\mathcal{U}]$. We say z_0 is a

- pole of order $d \in \mathbb{N}_0 \equiv f(z) \cdot (z - z_0)^d$ has an analytic continuation to $\mathcal{U} + z_0$.
- essential singularity otherwise

↳ pole of order 0 = removable singularity

Def: We say that $f^* : \mathcal{U}^* \to \mathbb{C}$ is an analytic continuation of $f : \mathcal{U} \to \mathbb{C}$ $\equiv$

1. $f \in \mathcal{H}[\mathcal{U}]$ and $f^* \in \mathcal{H}[\mathcal{U}^*]$
2. $\mathcal{U} \subseteq \mathcal{U}^*$ and $z \in \mathcal{U} \Rightarrow f^*(z) = f(z)$

Examples:

1. $f(z) = \frac{\sin(z)}{z} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

   removable singularity

   $\frac{\sin(z)}{z} = 1 - \frac{x^2}{3} + \frac{x^4}{5} + \dots \Rightarrow 0$ is a pole of order 0

2. $f(z) = \frac{z-3}{(z-2)^2 (z-1)^3} : \mathbb{C} \setminus \{1,2\} \to \mathbb{C} \Rightarrow$ 2 = pole of order 2, 1 = pole of order 3

3. $f(z) = \exp\left(\frac{1}{z}\right) : \mathbb{C} \setminus \{0\} \to \mathbb{C} \Rightarrow 0$ = essential singularity

   ↳ not obvious

Fact: If f has a pole of order $d \geq 1$ at z_0, then $\lim_{z \to z_0} |f(z)| = +\infty$

Theorem (Little Pickard theorem): f entire & non-constant $\Rightarrow f[\mathbb{C}] = \mathbb{C} \setminus K$, $|K| \leq 1$.

Intuition: Liouville: f entire & non-constant $\Rightarrow f$ unbounded

↳ Pickard says that it attains all values except at most one

Theorem (Great Pickard theorem): $f \in \mathcal{H}[\mathcal{U}]$ has essential singularity at $z_0 \in \mathcal{U}$, then for $\forall \varepsilon > 0$ denote $\eta := \mathcal{U} \cap \eta^*_{<\varepsilon}(z_0)$ and $f[\eta] = \mathbb{C} \setminus K$, $|K| \leq 1$.

Corollary: If z_0 is an isolated singularity of $f \in \mathcal{H}[U]$ and f is bounded on some $\eta^*_{<\varepsilon}(z_0)$, then z_0 is removable (pole of order 0).

Def: Let $f \in \mathcal{H}[U]$ and denote $U^* := U \cup \{z \in \mathbb{C} \setminus U \mid z \text{ is a pole of } f\}$.
Then we say that f is meromorphic on U^*.

Intuition: Holomorphic on U^* except for some isolated poles.

Example: Rational functions are meromorphic on $\mathbb{C}$

$\hookrightarrow f(z) = \frac{P(z)}{Q(z)}$, P, Q = polynomials

Fact: $f \in \mathcal{H}[U]$ is meromorphic on $U^* \iff f$ can be written as $f(z) = \frac{h(z)}{g(z)}$, where $g, h \in \mathcal{H}[U^*]$ and h is not identically zero on U^*.

Fact: If $f \in \mathcal{H}[U]$ is meromorphic on U^* and the set of poles $K = U^* \setminus U$ is finite, then $\exists$ rational function $r: \Omega \setminus K \to \mathbb{C}$ and $g \in \mathcal{H}[U^*]$ s.t.

$$f(z) = r(z) + g(z) \quad \text{for } \forall z \in U$$

Taylor and Laurent series

- **Taylor series:** f holomorphic on some $\eta_{<\varepsilon}(z_0)$, then on η

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

Cauchy $\oint$ formula: $a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}$, $\langle \gamma \rangle \subseteq \eta$, γ Jordan curve

- **Laurent series:** f holomorphic on some $\eta^*_{<\varepsilon}(z_0)$, then on η^*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}, \quad \langle \gamma \rangle \subseteq \eta^*, \ \gamma \text{ Jordan}$$

Fact: The Laurent series always exists and the terms (a_n) are unique.

Corollary: When z_0 is a removable singularity of f, then Laurent = Taylor

$\hookrightarrow$ Taylor is a valid L. series and it is unique

Remark: The negative powers part of a Laurent series is called its principle part

The Simple Residue Theorem

simple closed
piece-wise

Theorem: Let $\gamma \subseteq \mathbb{C}$ be a positively oriented (counterclockwise) Jordan contour. Suppose $f \in \mathcal{H}[U]$ has only finitely many isolated singularities inside γ, $z_1, \dots z_k \in \mathrm{Int}(\gamma)$ and denote $U^* = U \cup \{z_1, \dots z_k\}$. If

① $\langle \gamma \rangle \subseteq U$ ... γ doesn't go through any singularities of f

② $\mathrm{Int}(\gamma) \subseteq U^*$ ... f is defined on some region containing γ up to the singularities

then

$$\oint_\gamma f = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k),$$

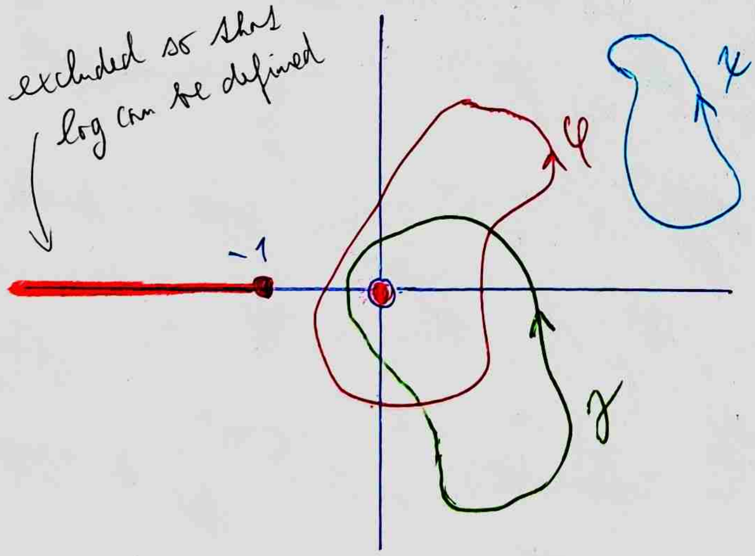
where $\mathrm{Res}(f, z_k)$ is the coefficient a_{-1} in the Laurent series expansion around z_k. Suppose f holomorphic on $\eta^*_{<\varepsilon}(z_k)$, then

$$\Rightarrow \mathrm{Res}(f, z_k) = \frac{1}{2\pi i} \oint_\varphi f, \quad \langle \varphi \rangle \subseteq \eta^*, \quad \varphi = \text{Jordan curve}$$

Calculating Residues

① $f(z) = \dfrac{\ln(1+z)}{z^3} = \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots\right) \cdot \frac{1}{z^3} \Rightarrow \mathrm{Res}(f, 0) = -\frac{1}{2}$

$U = \mathbb{C} \setminus (-\infty, -1] \setminus K, \quad K = \{0\}$



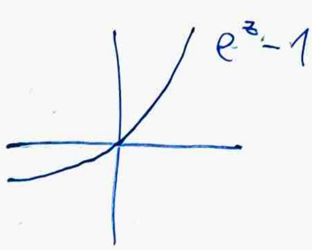
- $\int_\varphi f = \int_\gamma f = 2\pi i \cdot \left(-\frac{1}{2}\right) = -\pi i$
- $\int_\psi f = 0$

↳ but we also know this from the Cauchy-Goursat theorem

Proof: We will later proof a more general residue theorem formulation.

② $f(z) = \frac{1}{e^z - 1} \Rightarrow K = \{2\pi i \cdot k \mid k \in \mathbb{Z}\}$

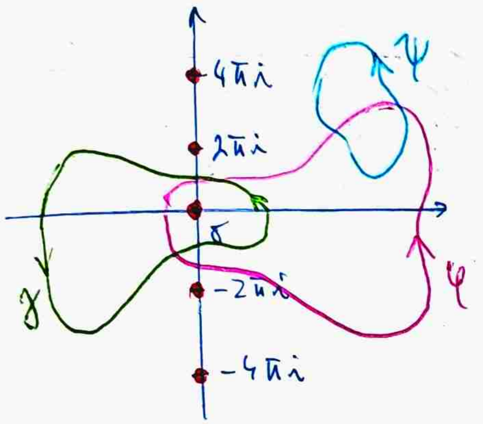
→ calculate $\text{Res}(f, 0)$



when $z \to 0$, $f(z) \to \frac{1}{z}$ $\because \lim_{z \to 0} \frac{e^z - 1}{z} = 1$

⇒ guess Laurent around 0 is $f(z) = \frac{1}{z} + B + Cz + Dz^2 + \dots$

because the first term is dominating when $z \to 0$ $\Rightarrow \text{Res}(f, 0) = 1$



$$\int_\gamma f = \int_\varphi f = 2\pi i$$

$$\int_\psi f = 0$$

<u>Theorem</u>: Suppose $f \in \mathcal{H}[U]$ and z_0 is a pole of f. What is $\text{Res}(f, z_0)$?

① removable singularity ⇒ Laurent = Taylor ⇒ $\text{Res}(f, z_0) = 0$

② pole of order $p \geq 1$ ⇒ $\exists\, g$ holomorphic on some $\mathcal{N}_{<\varepsilon}(z_0)$ s.t. $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - z_0)^p} \quad \text{and} \quad \text{Res}(f, z_0) = \frac{1}{(p-1)!} g^{(p-1)}(z_0)$$

<u>Proof</u>: Because g is holomorphic around z_0, it is analytic at z_0 and it has a Taylor expansion at z_0

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{n!} g^{(n)}(z_0)$$

The Laurent expansion of $f(z)$ around z_0 therefore is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-p} \Rightarrow [z^{-1}] f(z) = a_{p-1} = \frac{1}{(p-1)!} g^{(p-1)}(z_0) \qquad \blacksquare$$

<u>Special cases</u>:

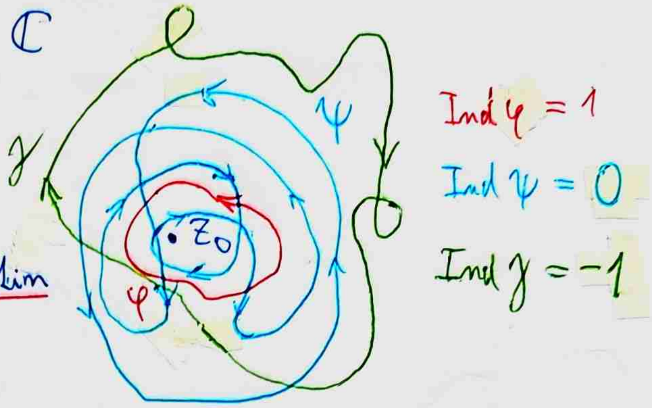
$$f(z) = \frac{g(z)}{z - z_0} \Rightarrow \text{Res}(f, z_0) = g(z_0)$$

$$f(z) = \frac{g(z)}{(z - z_0)^2} \Rightarrow \text{Res}(f, z_0) = g'(z_0)$$

The Winding Number

Def: Let $\gamma \subseteq \mathbb{C}$ be a contour of finite lenght and let $z_0 \in \mathbb{C} \setminus \langle \gamma \rangle$.
The windig number of γ about z_0 (or index of z_0 w.r.t. γ) is

$$\mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{1}{z - z_0} \in \mathbb{C}$$



Intuition: When γ is closed, Ind γ counts the # of times γ winds about z_0 in the positive (counterclockwise) direction

→ for simplicity suppose z_0 = origin
→ imagine the chage of $z = re^{i\theta}$ along γ
↳ what is dz? ... because dz is infinitely small : $dz = dz + dz$

$$\Rightarrow dz = dz + dz = \frac{dz}{dr} dr + \frac{dz}{d\theta} d\theta = e^{i\theta} dr + i r e^{i\theta} d\theta$$

$$\Rightarrow \frac{dz}{z} = \frac{dr}{r} + i\, d\theta$$

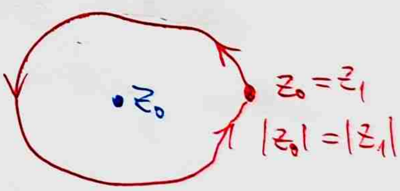
$$\Rightarrow \int_\gamma \frac{dz}{z} = \int_\gamma \frac{dr}{r} + i \int_\gamma d\theta$$

- $\int_\gamma \frac{dr}{r}$ = continuous change in $\frac{1}{r}$ along γ
- $\int_\gamma d\theta$ = continuous chage in θ along γ

} since γ is a contour ⇒ ind : $\mathbb{C} \setminus \langle\gamma\rangle \to \mathbb{C}$ is a continuous function

→ what if γ is closed?

$\oint_\gamma \frac{dr}{r} = 0$ ∵ the start and end $r = |z|$ is the same → total chge $= 0$



→ $\oint_\gamma \frac{dz}{z} = i \cdot$ The total change in Angle

⇒ that is why $\mathrm{Ind} = \frac{1}{2\pi i} \oint \frac{dz}{z}$ ... we want # trips about 0

Properties of Ind

① ind : $\mathbb{C} \setminus \langle\gamma\rangle \to \mathbb{C}$, $\mathrm{ind}(z) = \mathrm{Ind}_\gamma(z)$ is continuous

② γ closed ⇒ $\mathrm{Ind}_\gamma(z_0) \in \mathbb{Z}$

③ $\gamma = \varphi \oplus \psi$ ⇒ $\mathrm{Ind}_\gamma(z_0) = \mathrm{Ind}_\varphi(z_0) + \mathrm{Ind}_\psi(z_0)$

Def: The interior of a closed curve γ is $\mathrm{Int}(\gamma) := \{ z \in \mathbb{C} \mid \mathrm{Ind}_\gamma(z) \neq 0 \}$

Lemma: If γ is a closed contour, then $\mathrm{ind}_\gamma : \mathbb{C} \setminus \langle \gamma \rangle \to \mathbb{Z}$ is constant on the connected components of $\mathbb{C} \setminus \langle \gamma \rangle$.

Proof: Directly follows from properties ① and ② ■

Jordans Curve Theorem

→ let $\gamma \subseteq \mathbb{R}^2$ be Jordan curve $\equiv$ simple & closed

Theorem (weak J. thm): $\mathbb{R}^2 \setminus \langle \gamma \rangle$ is disconnected.

Theorem (Jordan theorem): $\mathbb{R}^2 \setminus \langle \gamma \rangle$ has exactly two connected components:

① a bounded region → the interior of γ $=: \mathrm{Int}(\gamma)$

② an unbounded region → the exterior of γ

and γ is their common boundary

Theorem (Jordan – Schoenflies thm): Moreover, the bounded component of $\mathbb{R}^2 \setminus \langle \gamma \rangle$ is homeomorphic to an open disk in $\mathbb{R}^2$.

$\Rightarrow \exists$ homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ which maps

- $\mathrm{Int}(\gamma)$ to the open unit disk $\{z \in \mathbb{R}^2 \mid |z| < 1\}$
- $\langle \gamma \rangle$ to the unit circle $\{z \in \mathbb{R}^2 \mid |z| = 1\}$

Recall: Homeomorphism is a bijection $f : X \to Y$ s.t. both f and f^{-1} are continuous.

Because $\mathbb{R}^2$ and $\mathbb{C}$ are homeomorphic, it doesn't matter where we work

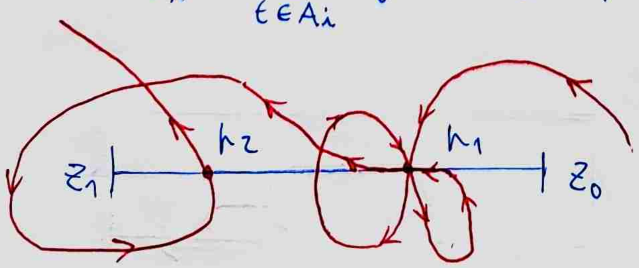
Theorem (Maříková věta): Let $\gamma : I \to \mathbb{C}$ be a closed curve and $z_0, z_1 \in \mathbb{C} \setminus \langle \gamma \rangle$ s.t. z_1 lies left of z_0, that is $\mathrm{Im}(z_0) = \mathrm{Im}(z_1)$, $\mathrm{Re}(z_0) > \mathrm{Re}(z_1)$.

Let γ intersect the segment $\overrightarrow{z_0 z_1}$ at $\langle \gamma \rangle \cap \langle z_0 z_1 \rangle = \{p_i\}_{i \in I}$. (could be infinite)

and denote $A_i := \{t \in I \mid \gamma(t) = p_i\}$. If $\forall t_0 \in A_i : \gamma'(t_0)$ is continuous, then

$$\mathrm{Ind}_\gamma(z_1) = \mathrm{Ind}_\gamma(z_0) + \sum_{i \in I} \delta_i \quad , \text{ where}$$

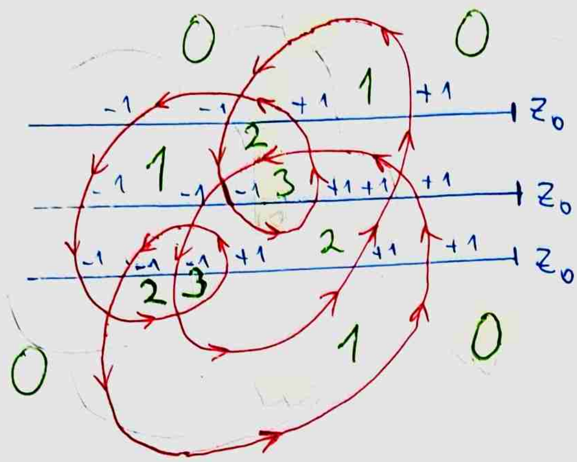
$$\delta_i := \sum_{t \in A_i} \mathrm{sign}(\mathrm{Im}(\gamma'(t))) = \sum \begin{pmatrix} +1 & \text{if } \gamma \text{ is going up at } \gamma(t) \\ -1 & \text{if } \gamma \text{ is going down at } \gamma(t) \end{pmatrix}$$

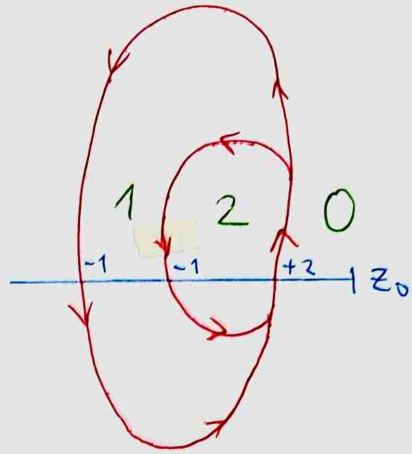


$\delta_1 = -1 - 1 + 0 = -2$

$\delta_2 = +1$

Example:





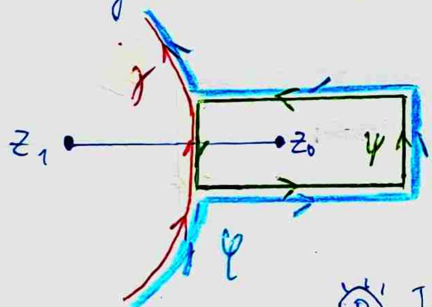
Proof: Induction by # connected components of $\langle z_0 z_1 \rangle \setminus \langle \gamma \rangle$

① if # components = 1, then γ doesn't cut $\overrightarrow{z_0 z_1}$ and so z_0 and z_1 are in the same component of $\mathbb{C} \setminus \langle \gamma \rangle$ (they are connected by $\overrightarrow{z_0 z_1}$)

$\Rightarrow \mathrm{Ind}_\gamma(z_0) = \mathrm{Ind}_\gamma(z_1)$ ... ind_γ is constant on the components of $\mathbb{C} \setminus \langle \gamma \rangle$

② WLOG assume # components = 2 ... bigger by induction

→ for simplicity assume that $\langle \gamma \rangle \cap \langle z_0 z_1 \rangle = \{p\}$ and that γ intersects $\overrightarrow{z_0 z_1}$ at p only once



→ we will create a simple closed Ψ encapsulating z_0 and going through p at the opposite direction

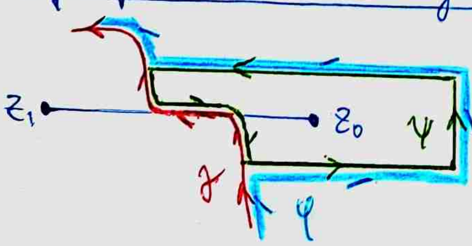
$\Rightarrow \varphi := \gamma \oplus \Psi$

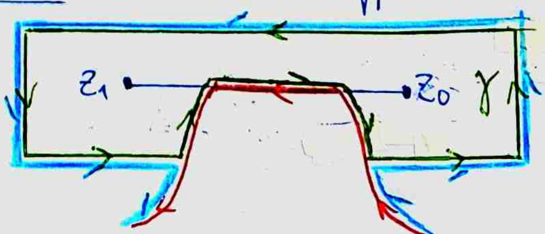
$\mathrm{Ind}_\varphi(z_1) = \mathrm{Ind}_\varphi(z_0)$ $\because$ z_0 and z_1 are in the same component of $\mathbb{C} \setminus \langle \varphi \rangle$

→ suppose that γ is going up through p

$$\left.\begin{aligned} \mathrm{Ind}_\varphi(z_1) &= \mathrm{Ind}_\gamma(z_1) + \mathrm{Ind}_\Psi(z_1) = \mathrm{Ind}_\gamma(z_1) + 0 \\ \mathrm{Ind}_\varphi(z_0) &= \mathrm{Ind}_\gamma(z_0) + \mathrm{Ind}_\Psi(z_0) = \mathrm{Ind}_\gamma(z_0) + 1 \end{aligned}\right\} \underline{\underline{\mathrm{Ind}_\gamma(z_1) = \mathrm{Ind}_\gamma(z_0) + 1}}$$

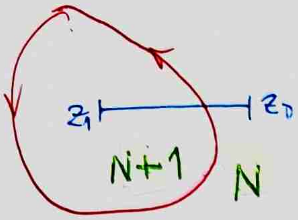
- if γ intersects $\overrightarrow{z_0 z_1}$ at p more than once
  $\Rightarrow$ simply repeat this process for each intersection
- if $\langle \gamma \rangle \cap \langle z_0 z_1 \rangle$ is a segment $\langle v_0 v_1 \rangle$ → create a different Ψ





Corollary: Jordan's weak theorem holds.

Proof: Let γ be a simple closed curve



Recall: $\operatorname{ind}_\gamma$ is constant on connected components of $\mathbb{C} \setminus \langle\gamma\rangle$

Marik: It is easy to see, that there $\exists\, z_0, z_1 \in \mathbb{C} \setminus \langle\gamma\rangle$, z_1 left of z_0 s.t. $\operatorname{Ind}(z_1) = \operatorname{Ind}(z_0) \pm 1$

$\Rightarrow$ Ind is not constant $\Rightarrow \mathbb{C} \setminus \langle\gamma\rangle$ is not connected $\blacksquare$

The General Residue Theorem

piece wise C^1

Theorem: Let $\gamma \subseteq \mathbb{C}$ be a closed contour with finite lenght.

Suppose $f \in \mathcal{H}[\mathcal{U}]$ has isolated singularities K and denote $\mathcal{U}^* := \mathcal{U} \cup K$. If

1. $\langle\gamma\rangle \subseteq \mathcal{U}$
2. $\mathcal{U}^*$ is simply connected = doesnt contain any "holes"
3. f has only finitely many singularities $z_1, \dots z_n$ s.t. $\operatorname{Ind}_\gamma(z_k) \neq 0$

then

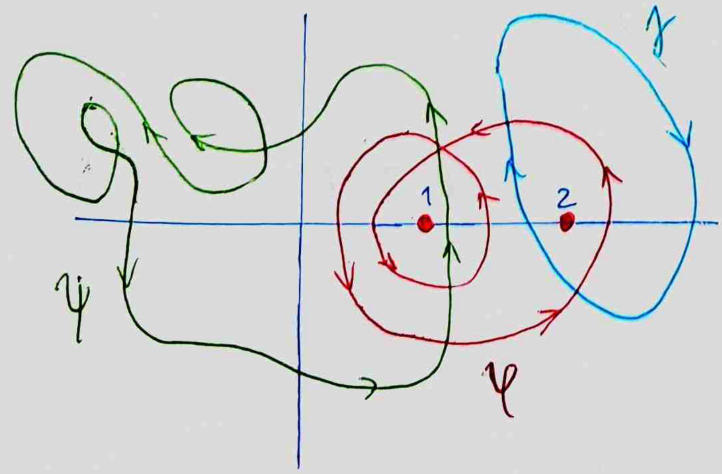
$$\oint_\gamma f = 2\pi i \sum_{k=1}^{n} \operatorname{Ind}_\gamma(z_k) \cdot \operatorname{Res}(f, z_k)$$

Example

$$f(z) = \frac{\sin(\frac{\pi}{4} z)}{(z-1)(z-2)^2} \quad \Rightarrow \text{poles} \begin{cases} z_1 = 1 & \text{of degree } 1 \\ z_2 = 2 & \text{of degree } 2 \end{cases}$$

$$\operatorname{Res}(f, 1) = \frac{\sin(\frac{\pi}{4} z)}{(z-2)^2}\Big|_{z=1} = \frac{\sin(\frac{\pi}{4})}{(1-2)^2} = \frac{\sqrt{2}}{2}$$

$$\operatorname{Res}(f, 2) = \frac{d}{dz} \frac{\sin(\frac{\pi}{4} z)}{z-1}\Big|_{z=2} = \frac{\frac{\pi}{4}\cos(\frac{\pi}{4} z)(z-1) - \sin(\frac{\pi}{4} z)}{(z-1)^2}\Bigg|_{z=2} = \frac{\pi}{4}\underbrace{\cos(\tfrac{\pi}{2})}_{=0} - \underbrace{\sin(\tfrac{\pi}{2})}_{=1} = -1$$



$$\oint_\gamma f = -(-1) = 1$$

$$\oint_\psi f = \frac{\sqrt{2}}{2}$$

$$\oint_\varphi f = 2 \cdot \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$

Generalized Cauchy-Goursat theorem

Def: For an open region $\mathcal{U} \subseteq \mathbb{C}$ we define its boundary as $\partial\mathcal{U} = \overline{\mathcal{U}} \setminus \mathcal{U}$.

Theorem (Standard C.-G.): Let Ω be a simply connected domain and $f \in \mathcal{H}[\Omega]$.
If $\gamma \subseteq \Omega$ is a Jordan contour, then $\oint_\gamma f = 0$.

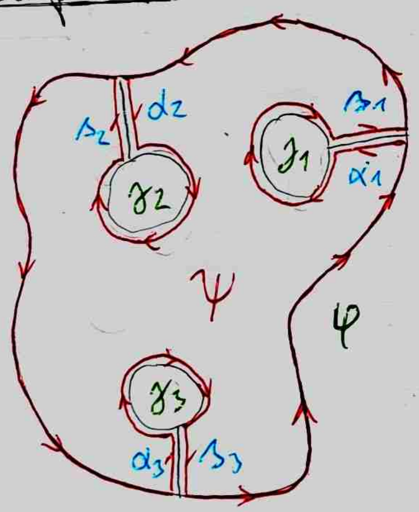
Theorem (Generalized C.-G.): Let Ω be a bounded domain whose boundary consists of finitely many positively oriented Jordan contours

- φ ... the outer boundary
- $\gamma_1, \dots \gamma_n$ ... the inner boundaries of "holes" in Ω

Let $\mathcal{U} \subseteq \mathbb{C}$ be an open region s.t. $\overline{\Omega} \subseteq \mathcal{U}$ and let $f \in \mathcal{H}[\mathcal{U}]$. Then

$$\oint_\varphi f = \sum_{i=1}^{n} \oint_{\gamma_i} f$$

Proof idea:



① from each γ_i we draw a smooth line to φ s.t. none of these lines intersect

② remove these lines from Ω $\rightsquigarrow \Omega'$

Ω' is now simply connected

③ create a new Jordan contour $\psi \subseteq \Omega'$ which traces the boundary of Ω' using $\varphi, \gamma_1, \dots \gamma_n$ and bridges α_i, β_i along the lines which created Ω'

$\hookrightarrow \Omega'$ is simply connected & $\psi \subseteq \Omega'$ is Jordan $\Rightarrow \oint_\psi f = 0$

$$\psi = \hat{\varphi} \oplus \sum_{i=1}^{n} (\alpha_i \oplus (-\hat{\gamma}_i) \oplus \beta_i)$$

→ where $\hat{\varphi}, \hat{\gamma}_i$ are φ, γ_i missing a little segment

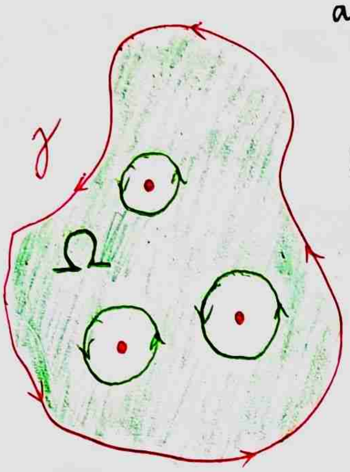
→ because $\overline{\Omega'} = \overline{\Omega}$, we can let the bridges α_i and β_i get arbitrarily close

$$\Rightarrow \hat{\varphi} \to \varphi, \ \hat{\gamma}_i \to \gamma, \ \beta_i \to -\alpha_i \quad \Rightarrow \psi \to \varphi \oplus \sum_{i=1}^{n} (-\gamma_i)$$

$$\Rightarrow \oint_\psi f = 0 \Rightarrow \oint_\varphi f - \sum_{i=1}^{n} \oint_{\gamma_i} f = 0 \Rightarrow \oint_\varphi f = \sum_i \oint_{\gamma_i} f \qquad \blacksquare$$

Proof of the Residue Theorem

→ for simplicity assume that the closed contour γ is simple (Jordan) and positively oriented



→ let $z_1, \dots z_m$ be the singularities of f inside $\mathrm{Int}(\gamma)$

① for each z_i, choose a small positively oriented Jordan γ_i around z_i s.t. $\langle \gamma_i \rangle \subseteq \mathrm{Int}(\gamma)$ and $\langle \gamma_i \rangle \cap \langle \gamma_j \rangle = \emptyset$

② let $\Omega := \mathrm{Int}(\gamma) \setminus \bigcup_{i=1}^{m} (\langle \gamma_i \rangle \cup \mathrm{Int}(\gamma_i))$

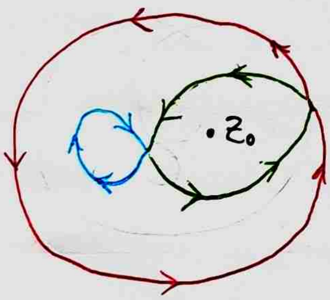
$f \in \mathcal{H}[\Omega]$ because we removed all of the singularities

③ use the generaralized Cauchy-Goursat theorem

$$\oint_\gamma f = \sum_{i=1}^{m} \oint_{\gamma_i} f = 2\pi i \cdot \sum_{i=1}^{m} \mathrm{Res}(f, z_i)$$ from the def. of residues

→ <u>what if γ is not simple?</u>

every closed contour γ of finite lenght can be decomposed into finitely many Jordan contours $\varphi_1, \dots \varphi_n$ s.t. $\gamma = \varphi_1 \oplus \dots \oplus \varphi_k$



→ let $z_1, \dots z_m$ be the singularities z_j of f for which $\exists\, \varphi_i$ s.t. $z_j \in \mathrm{Int}(\varphi_i)$

for each φ_i we have

$$\mathrm{Ind}_{\varphi_i}(z_j) = \begin{cases} 0, & z_j \notin \mathrm{Int}(\varphi_i) \\ +1, & z_j \in \mathrm{Int}(\varphi_i) \ \& \ \varphi_i \text{ is positively oriented} \\ -1, & z_j \in \mathrm{Int}(\varphi_i) \ \& \ \varphi_i \text{ is negatively oriented} \end{cases}$$

⇒ using the simple version we have

$$\oint_{\varphi_i} f = 2\pi i \sum_{j=1}^{m} \mathrm{Ind}_{\varphi_i}(z_j)\, \mathrm{Res}(f, z_j)$$

→ earlier we assumed positive orient. ⇒ $\mathrm{Ind} = 1$

→ because $\mathrm{Ind}_{\alpha \oplus \beta}(z) = \mathrm{Ind}_\alpha(z) + \mathrm{Ind}_\beta(z)$ and $\int_{\alpha \oplus \beta} f = \int_\alpha f + \int_\beta f$

$$\mathrm{Ind}_\gamma(z_j) = \sum_{i=1}^{k} \mathrm{Ind}_{\varphi_i}(z_j)$$

$$\oint_\gamma f = \sum_{i=1}^{k} \oint_{\varphi_i} f = 2\pi i \sum_{i=1}^{k} \sum_{j=1}^{m} \mathrm{Ind}_{\varphi_i}(z_j)\, \mathrm{Res}(f, z_j)$$

$$= 2\pi i \sum_{j=1}^{m} \mathrm{Res}(f, z_j) \sum_{i=1}^{k} \mathrm{Ind}_{\varphi_i}(z_j) = 2\pi i \sum_{j=1}^{m} \mathrm{Res}(f, z_j)\, \mathrm{Ind}_\gamma(z_j)$$

■

Poles and Zeroes

→ zeroes of a function are the dual concept of poles

Fact: f analytic on a domain Ω, $zero(f) := \{z \in \Omega \mid f(z) = 0\}$, then either $zero(f) = \Omega$ or $zero(f)$ is a discrete set $\equiv \forall z_0 \in zero(f)$ is isolated in $\mathbb{C}$.

Consequence: $f \not\equiv 0$ analytic at z_0 $\Rightarrow \exists m : \frac{f(z)}{(z-z_0)^m}$ nonzero on some $\eta_{<\varepsilon}(z_0)$

↳ let $A(x)$ be the series expansion of f and a_m the first non-zero coefficient

$$f(z) = \sum_{k=m}^{\infty} a_k (z-z_0)^k = (z-z_0)^m \underbrace{\sum_{k=0}^{\infty} a_{k+m}(z-z_0)^k}_{\text{analytic and nonzero on some } \eta_{<\varepsilon}(z_0)} \quad \blacksquare$$

Def: If $f \not\equiv 0$ is meromorphic on some $\eta^*_{<\varepsilon}(z_0)$, then there exists a unique $p \in \mathbb{Z}$ s.t. $(z-z_0)^p f(z)$ is holomorphic and nonzero on some $\eta_{<\delta}(z_0)$.

- $p = 0 \Rightarrow z_0$ is a removable singularity of f
- $p > 0 \Rightarrow z_0$ is a pole of order p of f
- $p < 0 \Rightarrow z_0$ is a zero of order $|p|$ of f

$|p| = 1$: simple pole / zero

Intuition:

$f(z) = \frac{h(z)}{(z-z_0)^p}$ ... z_0 is a pole of order p $\Leftrightarrow$ h is analytic and nonzero on some $\eta_{<\delta}(z_0)$

$f(z) = (z-z_0)^p g(z)$ ... z_0 is a zero of order p $\Leftrightarrow$ g is analytic and nonzero on some $\eta_{<\delta}(z_0)$

Example

$$f(z) = \frac{(z-1)(z+i)^2}{(e^z-1)(z-2)^4}$$

$\Rightarrow$ $z = 1$ ... zero of order 1

$z = -i$ ... zero of order 2

$z = 2$ ... pole of order 4

↓

$e^z = 1 \Leftrightarrow z = 2\pi i \cdot k \Rightarrow z = 2\pi i \cdot k$ ... pole of order 1, $k \in \mathbb{Z}$

↳ $e^z - 1 = z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots$

z_0 order p zero of $f(z)$ $\Leftrightarrow$ z_0 order p pole of $\frac{1}{f(z)}$

Laurent series around zeroes and poles

→ suppose z_0 is a zero / pole of $f(z)$ and consider the L. expansion around z_0

① z_0 order p zero $\Rightarrow$ $f(z) = \sum_{n=p}^{\infty} a_n (z-z_0)^n$, $a_p \neq 0$

② z_0 order p pole $\Rightarrow$ $f(z) = \sum_{n=-p}^{\infty} a_n (z-z_0)^n$, $a_{-p} \neq 0$

Calculating Residues at poles

Recall: z_0 order p pole of f $\Rightarrow \exists\, g$ holomorphic and nonzero on some $\eta_{<\varepsilon}(z_0)$ s.t.

$$f(z) = \frac{g(z)}{(z-z_0)^p} \quad \text{and} \quad \operatorname{Res}(f, z_0) = \frac{1}{(p-1)!} g^{(p-1)}(z_0)$$

Special cases:

→ suppose that g is holomorphic and nonzero on some $\eta_{<\varepsilon}(z_0)$

① $f(z) = \dfrac{g(z)}{z-z_0} \Rightarrow \operatorname{Res}(f, z_0) = g(z_0)$

② $f(z) = \dfrac{g(z)}{(z-z_0)^2} \Rightarrow \operatorname{Res}(f, z_0) = g'(z_0)$

→ suppose that $h(z)$ has a simple zero at z_0

③ $f(z) = \dfrac{1}{h(z)} \Rightarrow \operatorname{Res}(f, z_0) = \dfrac{1}{h'(z_0)}$

④ $f(z) = \dfrac{g(z)}{h(z)} \Rightarrow \operatorname{Res}(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}$

Proof of ④

$$f(z) = \frac{g(z)}{(z-z_0)\mu(z)} \Rightarrow \operatorname{Res}(f, z_0) = [z^{-1}] f(z) = [z^{-1}] \frac{g(z)}{(z-z_0)\mu(z)} = [z^0] \frac{g(z)}{\mu(z)}$$

→ because both $g(z)$ and $\mu(z)$ are analytic and nonzero at z_0, this is

$$\operatorname{Res}(f, z_0) = \frac{[z_0]\, g(z)}{[z_0]\, \mu(z)} = \frac{g(z_0)}{h'(z_0)} \qquad \blacksquare$$

Example:

$f(z) = \dfrac{z-i}{(e^z-1)(z-2)^2}$

$z_0 = 2 \xrightarrow{②} \operatorname{Res}(f, 2) = \left.\dfrac{e^z - 1 - (z-i)e^z}{(e^z-1)^2}\right|_{z=2}$

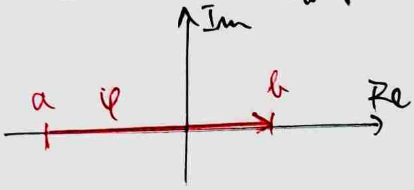
$z_0 = 2\pi i k \xrightarrow{④} \operatorname{Res}(f, 2\pi i k) = \left.\dfrac{z-i}{e^z (z-2)^2}\right|_{z=2\pi i k}$

Applications of the Residue Theorem

→ the Residue theorem is extremely powerful at evaluating certain kinds of <u>improper integrals</u>, even of non-elementary functions

Real integrals can be calculated using curve integrals

<u>Want:</u> $I = \int_a^b f(x)\,dx$



$\varphi : [a,b] \to \mathbb{C}$
$\varphi(t) = t$

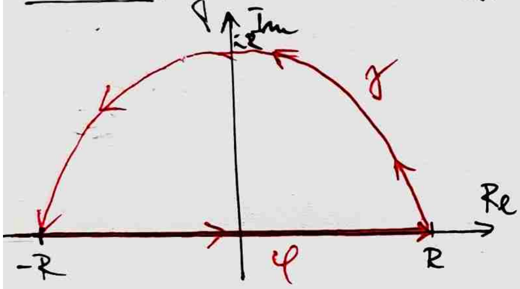
$\int_\varphi f(z)\,dz = \int_a^b f(t)\,dt = I$

<u>Recall: ML-equality:</u>

$\int_\varphi f \leq |\varphi| \cdot \sup_{z \in \langle\varphi\rangle} f(z)$

assume Q doesnt have any real roots

<u>Method:</u> f rational $= \frac{P(x)}{Q(x)}$ $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = ?$



<u>Residue theorem:</u> we can calculate $\oint_{\varphi \oplus \gamma} f$

$\oint_{\varphi \oplus \gamma} f = \int_\varphi f + \int_\gamma f$

→ <u>let $R \to \infty$</u>

- $\int_\varphi f = \int_{-R}^{R} f(x)\,dx \to \int_{-\infty}^{\infty} f(x)\,dx$ ← what we want
- $\int_\gamma f \leq |\gamma| \cdot \sup_{z \in \langle\varphi\rangle} |f(z)| = \pi \cdot R \cdot \sup_{z \in \langle\varphi\rangle} |f(z)|$

→ if $\deg(Q) \geq \deg(P) + 2$, then as $R \to \infty$:

$\int_\gamma f \leq \pi \cdot R \cdot \frac{1}{R^2} \longrightarrow 0$

$\Rightarrow \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \oint_\Gamma \frac{P(z)}{Q(z)}\,dz$, where $\Gamma =$  for Q no real roots, $\deg(Q) \geq \deg(P) + 2$

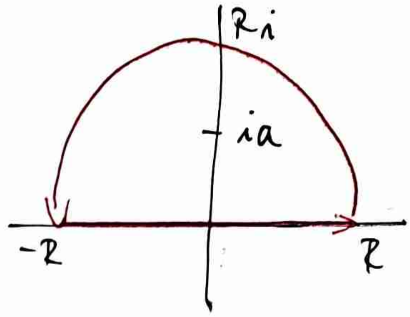
<u>Calculating Residues</u>

$\operatorname{Res}\left(\frac{1}{Q(z)}, z_0\right) = \frac{1}{Q'(z_0)}$

$\operatorname{Res}\left(\frac{P(z)}{Q(z)}, z_0\right) = \frac{P(z_0)}{Q'(z_0)}$

Examples

① $\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx = \frac{\pi}{a}$

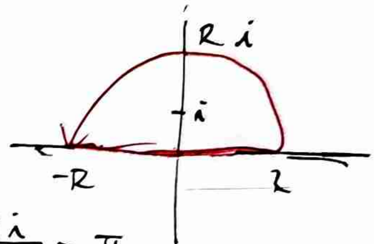


poles: $x^2 = -a^2 \Rightarrow x = \pm ia$ $\nearrow a > 0$

convergence: $\deg(x^2+a^2) \geq \deg(1) + 2$ ✓

$$I = \int_{\Gamma} \frac{1}{z^2+a^2} = 2\pi i \cdot \operatorname{Res}(f, ia) = 2\pi i \left.\frac{1}{2z}\right|_{ia} = 2\pi i \cdot \frac{1}{2ia} = \frac{\pi}{a}$$

② $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$

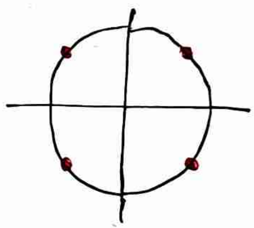


- convergence ✓
- poles: $z^2+1=0 \Rightarrow z = \pm i$, order $= 2$

$$I = \int_{\Gamma} \frac{dz}{(z^2+1)^2} = 2\pi i \cdot \operatorname{Res}(f, i) = 2\pi i \cdot \frac{-2}{(2i)^3} = \frac{-4\pi i}{-8i} = \frac{\pi}{2}$$

- $f(z) = \frac{1}{(z-i)^2} \cdot \frac{1}{(z+i)^2} \Rightarrow \operatorname{Res}(f,i) = \frac{d}{dz}\left.\frac{1}{(z+i)^2}\right|_{z=i} = \left.\frac{-2}{(z+i)^3}\right|_{z=i} = \frac{-2}{(2i)^3}$

③ $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$



poles: $z^4 = -1 \Rightarrow$ 4-th roots of unity

$\hookrightarrow e^{4i\theta} = e^{i(\pi + 2k\pi)}$

$\Rightarrow 4\theta = \pi(1+2k) \Rightarrow \theta = \frac{\pi}{4}(1+2k) = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\hookrightarrow$ upper half-plane: $\frac{\pi}{4}, \frac{3\pi}{4}$

- $\theta = \frac{\pi}{4} \Rightarrow z_1 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}(1+i)$
- $\theta = \frac{3\pi}{4} \Rightarrow z_2 = \frac{1}{\sqrt{2}}(-1+i)$

$\Big\}$ simple poles

$$\operatorname{Res}(f, z_1) = \left.\frac{1}{4z^3}\right|_{z_1} = \frac{1}{4e^{i\pi/4 \cdot 3}} = \frac{1}{4 \cdot \frac{1}{\sqrt{2}}(-1+i)} \cdot \frac{-1-i}{-1-i} = \frac{1}{2\sqrt{2}} \cdot \frac{-1-i}{1+1} = \frac{-1-i}{4\sqrt{2}}$$

$$\operatorname{Res}(f, z_2) = \left.\frac{1}{4z^3}\right|_{z_2} = \frac{1}{4e^{i3\pi/4 \cdot 3}} = \frac{1}{4 \cdot \frac{1}{\sqrt{2}}(1+i)} \cdot \frac{1-i}{1-i} = \frac{1-i}{2\sqrt{2}(1+1)} = \frac{1-i}{4\sqrt{2}}$$

$$I = 2\pi i \sum \operatorname{Res} = 2\pi i \cdot \left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right) = 2\pi i \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Picards Theorem (E+U)

Theorem: Let $x_0, y_0 \in \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ be continuous and suppose $\exists M > 0$ s.t.

for $\forall x, y_1, y_2 \in \mathbb{R}$: $|F(x,y_1) - F(x,y_2)| \leq M \cdot |y_1 - y_2|$ $\boxtimes$

Then the initial value problem

$\circledast$ $y(x_0) = y_0$ & $y'(x) = F(x, y(x))$

has a unique solution $y : U \to \mathbb{R}$ defined on some $U = [x_0 - \delta, x_0 + \delta]$, $\delta > 0$

Proof: Recall this result implied by the Fundamental Theorem of Calculus:

$y(x)$ solves $\circledast$ $\iff$ $y(x)$ solves $y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt$ $(**)$

→ let $I := [x_0 - \delta, x_0 + \delta]$, we will show that for sufficiently small δ, $(**)$ has a unique solution on I.

Def: $f : (M,d) \to (M,d)$ is a contracting mapping of (M,d) into itself $\equiv$

$\exists \alpha \in (0,1)$ s.t. $\forall a, b \in M$: $d(f(a), f(b)) \leq \alpha \cdot d(a,b)$

Theorem (Banach): Every contraction $f : M \to M$ (M complete) has a unique fixed point $f(a) = a$.

Lemma: The MS $(C[I], \|f-g\|_\infty)$ is complete. $\|f - g\| = \max_{x \in I} |f(x) - g(x)|$

→ back to the proof. Notice that $(**)$ defines a mapping $A : C[I] \to C[I]$

$$A : y \longmapsto y_0 + \int_{x_0}^{x} F(t, y(t))\, dt$$

→ plan: show that A is a contraction of $C[I]$ into itself $\xRightarrow{\text{Banach}}$ it has a unique fixed point

→ let $y_1, y_2 \in C[I]$ and δ be very small, then

$$d(A(y_1), A(y_2)) = \max_{x \in I} |A(y_1)(x) - A(y_2)(x)| = \max_{x \in I} \left| \int_{x_0}^{x} F(t, y_1(t))\, dt - \int_{x_0}^{x} F(t, y_2(t))\, dt \right|$$

$$= \max_{x \in I} \left| \int_{x_0}^{x} F(t, y_1(t)) + F(t, y_2(t))\, dt \right| \leq \max_{x \in I} \int_{x_0}^{x} |F(t, y_1(t)) - F(t, y_2(t))|\, dt$$

$$\boxtimes \leq \max_{x \in I} \int_{x_0}^{x} M \cdot |y_1(t) - y_2(t)|\, dt \leq M \cdot \max_{x \in I} \int_{x_0}^{x} \underbrace{\|y_1 - y_2\|_\infty}_{\in \mathbb{R}}\, dt$$

$$= M \cdot d(y_1, y_2) \cdot \overbrace{(x - x_0)}^{\leq \delta} \leq \delta M \cdot d(y_1, y_2)$$

→ we just set δ sufficiently small that $\delta M < 1$ and we have a contraction