Graph Cycle and Cut Spaces

Def: A tour in a graph is a sequence of vertices and edges $v_1 e_1 v_2 \dots e_k v_{k+1}$ where no two edges repeat.

Def: An Euler tour is a tour which covers all of the edges.

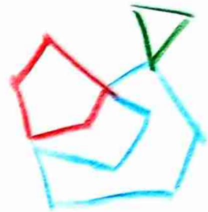
Theorem: A connected graph G has a closed Euler tour $\iff$ G has all degrees even.

Pf: ↳ closed tour := start node = end node

$\Rightarrow$: The tour is closed, so every time it enters a vertex it also exits it and because it cannot reuse edges: 1 visit $\Rightarrow$ degree $+2$

$\Leftarrow$:

Def: $E' \subseteq E(G)$ is even $\equiv G' = (V, E')$ has all degrees even.



Lemma: Let $G = (V, E)$ where $E \neq \emptyset$ is even. Then G has a cycle.

Pf: Since $E \neq \emptyset$ there is a component of G with some edges

↳ lets focus on this component only

↳ if it didnt contain a cycle, then it would be a tree $\Rightarrow E$ not even ↯

Corollary: Every graph with all degrees even is an edge-disjoint union of cycles.

↳ simply apply the previous lemma, find a cycle

↳ remove the cycle → all degrees still even

↳ repeat untill there are no edges left

→ it is easy to create the Euler tour from these cycles by induction by starting with no edges and adding one cycle at a time

↳ the next cycle always has to be conected to what we already have

↳ but G is connected so that is not an issue ∎

This algorithm is polynomial $O(|V| \cdot (|V| + |E|))$

Fact: The following generalization of this problem can also be solved polynomially

$G = (V, E)$, $w: E \to \mathbb{Q}$, find even $E' \subseteq E$ s.t.

$$w(E') := \sum_{e \in E'} w(e) \text{ is maximal}$$

Fact: The problem of finding a closed path covering all vertices (Hamiltonian cycle) cannot be solved polynomially → its NP complete

- Cycle space

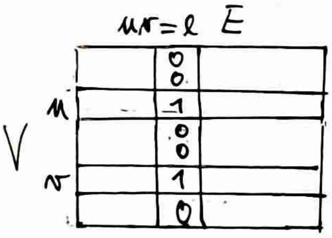
Def: The cycle space of a graph $G=(V,E)$ is the set of all of its even-degree subgraphs over the finite field $\mathbb{F}_2=\{0,1\}$.

- vectors $=\{(V,E') \mid \text{even } E'\subseteq E\}$
- zero vector $=(V,\emptyset)$ → symmetric difference $A\Delta B=(A\cup B)\setminus(A\cap B)$
- sum of two vectors: $(V,E_1)\oplus(V,E_2):=(V,E_1\Delta E_2)$

- opposite vector: $-(V,E)=(V,E)$ $\because$ $(V,E)\oplus(V,E)=(V,\emptyset)$
- scalar multiplication: $0\cdot(V,E)=(V,\emptyset)$
  $1\cdot(V,E)=(V,E)$

Theorem: The kernel of the incidence matrix of G over $\mathbb{F}_2$ is the cycle space of G.

Def: The incidence matrix of $G=(V,E)$ is



$$(I_G)_{ij}:=\begin{cases}1, & v_i\in e_j\\ 0, & v_i\notin e_j\end{cases}\qquad I_G\in\mathbb{F}_2^{|V|\times|E|}$$

Pf: $\ker_{\mathbb{F}_2}(I_G)=\{x\in\mathbb{F}_2^{|E|}\mid I_G\cdot x=0\}$

↳ x determines which edges will be used

$\Rightarrow x=\chi_{E'}$ = characteristical vector for some $E'\subseteq E$

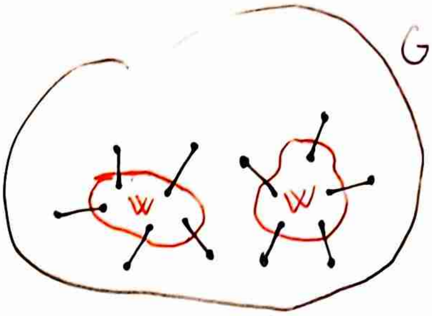
the i^{th} element of $I_G\cdot\chi_{E'}$ is $\deg(v_i)$ in (V,E')

$\Rightarrow$ we are in $\mathbb{F}_2$ $\Rightarrow$ $I_G\cdot\chi_{E'}=0 \iff$ all degrees in (V,E') are even

$\Rightarrow \ker_{\mathbb{F}_2}(I_G)=\{\chi_{E'}\mid E'\subseteq E \text{ is even}\}$ $\square$

Corollary: Since $\ker I_G$ = cycle space and ker is a vector space, it follows that cycle space is really a vector space too.

• Cut Space



Def: An edge cut in $G=(V,E)$ is any $E' \subseteq E$ s.t.

$$\exists W \subseteq V : E' = \{ e \in E \mid |e \cap W| = 1 \}$$

Def: The cut space of $G=(V,E)$ is the set of all of its edge cuts over the field $\mathbb{F}_2$.

- vectors $= \{ E' \subseteq E \mid E'$ is an edge cut in $G \}$
- zero vector $= \emptyset$
- vector addition: $E_1 \oplus E_2 := E_1 \Delta E_2$ ... symmetric difference
- opposite vector: $-E' = E'$ $\because E' \Delta E' = \emptyset$

Lemma: The symmetric difference between two edge cuts is an edge cut.

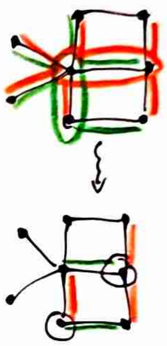
Pf: E_1 with W_1 ... $e \in E_1 \Leftrightarrow |e \cap W_1| = 1$

E_2 with W_2 ... $e \in E_2 \Leftrightarrow |e \cap W_2| = 1$

$\rightarrow E_1 \Delta E_2$ with $W_1 \Delta W_2$ ...

↳ $e \in E_1 \Delta E_2 \Longleftrightarrow |e \cap (W_1 \Delta W_2)| = 1$

$e \in E_1 \cup E_2$ & $e \notin E_1 \cap E_2 \Longleftrightarrow \underbrace{|e \cap (W_1 \cup W_2)|}_{2 \text{ or } 1} - \underbrace{|e \cap (W_1 \cap W_2)|}_{1 \text{ or } 0} = 1$



Lemma: Each edge cut $E' \subseteq E$ with $W \subseteq V$ is the symmetric difference of elementary edge cuts $N(v)$ defined by $v \in W$. $N(v) = \{ e \in E \mid v \in e \}$

Pf: Induction by adding vertices to W.

Corollary: $\{ N(v) \mid v \in V \}$ generates $G=(V,E)$.

Theorem: The row space of I_G is the cut space of G.

Def: The row space of $A \in \mathbb{K}^{n \times m}$ is $R_{\mathbb{K}}(A) = \{ A^T \tilde{x} \in \mathbb{K}^m \mid \tilde{x} \in \mathbb{K}^n \}$

↳ the space generated by the rows of A

Pf: Row i of I_G = characteristic edge vector of $N(v_i) \subseteq E$

↳ elementary edge cut

$\Rightarrow$ the rows of I_G form a basis for the cut space of G

$\Rightarrow R(I_G) =$ cut space of G

Fact: The problem of finding a max. cut in a graph is NP complete

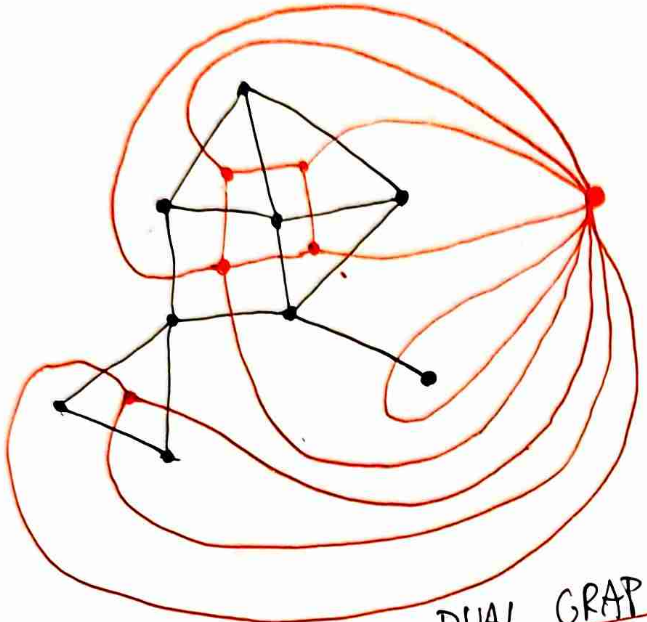
- **Geometric Duality**

dual graph

Fact: For planar graphs: CutSpace(G) ~ CycleSpace(G^*).

→ it is very difficult to geometrically define what the dual of a planar graph is

⇒ we will use an abstract definition



→ faces of G (including the outer face) become the vertices of G^*

→ each edge from G touches 1 or 2 faces of G (vertices of G^*) and therefore defines an "edge" of G^*

Def: Let F be the faces of a planar graph $G=(V,E)$.

$$G^* := (F, f[E]) \quad , \quad f: E \to \binom{F}{2} \cup F$$

Where $f(e)$ is the set of faces touching e.

Corollary: G planar ⇒ find max. cut is polynomial

↳ convert G to G^*, work in cycle space of G^* (polynomial), convert back

- **Ising Model**

→ some simple historical model of something in modern physics

~ named by the scientist Ising

$G=(V,E)$, $w: E \to \mathbb{Q}$

State: $s: V \to \{1,-1\}$

State energy: $e(s) := \sum_{uv \in E} s(u)s(v)w(uv)$

→ goal: find some state with minimum energy

each state defines an edge cut

$\varphi(s) = \{uv \in E \mid s(u) \neq s(v)\}$ with $W = \{v \mid s(v)=1\}$

$w(E) - e(s) = 2 \cdot w(\varphi(s)) \Rightarrow e(s) = -2 \cdot (\text{weight of cut}(s)) + \text{constant}$

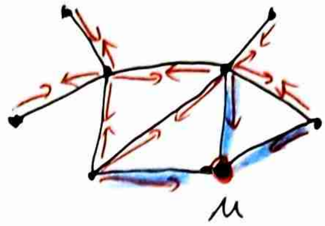
⇒ to find min. energy we need to find max. cut

• More on Cycle and Cut Spaces

① What is dim(Cut Space)? Find a basis.

→ Suppose G is connected

→ we know that $\{N(v) \mid v \in V\}$ generates G



👁 for any $u \in V$: $N(u) = \triangle_{\substack{v \in V \\ v \neq u}} N(v)$

👁 if we were to remove another vertex, we would not be able to generate it

⇒ $\{N(v) \mid u \neq v \in V\}$ is a minimal, lin-ind set → basis ... dim $= |V| - 1$

→ Suppose G is not connected ⇒ we handle each component separately

Theorem: Let G be a graph with k components and n vertices.

Then $\dim(\text{CutSpace}(G)) = n - k$.

② What is dim(CycleSpace)? Find a basis.

→ Suppose $G = (V, E)$ is connected and consider a spanning tree $T = (V, F)$

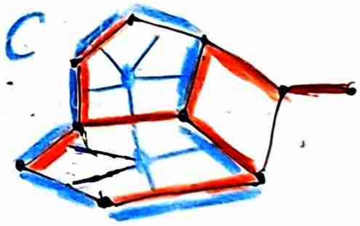
recall: $E' \in$ Cycle Space $\equiv$ (V, E') has all degrees even

⇒ E' = edge-disjoint union of cycles

⇒ we need to generate all cycles in G

👁 Adding any edge $e \in E \setminus F$ to T creates a single cycle C_e

⇒ $\mathcal{B} := \{C_e \mid e \in E \setminus F\}$



claim: $\mathcal{B}$ generates $\forall$ cycle $C \subseteq E$ as $C = \triangle_{e \in C \setminus F} C_e$

↳ lets show $X := C \,\triangle \underbrace{\left(\triangle_{e \in C \setminus F} C_e\right)}$ $= \emptyset$

👁 $X \subseteq F$, & C even $\wedge$ • even ⇒ X even

→ x is a union of cycles & $x \subseteq F$ ⇒ $X = \emptyset$

claim: $\mathcal{B}$ is lin. ind. and minimal ⇒ basis

↳ C_e is the only cycle in $\mathcal{B}$ containing e

→ $|\mathcal{B}| = ?$ ... $|V| = n$ ⇒ $|F| = n - 1$ ⇒ $|E \setminus F| = m - n + 1$

→ for G non-connected we do each component separately and together $|F| = n - k$

Theorem: Let G be a graph with k components, $m := |E|$, $n := |V|$.

Then $\dim(\text{CycleSpace}(G)) = m - n + k$.

(3) Let $G=(V,E)$ be 2-connected planar and $G^*=(V^*,E^*)$ its geometric dual.
Prove $\mathrm{CutSpace}(G) \cong \mathrm{CycleSpace}(G^*)$

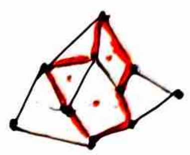
Def: Let U, V be vector spaces with bases B_U, B_V. We say that $\to$ field $\mathbb{K}$
U and V are isomorphic $U \cong V \equiv \exists$ linear bijection $f: U \to V$

Sufficient: $\exists$ bijection $B_U \leftrightarrow B_V \Rightarrow$ we can extend it to be a linear bijection $U \leftrightarrow V$
$\hookrightarrow (\forall a \in \mathbb{K})(\forall u, v \in B_U)$: $f(a \cdot v) = a \cdot f(v)$, $f(u+v) = f(u) + f(v)$

Theorem: The inner faces of a 2-connected planar G form a basis of $\mathrm{CycleSpace}(G)$.

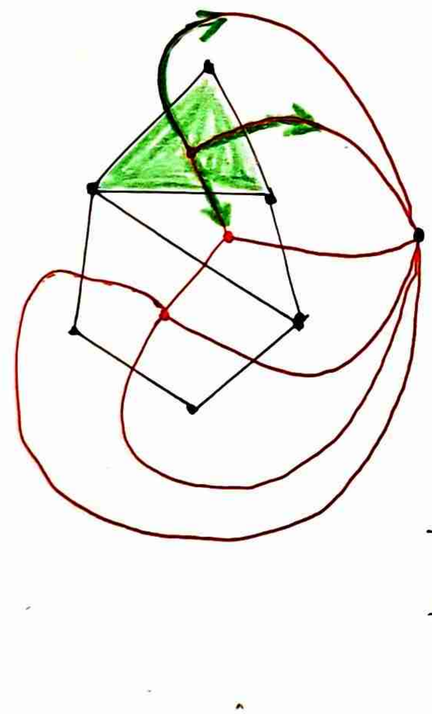
Pf: 1) they generate all cycles $C \subseteq E(G)$

$C = \triangle \{F \mid F \text{ is a face inside } C\}$



2) they are a minimal generating set
$\hookrightarrow$ a basis of G has size $m-n+1$ ... last theorem
$\hookrightarrow$ Eulers formula: $n - m + f = 2 \Rightarrow f = m-n+2$
$\hookrightarrow$ we are using only inner faces $\Rightarrow |B| = m-n+1$ ∎

Theorem: G 2-connected planar $\Rightarrow \mathrm{CutSpace}(G) \cong \mathrm{CycleSpace}(G^*)$



G 2-connected $\Rightarrow$ we will not get things as

$\mathrm{Basis}(\mathrm{CycleSpace}(G)) =$ inner faces of G
$\mathrm{Basis}(\mathrm{CutSpace}(G^*)) = \{N_{G^*}(F) \mid F \text{ is inner face of } G\}$

$\to$ there is a clear bijection between the basis

$\Rightarrow$ we can extend it to get a linear bijection between $\mathrm{CycleSpace}(G)$ and $\mathrm{CutSpace}(G^*)$ ∎

Conclusion: The isomorphism maps an inner face of G to the edges of G^* which intersect its boundary
$\Rightarrow$ equivalently it maps an edge of G to the corresponding edge in G^*
$\hookrightarrow$ this is the general isomorphism which works for all planar graphs

Matroids

Def: $M = (X, \varphi)$, $\varphi \subseteq 2^X$ is a <u>matroid</u> $\equiv$ (X → ground set)

1. $\emptyset \in \varphi$ ... nonempty
2. $A \in \varphi$ & $A' \subseteq A \Rightarrow A' \in \varphi$ ... hereditary

} <u>hereditary system</u>

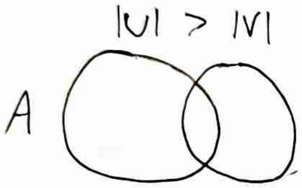
3. $U, V \in \varphi$, $|U| > |V| \Rightarrow \exists x \in U \setminus V : (V \cup \{x\}) \in \varphi$ ... <u>exchange axiom</u>

3'. $Y \subseteq X \Rightarrow$ all $\max_{\subseteq} \{A \subseteq Y \mid A \in \varphi\}$ have the same cardinality (→ bases of Y)

→ The elements of φ are called <u>independent sets</u>

Lemma: (3) $\Longleftrightarrow$ (3')

Pf: $3' \Rightarrow 3$: We have $U, V \in \varphi$ s.t. $|U| > |V|$ → define $A := U \cup V$



↳ all $\max_{\subseteq}$ subsets of A have the same cardinality

↳ $|U| > |V|$ → we can add something from $A \setminus V$ to V

$3 \Rightarrow 3'$: for contradiction assume two maximal $|U| > |V|$ – different sizes

↳ by (3) $\exists x \in U \setminus V$: $(V \cup \{x\}) \in \varphi \Rightarrow V$ is not maximal ↯ $\blacksquare$

Def: $M = (X, \varphi)$ matroid, for $Y \subseteq X$ we define

<u>$\text{rank}(Y)$</u> $:= |\max_{\subseteq} \{A \subseteq Y \mid A \in \varphi\}|$ ... this is OK because (3') guarantees the same size of all maxims

👁 $\text{rank}(Y) = \max \{|A| \mid A \subseteq Y \ \& \ A \in \varphi\}$

↳ this is a weaker definition which doesnt require ↗

Corollary: Matroids are <u>exactly</u> the hereditary set systems, where rank can be correctly defined.

→ max ind. subsets of $Y \subseteq X$ are called the bases of Y

Def: Max independent subsets of $M = (X, \varphi)$ are called the <u>bases</u> of M.

↳ i.e. elements of φ which are max with respect to inclusion $\subseteq$.

Def: The <u>rank</u> of $M = (X, \varphi)$ is the $\text{rank}(M) := \text{rank}(X) = \text{rank}(\text{base of } M)$

👁 The bases of $M = (X, \varphi)$ determine φ

↳ $\forall A \subseteq X$: $A \in \varphi \Longleftrightarrow \exists$ basis B of M s.t. $A \subseteq B$

👁 <u>$A \in \varphi \Longleftrightarrow \text{rank}(A) = |A|$</u> $\Rightarrow$ rank determines φ

Ex: $\rightarrow$ also sometimes column matroid

- **Vectorial Matroid**

  $N =$ matrix over a field $\mathbb{K}$

  $\hookrightarrow M = (X, \mathcal{I})$ ... $X =$ columns of N
  $A \in \mathcal{I} \equiv A$ is lin. ind.

  ① $\emptyset \in \mathcal{I}$ ✓

  ② A lin. ind. & $B \subseteq A \Rightarrow B$ lin. ind. ✓

  ③ implied by the Steinitz exchange theorem

  👁 A is a basis of ColSpace(N) $\Leftrightarrow$ A is a basis of M

- **Cycle Matroid**

  E' is a forrest

  $G = (V, E) \leadsto M_G := (E, \{E' \subseteq E \mid E' \text{ is acyclic}\})$

  ① $\emptyset$ is acyclic ✓

  ② E_1 acyclic & $E_2 \subseteq E_1 \Rightarrow E_2$ acyclic ✓

  ③' $E' \subseteq E \Rightarrow$ all $\max_{\subseteq} \{F \subseteq E' \mid F \text{ acyclic}\}$ have the same cardinality

  👁 these $\max_{\subseteq}$ forrests are the spanning forrests of E' $\Rightarrow$ same size

Ex: operations with matroids

- **deletion**: $M = (X, \mathcal{I})$, $Y \subseteq X \leadsto M - Y := (X \setminus Y, \overbrace{\{A \setminus Y \mid A \in \mathcal{I}\}}^{\mathcal{I}'})$ & $r'(A) = r(A)$

  ① $\emptyset \in M - Y$

  ② $A \in \{B \setminus Y \mid B \in \mathcal{I}\}$ & $A' \subseteq A \overset{?}{\Rightarrow} A' \in \{B \setminus Y \mid B \in \mathcal{I}\}$ $\leftarrow$ $A' \cap Y = \emptyset$ ✓

  $\hookrightarrow (\exists \Delta \subseteq Y): A \dot{\cup} \Delta \in \mathcal{I}$ & $A' \subseteq (A \cup \Delta) \Rightarrow A' \in \mathcal{I}$

  ③' $Z \subseteq X \setminus Y$ ... all $\max_{\subseteq} \{B \subseteq Z \mid B \in \mathcal{I}'\}$ have the same size

  $\hookrightarrow$ we want to use ③' of M $\rightarrow$ need $(\forall B \in Z)$: $B \in \mathcal{I}' \Leftrightarrow B \in \mathcal{I}$

  $\hookrightarrow B \in \mathcal{I}' \Leftrightarrow \exists A \in \mathcal{I}: B = A \setminus Y \Rightarrow B \subseteq A \overset{②}{\Rightarrow} B \in \mathcal{I}$

  $\hookrightarrow$ other direction: $B \in \mathcal{I}$ & $B \in Z$ ... $B \cap Y = \emptyset \Rightarrow B \in \mathcal{I}'$

- **direct sum**

  $\hookrightarrow$ matroids $M_1 = (X_1, \mathcal{I}_1)$, $M_2 = (X_2, \mathcal{I}_2)$ s.t. $X_1 \cap X_2 = \emptyset$

  $$M_1 + M_2 := (X_1 \dot{\cup} X_2, \{Y_1 \dot{\cup} Y_2 \mid Y_1 \in \mathcal{I}_1, Y_2 \in \mathcal{I}_2\})$$

Theorem: $r: 2^X \to \mathbb{N}$ is the rank function of a matroid $\Leftrightarrow$ $\forall Y \subseteq X$:

(R1) $r(\emptyset) = 0$

(R2) $r(Y) \leq r(Y \cup \{y\}) \leq r(Y) + 1$

(R3) $r(Y \cup \{y\}) = r(Y \cup \{z\}) = r(Y) \Rightarrow r(Y) = r(Y \cup \{y, z\})$

Pf: $\Rightarrow$: Let $M = (X, \mathcal{Y})$ and r its rank function. It satisfies:

(R1) $r(\emptyset) = 0$ ✓

(R2) $r(Y)$ = size of max. independent subset of Y

$r(Y \cup \{y\})$ ... adding 1 element can not increase it by more than 1

(R3) let max. ind. subset of Y be $B \Rightarrow r(Y) = |B|$

↳ since $r(Y \cup \{y\}) = r(Y \cup \{z\}) = r(Y)$, B can not be extended by y, z

$\Rightarrow Y \cup \{y\}$ and $Y \cup \{z\} \notin \mathcal{Y}$

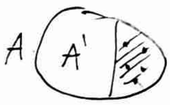
$\Rightarrow$ therefore $r(Y) = r(Y \cup \{y, z\})$ ... if $r(Y \cup \{y, z\}) > r(Y)$, then Y can be extended by y or z ↯

$\Leftarrow$: Define $\mathcal{Y} := \{A \subseteq X \mid |A| = r(A)\}$

claim: $(X, \mathcal{Y})$ is a matroid with rank function r

① $\emptyset \in \mathcal{Y}$ ... because $r(\emptyset) = 0$ (R1)

② $|A| = r(A)$ & $A' \subseteq A \overset{?}{\Rightarrow} |A'| = r(A')$



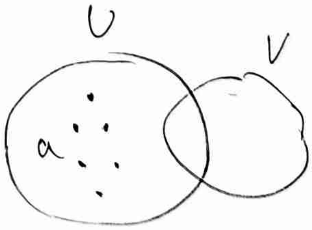
$r(A) = |A|$ → use (R2) repeatedly

$r(A') \leq |A'|$ ∵ A' is created from $\emptyset$ by adding $|A'|$ elements

Note: A is created from A' by adding $k := |A \setminus A'|$ elements

$r(A) \leq r(A') + k \Rightarrow |A| \leq r(A') + |A \setminus A'| \Rightarrow r(A') \geq |A| - |A \setminus A'| = |A'|$

③ for contradiction assume $U, V \in \mathcal{Y}$, $|U| > |V|$ such that $\forall a \in U \setminus V$: $V \cup \{a\} \notin \mathcal{Y}$



↳ $\forall a \in U \setminus V$: $r(V \cup \{a\}) \neq |V \cup \{a\}|$

↳ using (R2) we have $r(V \cup \{a\}) = r(V)$

→ by repeatedly using (R3) we can add every $a \in U \setminus V$ to V and the r never increases

$\Rightarrow$ we arrive at $r(V \cup (U \setminus V)) = r(V) \Rightarrow r(U \cup V) = r(V)$ ↯

Note: $r(V) = |V|$, $U \subseteq U \cup V \Rightarrow r(U \cup V) \geq r(U) = |U| \Rightarrow$ (something $\geq |U|$) $= |V|$

Submodularity of rank

Def: $f: 2^X \to \mathbb{R}$ is submodular $\equiv$

$$\forall A, B \subseteq X \quad : \quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

Theorem: $r: 2^X \to \mathbb{N}_0$ is the rank function of a matroid $\iff \forall Y, Z \subseteq X$:

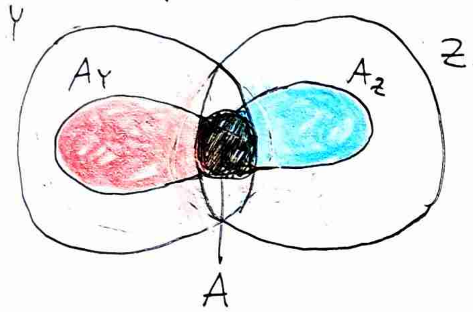
(R1') $\quad 0 \leq r(Y) \leq |Y|$

(R2') $\quad Z \subseteq Y \implies r(Z) \leq r(Y)$ ... monotonicity

(R3') $\quad r(Y) + r(Z) \geq r(Y \cup Z) + r(Y \cap Z)$ ... rank is submodular !

Pf: $\implies$: (R1') and (R2') trivially hold for all matroids

→ for (R3') consider the following: let $Y, Z \subseteq X$ be arbitrary



$A :=$ max independent set of $Y \cap Z$

$A_Y :=$ max ind. in Y containing A

$A_Z :=$ max ind in Z containing A

$$\text{rank}(Y) + \text{rank}(Z) = |A_Y| + |A_Z| = |A_Y \cup A_Z| + |\overbrace{A_Y \cap A_Z}^{A}| = |A_Y \cup A_Z| + \text{rank}(Y \cap Z)$$

need: $\text{rank}(Y \cup Z) = |\text{max ind. in } Y \cup Z| \leq |A_Y \cup A_Z|$

↳ we want to find max ind in $Y \cup Z$

$\implies$ start with A_Y (max in Y) and add stuff to it from $Z \setminus Y$

claim: let $W \subseteq Z \setminus Y$ s.t. $A_Y \cup W$ is max ind. in $Y \cup Z$

↳ then $|W| \leq |A_Z \setminus A| \implies |A_Y \cup W| \leq |A_Y| + |A_Z \setminus A| = |A_Y \cup A_Z|$

→ if (for contradiction) $|W| > |A_Z \setminus A|$, then $|W \cup A| > |A_Z|$

and since $A_Y \cup W$ is independent, $(W \cup A) \subseteq (A_Y \cup W)$ is also independent

$\implies A_Z$ is not max. independent in Z ↯

$\impliedby$: We will show (R1' & R2' & R3') $\implies$ (R1 & R2 & R3)

(R1) $r(\emptyset) = 0$ ... ∵ R1'

(R2) let $y \notin Y$ otherwise its trivial. $Y \subseteq Y \cup \{y\} \overset{R2'}{\implies} r(Y) \leq r(Y \cup \{y\})$

submodularity: $r(Y \cup \{y\}) + \overbrace{r(\emptyset)}^{0} \overset{R3'}{\leq} r(Y) + r(\{y\}) \leq r(Y) + 1$

(R3) let $y, z \notin Y$ s.t. $r(Y \cup \{y\}) = r(Y \cup \{z\}) = r(Y)$. We have $r(Y \cup \{y, z\}) \overset{R2'}{\geq} r(Y)$

$r(Y \cup \{y\}) + 0 \overset{R3'}{\leq} r(Y) + r(\{y\})$ & $r(Y \cup \{y\}) = r(Y) \implies r(\{y\}) = 0$ ... similarly $r(\{z\}) = 0$

$\implies r(Y \cup \{y, z\}) + \underbrace{r(\emptyset)}_{0} \overset{R3'}{\leq} r(Y) + \underbrace{r(\{y, z\})}_{0}$, but $r(\{y, z\}) + 0 \overset{R3'}{\leq} 0 + 0 \implies r(\{y, z\}) = 0$ ▨

Corollary: Matroids are exactly the hereditary set systems where rank is monotone and submodular

Recall: $f: 2^X \to \mathbb{R}$ is submodular $\equiv f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$

Notation: For $x \in X$ define $\Delta f_x : 2^X \to \mathbb{R}, \quad T \mapsto f(T \cup \{x\}) - f(T)$

Theorem: $f: 2^X \to \mathbb{R}$ is submodular $\iff (\forall x \in X)$: Δf_x is nonincreasing.

Corollary: Submodular functions model utility well.

↳ $\Delta f_x(T)$ = increase in utility after adding x to T

↳ we want $S \subseteq T \Rightarrow \Delta f_x(S) \geq \Delta f_x(T)$

↳ more significant utility increase when added to a smaller set

Proof:

$\Rightarrow$: Have: $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ $\quad T + x := T \cup \{x\}$

Want: $S \subseteq T \Rightarrow f(S+x) - f(S) \geq f(T+x) - f(T)$

$$\underbrace{f(S+x)}_{A} + \underbrace{f(T)}_{B} \geq \underbrace{f(T+x)}_{A \cup B} + \underbrace{f(S)}_{A \cap B}$$

$\Leftarrow$: Have: $\forall x, S \subseteq T$: $f(S+x) - f(S) \geq f(T+x) - f(T)$

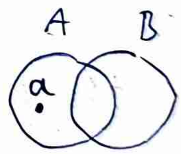
Want: $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

→ by induction on $|A \Delta B| = |\ |$

① **Base:** $|A \Delta B| = 0 \Rightarrow A = B \Rightarrow$ trivially holds

② $|A \Delta B| = k$

Want: $f(B) - f(A \cap B) \geq f(A \cup B) - f(A)$ (*)



$|A \Delta B| > 0 \Rightarrow$ WLOG assume $\exists a \in A \setminus B$

let $A' = A - a$, then $|A' \Delta B| = k - 1 < k$

$\Rightarrow$ by induction hypothesis we have

$f(B) - f(A' \cap B) \geq f(A' \cup B) - f(A')$

$f(B) - f(A \cap B) \geq f(A \cup B - a) - f(A - a)$

this together gives (*)

claim: $f(A \cup B - a) - f(A - a) \geq f(A \cup B) - f(A)$

↳ $\underbrace{f(A)}_{S+a} - \underbrace{f(A-a)}_{S} \geq \underbrace{f(A \cup B)}_{T+a} - \underbrace{f(A \cup B - a)}_{T}$

↳ use $S = A - a$, $T = A \cup B - a$, $S \subseteq T$, $x = a$

Simple Matroids

Def: $M=(X,\mathcal{U})$ is <u>simple</u> $\equiv \forall A\subseteq X:\ |A|\le 2 \Rightarrow r(A)=|A|$

$\hookrightarrow$ meaning $|A|\le 2 \Rightarrow A\in\mathcal{U}$.

Def: Let $M=(X,\mathcal{U})$. $A\subseteq X$ is <u>closed</u> $\equiv \forall x\in X\setminus A:\ r(A)<r(A\cup\{x\})=r(A)+1$

Each <u>simple matroid</u> $M=(X,\mathcal{U})$ <u>of rank 3</u> $(r(X)=3)$ is determined by

$$L(M):=\{A\subseteq X \mid r(A)=2,\ |A|>2,\ A \text{ closed}\}.$$

This set determines the rank function and therefore $\mathcal{U}$. Let $A\subseteq X$.

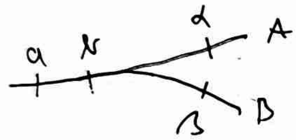
(*)
1) $|A|\le 2 \Rightarrow r(A)=|A|$
2) $\exists B\in L(M):\ A\subseteq B \Rightarrow r(A)=2$
3) otherwise $r(A)=3$

$\downarrow$

$\mathcal{U}=\{A\subseteq X \mid r(A)=|A|\}$

Lemma: $L(M)$ is almost disjoint: $A,B\in L(M),\ A\ne B \Rightarrow |A\cap B|\le 1$.

Pf: Suppose $|A\cap B|\ge 2$; $\alpha\in A,\ \alpha\notin B$ & $\beta\in B,\ \beta\notin A$



α,β exist, else $A\subset B$ or $B\subset A$ ↯ with closed

$\forall S\subseteq A,\ |S|\ge 2:\ r(S)=2$

$$2=r(A\cap B)=r(A\cap B+\alpha)=r(A\cap B+\beta)\overset{(R3)}{=}r(A\cap B+\alpha+\beta)$$

Now suppose that some additional $\gamma\in A\setminus\{a,b,\alpha\}$, $\gamma\notin B$.

By the same argument: $r(A\cap B+\gamma+\beta)=2$

$\Rightarrow$ using (R3) once more: $r(A\cap B+\alpha+\gamma+\beta)=2$

We can add all elements of $A\setminus B$ to this set and end up with

$$r((A\cap B)\cup(A\setminus B)+\beta)=r(A+\beta)=2$$

Recall $\beta\notin A$, and A is closed, therefore $r(A+\beta)=3$ ↯ $\blacksquare$

Def: $\mathcal{C}\subseteq 2^X$ is a <u>configuration</u> $\equiv$

1. $A\in\mathcal{C}\Rightarrow |A|\ge 3$
2. $A,B\in\mathcal{C}\Rightarrow |A\cap B|\le 1$

Theorem: $\mathcal{C}\subseteq 2^X$ is a <u>configuration</u> $\iff \mathcal{C}=L(M)$ for some <u>simple rank 3 matroid</u>.

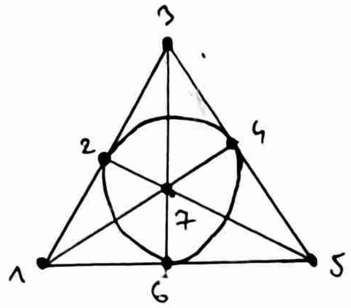
Pf: $\Leftarrow$: using previous lemma and by definition of $L(M)$

$\Rightarrow$: define rank

$$(*)\quad r_M(A):=\begin{cases}|A|, & |A|\le 2\\ 2, & A\subseteq B\in\mathcal{C}\\ 3, & \text{otherwise}\end{cases}$$

$\dots\ M=(X,\{A\subseteq X\mid r(A)=|A|\})$, : $L(M)=\mathcal{C}$ $\blacksquare$

Ex: The Fano Matroid $F_7 = (X, \varphi)$



$X =$ points of the Fano projective plane

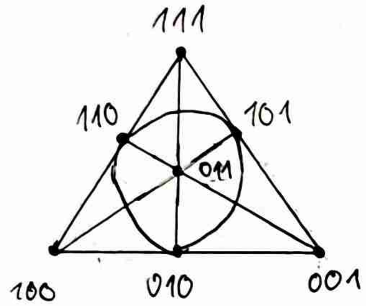
Bases of F_7 = lines

↳ $A \subseteq X \in \varphi \iff \exists$ basis B s.t. $A \subseteq B$

👁 F_7 is a simple matroid of rank 3 $\qquad 246 = \bigcirc$

👁 Configurations $= \{1245, \cancel{1246}, 1267, 1346, \cancel{2346}, 2347, 2356\}$

👁 F_7 is isomorphic to a vectorial matroid of $\mathbb{F}_2^{3\times 7}$



$$V = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3\times 7}$$

$M_V = (X, \varphi)$ ... X = Columns of V

$A \subseteq X \in \varphi \iff A$ is lin. ind.

👁 bases of M_V = bases of Col-space of V

↳ every line = basis of V

Def: Let $M_1 = (X_1, \varphi_1)$, $M_2 = (X_2, \varphi_2)$ be matroids. The bijection $f: X_1 \to X_2$ is an isomorphism of M_1 and M_2 $\equiv$

$$\forall A \subseteq X_1: \quad A \in \varphi_1 \iff f[A] \in \varphi_2$$

Note: Equivalent conditions (*)

- bases: $\forall B \subseteq X_1$: B is a basis of $M_1 \iff f[B]$ is a basis of M_2
- rank: $\forall A \subseteq X_1$: $\mathrm{rank}_1(A) = \mathrm{rank}_2(f[A])$

Def: The matroid M is representable over a field $\mathbb{F}$

$\equiv M$ is isomorphic to a vectorial matroid over $\mathbb{F}$

Def: The matroid M is binary $\equiv$ it is representable over $\mathbb{F}_2$

Def: The matroid M is graphic $\equiv \exists$ graph G s.t. $M \cong$ Cycle Matroid(G)

↳ isomorphic

↳ recall: CycleMatroid(G) = forests of G

Example: F_7 is binary.

(*) Also circuits - defined later

Operations with Matroids

① **Direct sum**: $M_1 = (X_1, \varphi_1)$, $M_2 = (X_2, \varphi_2)$ & $X_1 \cap X_2 = \emptyset$

$$M_1 + M_2 := (X_1 \dot\cup X_2,\ \{Y_1 \dot\cup Y_2 \mid Y_1 \in \varphi_1,\ Y_2 \in \varphi_2\}),\quad r(A) = r_1(A_1) + r_2(A_2)$$

↳ $A = A_1 \dot\cup A_2$, $A_1 \subseteq X_1$, $A_2 \subseteq X_2$

② **Uniform matroid**: $[n] := \{1, 2, \dots n\}$

$U(n, k) := \left([n], \binom{[n]}{\leq k}\right)$ ... obviously matroid, bases $= \binom{[n]}{k}$

③ **Partition matroid**: $X = \{X_1, \dots X_n\}$ partition of E → $\forall X_i \neq \emptyset$, $X_i \cap X_j = \emptyset$, $\bigcup X = E$

$P(X) := (E, \{F \subseteq E \mid \forall i: |F \cap X_i| \leq 1\})$

$P(X_1, \dots X_n) = \sum_{i=1}^{n} U(|X_i|, 1)$ ... direct sum of matroids is a matroid

④ **Restriction to T**: $M = (X, \varphi)$, $T \subseteq X$

$M|T := (T, \{A \in \varphi \mid A \subseteq T\})$, $r'(A) = r(A)$ ... $A \subseteq T$

⑤ **Deletion of T**: $M = (X, \varphi)$, $T \subseteq X$

$M - T := (X - T, \{A \in \varphi \mid A \subseteq X - T\})$ → note: $\varphi' = \{A - T \mid A \in \varphi\}$

$M - T = M | (X - T) \Rightarrow r'(A) = r(A)$ ... $A \subseteq X - T$

⑤ **Contraction of T**: $M = (X, \varphi)$, $T \subseteq X$ → $T \in \varphi \Rightarrow B = T$

→ let B be a basis of T ... (max ind. in T) ... $B \in \max_{\subseteq} \{J \subseteq T \mid J \in \varphi\}$

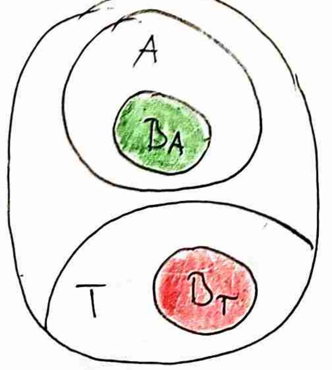
$M/T := (X - T, \{A \in \varphi \mid A \subseteq X - T \ \&\ A \dot\cup B \in \varphi\})$ → note: $A \dot\cup B \in \varphi$ & $A \subseteq A \dot\cup B \Rightarrow A \in \varphi$

Theorem: M/T is a matroid with $r': X \setminus T \to \mathbb{N}_0$, $r'(A) = r(A \dot\cup T) - r(T)$.

Corollary: Even though the definition of contraction has two parameters (T, B), the rank of M/T depends only on T

⇒ M/T is uniquely determined only based on T

Pf:



Let $A \subseteq X \setminus T$ and denote

- B_T ... basis of T in M (used to construct M/T)
- B_A ... basis of A in M/T

$B_A \cup B_T$ is a basis of $A \cup T$ in M

↳ can not extend $B*$ because then $*$ is not maximal

$r'(A) = |B_A| = |B_A \dot\cup B_T| - |B_T| = r(A \cup T) - r(T)$ ∎

Examples:

① $U(3,2) = \left([3], \binom{[3]}{\leq 2}\right)$ is binary $\hat{=}$ representable over $\mathbb{F}_2$

→ bases = subsets of size 2, 3 elements in total

$V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. $\{1,2\} \leftrightarrow \left\{\binom{1}{0}, \binom{0}{1}\right\}$, $\{1,3\} \leftrightarrow \left\{\binom{1}{0}, \binom{1}{1}\right\}$, $\{2,3\} \leftrightarrow \left\{\binom{0}{1}, \binom{1}{1}\right\}$

② $U(4,2) = \left(\{1,2,3,4\}, \binom{[4]}{\leq 2}\right)$ is not binary

→ bases = subsets of size 2, 4 elements in total

$V = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}$ need: every pair of vectors is lin. ind.
every triplet of vectors is lin. dep.

(*) equivalently need: every pair of vectors is a basis of ColSpace(V)

$\Rightarrow \{v_1, v_2\}$ basis and v_3, v_4 are generated from $\{v_1, v_2\}$

→ we are over $\mathbb{F}_2$ so the only possible lin. comb. of $\{v_1, v_2\}$ are

$\mathrm{Span}(\{v_1, v_2\}) = \{0, v_1, v_2, v_1 + v_2\}$ $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow 0$ can not be in V ... (*) would be broken

$\Rightarrow \{v_3, v_4\} \subseteq \{v_1, v_2, v_1 + v_2\}$ ... for example: $v_3 = v_1 + v_2$, $v_4 = v_2$

$\Rightarrow$ either v_1 or v_2 (or both) is repeated in V

$\Rightarrow$ the pair of these two identical vectors is not a basis $\Rightarrow$ (*) $\lightning$

Proposition: All graphic matroids are binary.

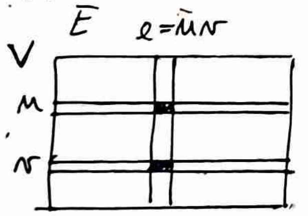
Pf: Let $G = (V, E)$ be a graph and consider its cycle matroid

$M_G = (E, \{E' \subseteq E \mid E' \text{ is acyclic}\})$ .... $|\text{Ground set}| = |E|$

want: $V \in \mathbb{F}_2^{k \times |E|}$ s.t $M_V \cong M_G$ with isomorphism $f: 2^E \to$ Colums of V

↳ $E' \subseteq E$: E' is acyclic $\Leftrightarrow f[E']$ is lin. ind.
E' has a cycle $\Leftrightarrow f[E']$ is lin. dep.

The incidence matrix of G meets our criteria



Assume E' has a cycle $C \subseteq E$

↳ $C \in$ Cycle Space$(G) \cong \ker(I_G)$

$\Rightarrow$ char. vector of C, $\chi_C \in \ker(I_G)$

$\Rightarrow$ the columns corresponding to C are lin. dep. ∎

Duality of Matroids

Motivation: $G=(V,E) \rightsquigarrow M_G = (E, \{E' \subseteq E \mid E' \text{ is a forrest}\})$

↳ $M_G^* := (E, \{E' \subseteq E \mid G - E' \text{ has the same \# components as } G\})$ ↗ show $M_G^* =$ matroid

$\text{rank}_M(E) =$ size of a spanning forrest of G

$E' \in M_G^* \iff$ removing edges E' doesnt create new components

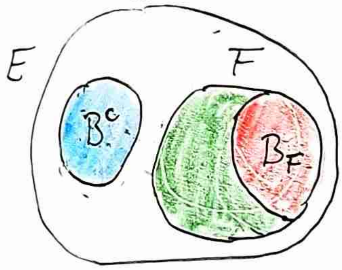
$\iff$ the size of a spanning tree of G doesnt change by removing E'

$\iff \text{rank}(E - E') = \text{rank}(E)$

Theorem: M_G^* is a matroid with rank $r^*(F) = |F| - r(E) + r(E-F)$

Pf: ① $\emptyset \in M_G^*$ ② $E' \in M_G^*$ & $F \subseteq E' \Rightarrow F \in M_G^*$

③' we will show that rank is well defined



→ let $F \subseteq E$ and let

- B_F be a basis of F in M^* ... largest removable part
- B^c be a basis of $E-F$ in M ... some forrest

→ now expand B^c with edges from $E - B_F$ to form a spanning forrest of G

↳ we can ∵ excluding B_F does not change the # of components

→ formally we made $B \subseteq E - B_F$ s.t. $B^c \subseteq B$ and B is a basis of M and we have used $B_F \in \varphi^* \Rightarrow r(E - B_F) = r(E)$

$B_F = F - B$ ... all edges from $F - B_F$ have been used

↳ if $\exists e \in F - B - B_F$ ... unused edge which isnt forbidden by B_F then this e can be added to B_F and B is still a spaning forrest

$\Rightarrow B_F$ would not be maximal independent subset of F ↯

$B = B^c \dot\cup (F - B_F)$ ... else B^c wouldnt be max. ind. in $E-F$

$|B_F| = |F| - |F \cap B| = |F| - |F - B_F| = |F| - (|B| - |B^c|) = |F| - r(E) + r(E-F)$ ∎

Def: The dual of a matroid $M = (X, \varphi)$ is the matroid

$$M^* = (X, \{A \subseteq X \mid r(X-A) = r(X)\}) \quad \text{with} \quad r^*(A) = |A| - r(X) + r(X-A)$$

B is a basis of $M \iff X - B$ is a basis of M^* ← complementary bases

Corollary: $(M^*)^* = M$, $\quad \text{rank}(M) + \text{rank}(M^*) = |X|$

Deletion and Contraction are dual operations

Theorem: $M=(X,\mathcal{I})$, $T\subseteq X$. Then $(M-T)^* = M^*/T$.

Pf: Denote rank of $M, M^*, (M-T)^*, M^*/T$ as r, r^*, r_D, r_C respectively.
Let $A\subseteq X-T$. We will show $r_D(A) = r_C(A)$

- LHS: $M-T$ ... $\text{rank}_{M-T}(A) = r(A)$

$$\Rightarrow r_D(A) = r^*_{M-T}(A) = |A| - r(X-T) + r(X-T-A)$$

- RHS: M^* ... $r^*(A) = |A| - r(X) + r(X-A)$

recall: M with rank r $\Rightarrow$ M/T has rank $r'(A) = r(A\dot\cup T) - r(T)$

$$\Rightarrow r_C(A) = r'_{M^*}(A) = r^*(A\dot\cup T) - r^*(T)$$
$$= |A\dot\cup T| - r(X) + r(X-(A\dot\cup T)) - |T| + r(X) - r(X-T)$$
$$= |A| + r(X-A-T) - r(X-T) \quad \blacksquare$$

Theorem: $M=(X,\mathcal{I})$, $T\subseteq X$. Then $(M/T)^* = M^*-T$.

Pf: Let $A\subseteq X-T$. We will show $r_C(T) = r_D(T)$

- RHS: M^* ... $r^*(A) = |A| - r(X) + r(X-A) = r_D(A)$
- LHS: M/T ... $r'(A) = r(A\dot\cup T) - r(T)$

$$\Rightarrow r_C(A) = |A| - r'(X-T) + r'(X-T-A)$$
$$= |A| - r(X) + r(T) + r(X-A) - r(T) = |A| - r(X) + r(X-A) \quad \blacksquare$$

Duality of Graphic Matroids

Def: The circuits of $M=(X,\mathcal{I})$ are its minimum dependent subsets.
That is $C\subseteq X$ s.t. $C\notin\mathcal{I}$ & $A\subset C \Rightarrow A\in\mathcal{I}$.

Circuits of a graphic matroid are cycles in the corresponding graph.

Terminology: When talking about a matroid property and prepend "co" $\Rightarrow$ dual
- ↳ cocircuits, cobases = circuits, bases of the dual matroid
- ↳ M is cographic = M^* is graphic

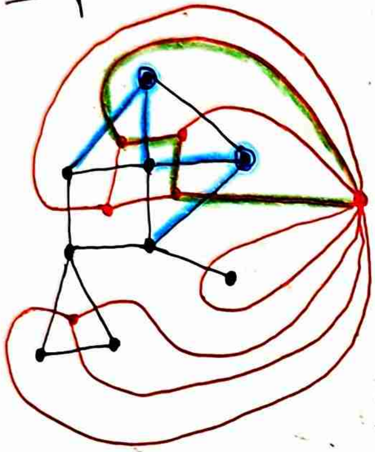
Cocircuits of a graphic matroid are minimum edge cuts in the corresp. graph.
- ↳ F is a cocircuit of M_G $\Leftrightarrow$ $F\notin\mathcal{I}^*$ & $A\subset F \Rightarrow A\in\mathcal{I}^*$
- ↳ $F\notin\mathcal{I}^*$ $\Leftrightarrow$ deleting F reduces $r(M)$ $\Leftrightarrow$ creates new components $\Leftrightarrow$ is edge cut
- ↳ $A\subset F \Rightarrow A\in\mathcal{I}^*$ $\Leftrightarrow$ A is not edge cut $\Leftrightarrow$ F is minimal

Theorem: $M = (X, \mathcal{I})$ is completely characterized by any of the following.

1. rank function r : $A \in \mathcal{I} \iff r(A) = |A|$
2. dependent sets $\mathcal{D}$: $A \in \mathcal{I} \iff A \notin \mathcal{D}$
3. bases $\mathcal{B}$: $A \in \mathcal{I} \iff \exists B \in \mathcal{B} : A \subseteq B$ ... subset of a basis
4. circuits $\mathcal{C}$: $A \in \mathcal{I} \iff \nexists C \in \mathcal{C} : C \subseteq A$ ... doesn't contain a circuit

Theorem: Let $G = (V, E)$ be a planar graph. Then $M^*(G) \cong M(G^*)$ G^* = geom. dual

Proof: We will show that $M^*(G)$ and $M(G^*)$ have the same circuits.



- Circuits of $M^*(G)$ = min. edge cuts of G
- Circuits of $M(G^*)$ = cycles of G^*

there is a very clear bijection between the edges of G and the edges of G^* ... call it f

know (*): f isomorphism of $CutSpace(G)$ and $CycleSpace(G^*)$

want: f isomorphism of $M^*(G)$ and $M(G^*)$

$A \subseteq E(G)$ edge cut in $G \overset{(*)}{\iff} f[A] \subseteq E(G^*)$ even subgraph of G^*:

→ isomorphisms preserve relative properties $\nexists$ even $F \subset f[A]$

$\Rightarrow$ A min. edge cut in $G \iff f[A]$ min. even subgraph of G^*

$\iff f[A]$ cycle in G^* ∎

Planar Graphs Recap

Def: The graph H is a minor of the graph G, $H \leq G$

$\equiv H$ can be obtained from G using a sequence of operations

1. vertex deletion 2. edge deletion 3. edge contraction

G planar & $H \leq G \Rightarrow H$ planar

K_5 and $K_{3,3}$ are not planar

Theorem (Kuratowski–Wagner): G planar $\iff K_5 \nleq G$ & $K_{3,3} \nleq G$

→ we will define a matroid minor and show that there is a strong connection between the minors of graphic matroids and graph minors

Matroid Minors

<u>Def</u>: The matroid N is a <u>minor</u> of the matroid M, $N \leq M$

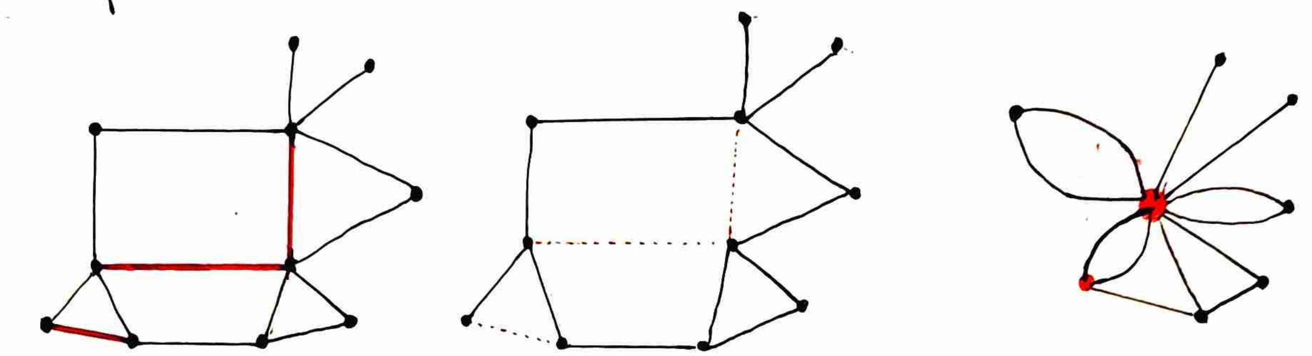
$\equiv$ N can be obtained from M using a sequence of operations

① <u>deletion of T</u> ①' restriction to T ② <u>contraction of T</u>

<u>Note</u>: Recall that $M-T = M|(X-T)$.

(multigraph)

If M is graphic, then ① ~ deletion of edges and ② ~ contraction of edges



<u>Theorem</u>: $G=(V,E)$, $F \subseteq E$. Then $\underline{M(G-F) \cong M(G)-F}$ and $\underline{M(G.F) = M(G)/F}$.

↳ $M(G) = (E, \{E' \subseteq E \mid E' \text{ acyclic in } G\})$

① <u>deletion of $F \subseteq E$</u>

$M(G-F) = (E-F, \{E' \subseteq E-F \mid E' \text{ is acyclic}\})$

$M(G)-F \doteq M(G)|(E-F) = (E-F, \{E' \subseteq E-F \mid \overbrace{E' \in \mathcal{I}(M_G)}^{E' \text{ acyclic}}\})$

② <u>contraction of $F \subseteq E$</u>

$M(G.F) = (E-F, \{E' \subseteq E-F \mid E' \text{ is acyclic in } G.F\})$

→ let K be a spanning forest of F

$M(G)/F = (E-F, \{E' \subseteq E-F \mid E' \text{ acyclic in } G \;\&\; E' \dot\cup K \text{ acyclic in } G\})$

→ need to show $M(G.F) \cong M(G)/F$

- <u>$E' \subseteq E-F$ acyclic in $G.F$</u> → want $E' \dot\cup K$ acyclic in G
  - ↳ replacing a contracted vertex by its spanning tree doesn't introduce cycles
  - → if there was a cycle in $E' \dot\cup K$ in G then there must have been a cycle in E' in $G.F$ because trees are acyclic
- <u>$E' \subseteq E-K$ acyclic in G & $E' \dot\cup K$ acyclic in G</u> → want E' acyclic in $G.F$
  - → if there was a cycle in E' in $G.F$, then there must have been a cycle in $E' \dot\cup K$ in G (we replace contracted vertices by their spanning trees) because trees are connected ∎

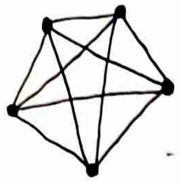
← other direction tricky because of the multiedges

<u>Corollary</u>: $\underline{H \leq G \Rightarrow M(H) \leq M(G)}$.

① $G-e \Rightarrow M(G)-e$ ② $G-v \Rightarrow M(G) - \{e \in E \mid v \in e\}$ ③ $G.e \Rightarrow M(G)/e$ & reduce multiedges to simple edges

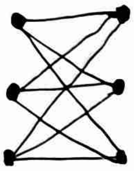
Lemma: $M(K_5)$ and $M(K_{3,3})$ are <u>not cographic.</u> $\rightarrow$ $M^*(K_5)$ and $M^*(K_{3,3})$ are not graphic

Pf: Suppose $M(K_5)$ is cographic i.e. $\exists H$ s.t. $M^*(K_5) \cong M(H)$ i.e. $M^*(H) \cong M(K_5)$



<u>recall</u>: cocircuits of $M(H)$ = circuits of $M^*(H) \cong M(K_5)$ = cycles of K_5
= min. edge cuts of H $(*)$

- The ground sets of $M(K_5)$ and $M^*(K_5) \cong M(H)$ are edges of K_5.
$\Rightarrow$ 10 elements in total, rank of $M(K_5) = 4$ (size of spanning tree)
$\Rightarrow$ rank of $M(H) = 10 - 4 = 6$
rank of $M(H)$ = size of spanning forest of H = $|V(H)| - k$ ← # components
$\Rightarrow |V(H)| = \text{rank} + k = 6 + k \geq 7$
$\Rightarrow$ average degree of $H = \dfrac{2 \cdot |E(H)|}{|V(H)|} \leq \dfrac{2 \cdot 10}{7} < 3$
$\Rightarrow \exists v \in V(H)$ with $\deg(v) \leq 2$ $\Rightarrow$ min edge cut of H with size ≤ 2
$(*)$ $\Rightarrow$ $\exists$ cycle in K_5 with size ≤ 2 which is obviously nonsense $\lightning$



Using the same approach for $K_{3,3}$ we have
rank $(M(K_{3,3})) = 5 \Rightarrow \text{rank}(M(H)) = |E(K_{3,3})| - 5 = 9 - 5 = 4$
$\Rightarrow |V(H)| = \text{rank} + k = 4 + k \geq 5$

① $k = 1$: $|V(H)| = 5$ & $|E(H)| = 9$ $\Rightarrow H \cong K_5$ without one edge
<u>circuits of $M(H)$</u> = cycles in H = min edge cuts in $K_{3,3}$
$K_{3,3}$ has 6 edge cuts of size 3
H has 7 cycles of size 3 ... $\binom{5}{3} = 10$ in K_5 -3 $\lightning$

② $k \geq 2$: $|V(H)| \geq 6$ & $|E(H)| = 9$ $\Rightarrow$ avg degree $\leq \frac{2 \cdot 9}{6} = 3$
$\Rightarrow$ $\exists v \in H$ with degree ≤ 3 $\Rightarrow$ $\exists$ min edge cut of H of size ≤ 3
$\Rightarrow$ $\exists$ cycle in $K_{3,3}$ of size ≤ 3 $\lightning$ ... smallest cycle has size 4 ■

M graphic & $N \leq M$ $\Rightarrow$ N graphic

Lemma: <u>$N \leq M \iff N^* \leq M^*$.</u>

Pf: We will use the duality of the deletion and contraction operations.
We will show "$\Rightarrow$", the other direction is similar. From the sequence of operations $M \rightarrow N$, we will construct a series of operations $M^* \rightarrow N$.
$$M = M_1 \geq M_2 \geq \dots \geq M_n = N \quad \rightsquigarrow \quad M^* = M_1' \geq M_2' \geq \dots \geq M_n' = N^*$$
<u>recall</u>: $(M - T)^* = M^*/T$ & $(M/T)^* = M^* - T$
<u>induction</u>: $M_1' = M^*$, now assume $M_k = M_{k-1} - T$. (contraction similar)
$\Rightarrow M_k' := M_{k-1}'/T = M_{k-1}^*/T = (M_{k-1} - T)^* = M_k^*$ ■

Whitney's Planarity Criterion

Theorem (Whitney): G planar $\iff$ $M(G)$ cographic. $\equiv M^*(G)$ graphic

Pf: $\Rightarrow$: G planar $\Rightarrow \exists$ geometric dual G^* and $M(G^*) \cong M^*(G)$

$\hookrightarrow M(G^*)$ is graphic (by definition) $\Rightarrow M^*(G)$ is graphic

$\Leftarrow$: Suppose G is nonplanar. Then by the K-W theorem: $K_5 \leq G$ or $K_{3,3} \leq G$.

recall: $H \leq G \Rightarrow M(H) \leq M(G) \ldots \Rightarrow M(K_5) \leq M(G)$ or $M(K_{3,3}) \leq M(G)$

duality: $M^*(K_5) \leq M^*(G)$ or $M^*(K_{3,3}) \leq M^*(G)$

recall: $M^*(K_5)$ and $M^*(K_{3,3})$ are not graphic

$\rightarrow M^*(G)$ is graphic and minors of graphic matroids are also graphic ↯ $\blacksquare$

Lemma: Let G, H be graphs. If H is simple (no loops or multiedges), then

$$H \leq G \iff M(H) \leq M(G)$$

Pf: We know "$\Rightarrow$". Now lets assume $M(H) \leq M(G)$ and H is simple.

$$M(G) = M(G_1) \geq M(G_2) \geq \cdots \geq M(G_m) = M(H)$$

We want to create a sequence of operations to turn G into H.

Deletion is straightforward: $M(G_i) - T \rightsquigarrow$ delete edges of T from G_i

Matroid contraction is problematic because it creates multiedges.

(*) { Each contraction is followed by a deletion which turns all newly created multiedges into simple edges

(**) { The graph corresponding to the final matroid $M(H)$ has no multiedges

(*) $\Rightarrow H \leq G$ ... simply contract the corresponding edges

claim: $M(H) \leq M(G)$ & (**) $\Rightarrow \exists$ construction of $M(H)$ from $M(G)$ satisfying (*)

$\hookrightarrow$ suppose a contraction $M(G)/T$ created an multiedge

$\rightarrow H$ doesnt have the multiedge, so it must have been deleted / reduced

1) one of the edges in the multiedge contracted later

we can contract it now – edges outside this multiedge cont care

2) the multiedge was deleted / reduced to a simple edge

we can do that now

$\blacksquare$

Forbidden Minor Characterization

Theorem (Tutte): M is graphic $\iff$ $M^*(K_5) \not\preceq M$ & $M^*(K_{3,3}) \not\preceq M$.

Pf: M graphic $\iff$ M^* cographic $\iff$ $M^* = M(G)$ for some planar G

G planar $\iff K_5 \not\preceq G$ & $K_{3,3} \not\preceq G$ ← Kuratowski – Wagner

$\iff M(K_5) \not\preceq M(G) = M^*$ & $M(K_{3,3}) \not\preceq M(G) = M^*$ ← previous lemma

$\iff M^*(K_5) \not\preceq M$ & $M^*(K_{3,3}) \not\preceq M$ ← taking duals ■

Def: M is binary $\equiv$ it is representable over $\mathbb{F}_2$. ← $M \cong$ Vectorial Matroid $(V \in \mathbb{F}_2^{m \times n})$

Recall: All graphic matroids are binary. $U(4,2)$ is not binary

Def: M is regular $\equiv$ it is representable over every field.

Fact: M regular $\iff$ M^* regular. Minors of a regular matroid are regular.

Theorem (Tutte): M is regular $\iff$ it is representable with an totally unimodular matrix.

Corollary: M graphic $\Rightarrow$ M regular.

Pf: $M(G)$ can be represented by the directed incidence matrix of G.

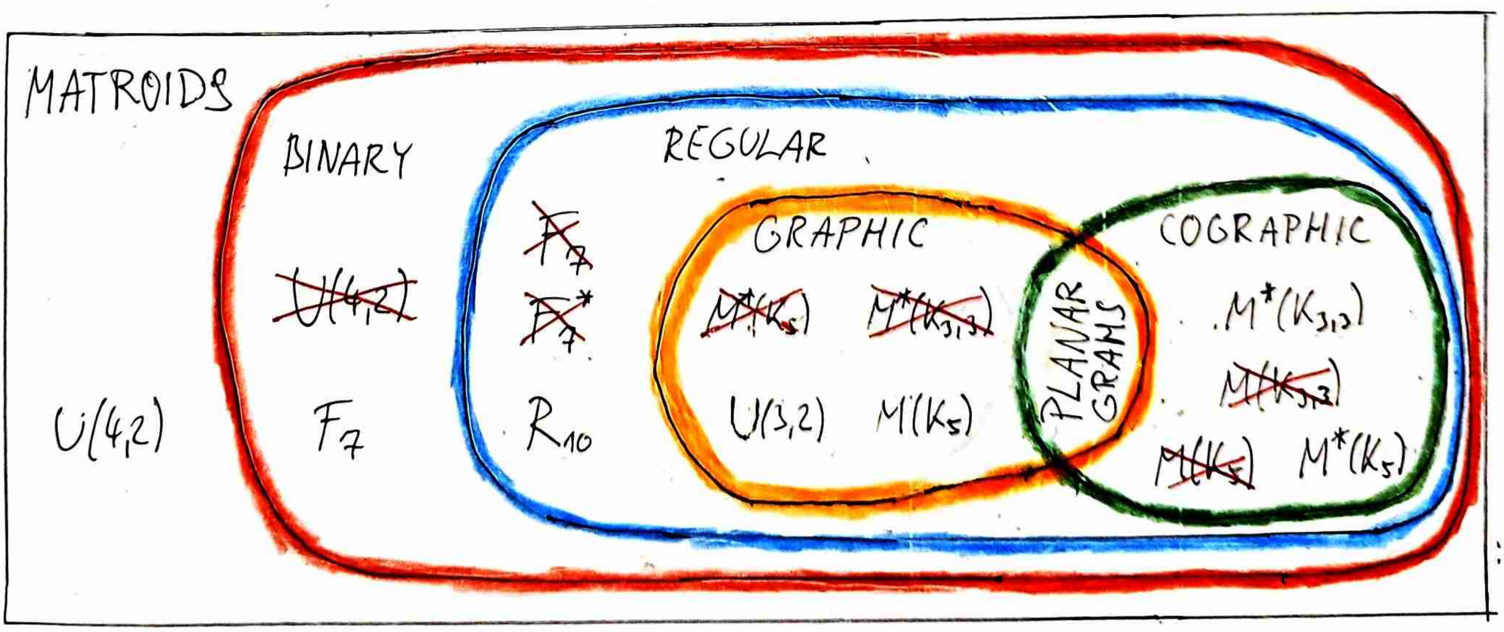
Recall: Incidence matrices for directed graphs are allways totally unimodular. ■

Theorem (Tutte): We can characterize certain minor-closed matroid classes using a finite set of forbiden minors.

1. binary $\iff$ not contains $U(4,2)$ ← Fano matroid
2. regular $\iff$ not contains $U(4,2)$, F_7, F_7^*
3. graphic $\iff$ not contains $M^*(K_5)$, $M^*(K_{3,3})$

Note: This is not as strong as the graph minor theorem.

↳ matroid minors are not a well-quasi ordering.



Greedy Algorithm

Consider the hereditary system $(X, \mathcal{C} \subseteq 2^X)$ with weight $w: X \to \mathbb{Q}$.

↳ let $|X| = n$ and number the elements $x_1, \dots x_n$, $w_i := w(x_i)$

↳ WLOG let $w_1 \geq w_2 \geq \dots \geq w_m \geq 0 \geq \dots \geq w_n$

We want to find $J \in \mathcal{C}$ s.t.

$$w(J) := \sum_{x \in J} w(x) \quad \text{is maximal} \quad \dots \quad w(J) = \max_{A \in \mathcal{C}} w(Y)$$

The greedy algorithm tries to find J like this:

Algorithm: $GA(X, \mathcal{C}, w) \to J$

1. $J \leftarrow \emptyset$
2. for $i = 1, \dots n$ do
3. if $J \cup \{x_i\} \in \mathcal{C}$ then $J \leftarrow J \cup \{x_i\}$
4. return J

always add heaviest element if possible

We will show that this works $\Leftrightarrow$ $(X, \mathcal{C})$ is a matroid

Theorem: Let $(X, \mathcal{C})$ be a nonempty hereditary system ... $A \in \mathcal{C}$ & $A' \subseteq A \Rightarrow A' \in \mathcal{C}$.

The GA works for every weight function w $\Leftrightarrow$ $(X, \mathcal{C})$ is a matroid

Proof:

$\Rightarrow$: For contradiction assume that $(X, \mathcal{C})$ is not a matroid

$$\Rightarrow \exists U, V \in \mathcal{C}, \ |U| > |V| \ \& \ \forall x \in U \setminus V: V \cup \{x\} \notin \mathcal{C}$$

→ Define $w(x) := \begin{cases} 1 + \varepsilon, & x \in V \\ 1, & x \in U \setminus V \\ 0, & \text{else} \end{cases}$

→ G.A. will return exactly V because elements from V have best w and it isnt possible to extend V from $U \setminus V$ & the rest are zeroes.

→ Since $|V| < |U|$ we can choose ε small enough s.t. $w(U) > w(V)$ ↯

$\Leftarrow$: Let $(X, \mathcal{C})$ be a matroid, $w: X \to \mathbb{Q}$ and suppose that GA failed

$J = \{g_1, g_2, \dots g_k\}$, $w(g_1) \geq \dots \geq w(g_k)$ ... output of GA

$M = \{o_1, o_2, \dots o_l\}$, $w(o_1) \geq \dots \geq w(o_l)$ ... optimal solution

Define j as the first i where $w(g_i) < w(o_i)$

↳ if $\forall i = 1, \dots k$ we have $w(g_i) \geq w(o_i)$, then $l > k$ and $j := k+1$

GA in step j didnt add any of $o_1, \dots o_j$, but either nothing or g_j, which has smaller weight than o_j and so than all $o_1, \dots o_j$.

This means that one of the two following happened

$\circledast\ g_1 \geq g_2 \geq \dots \geq g_{j-1} \geq g_j$ $\qquad \circledast$ = already added

$\wedge$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{j-1} \geq \sigma_j$ $\qquad g_j$ picked over $\sigma_1, \dots \sigma_j$

① All $\sigma_1, \dots \sigma_j$ were already added previosly

↳ impossible ∵ we only added $j-1$ elements at this point ↯

② The set $\{g_1, \dots g_{j-1}\}$ cannot be extended by any $\sigma_i \in \{\sigma_1, \dots \sigma_j\} \setminus \{g_1, \dots g_{j-1}\}$

↳ this contradicts the exchange axiom

$$\left.\begin{array}{l} U := \{\sigma_1, \dots \sigma_j\} \in \mathcal{C} \because M \in \mathcal{C} \\ V := \{g_1, \dots g_{j-1}\} \in \mathcal{C} \because J \in \mathcal{C} \end{array}\right\} |U| > |V| \ \& \ \nexists x \in U \setminus V : V + x \in \mathcal{C} \ ↯$$

■

Examples

① If we set $\forall x \in X : w(x) = 1$, then the GA finds a basis of $(X, \mathcal{C})$

② If $\forall x : w(x) > 0$, then the GA finds the heaviest basis

- max. weight spanning forest

↳ because the basis of the cycle matroid of a graph are spanning forests, the GA works for finding the max. weight spanning forest

Matroid Polytope Theorem

Let $(X, \mathcal{C})$ be a matroid with rank r, $X = \{x_1, x_2, \dots x_m\}$.

- for $A \subseteq X$ define characteristic vector $\chi^J \in \{0,1\}^m$, $\chi^J_i = \begin{cases} 1, & x_i \in A \\ 0, & x_i \notin A \end{cases}$
- for $x \in \mathbb{R}^m$, $B \subseteq X$ define $x(B) := \sum_{x_i \in B} x_i$

👁 if $J \in \mathcal{C}$, $A \subseteq X$, then $\chi^J(A) = |J \cap A| \leq r(A)$ rank

↳ $J \cap A$ is an ind. subset of A ... not bigger than the max ind. subset

Now consider the following LP: Let $w : X \to \mathbb{Q}$ and find $x \in \mathbb{R}^m$...

$$\max \sum_{i=1}^{m} w_i \cdot x_i \quad \text{s.t.} \quad \forall A \subseteq X : x(A) \leq r(A) \quad \& \quad \forall i : x_i \geq 0$$

Clearly for $\forall J \in \mathcal{C}$, χ^J is a feasible solution (observation above)

Theorem (Edmons): The char. vector of the set J found by the GA is an optimal sol.

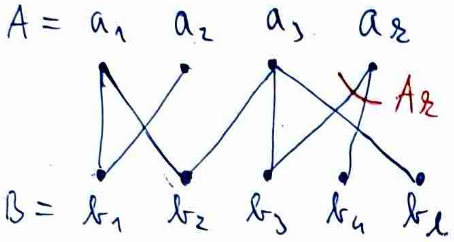
Theorem: $\text{ConvexHull}(\{\chi^J \mid J \in \mathcal{C}\}) = \{x \in \mathbb{R}^m \mid \forall A \subseteq X : x(A) \leq r(A) \ \& \ x \geq 0\}$

Matroid Intersection

- classic greedy: find max. weight ind. set of matroid M.
- intersection: given matroids $M_1=(X,\mathcal{I}_1)$, $M_2=(X,\mathcal{I}_2)$
  find max. weight common ind. set

→ why is this useful?

① Max matching in bipartite graph $G=(A\cup B, E)$



$\forall a_i$ define $A_i := \{e\in E \mid a_i \in e\}$

$\forall b_i$ define $B_i := \{e\in E \mid b_i \in e\}$

👁 both $\{A_i\}_i$ and $\{B_i\}_i$ partition E.

Recall: Partition matroid

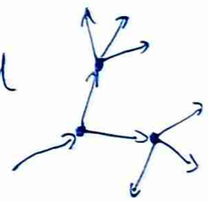
$M_1 := P(A_1,\dots A_k) = (E, \{F\subseteq E \mid \forall i: |F\cap A_i|\le 1\})$

$M_2 := P(B_1,\dots B_\ell) = (E, \{F\subseteq E \mid \forall i: |F\cap B_i|\le 1\})$

👁 Common independent sets of M_1 and M_2 are exactly the matchings of G

② Max branching in directed graph $G=(V,E)$

Def: A branching is a subgraph $B\subseteq G$ s.t. it is a forrest and each node in B has indegree ≤ 1.



→ we need forrest & restricted indegrees

$M_1 := (E, \{F\subseteq E \mid F \text{ is a forrest}\})$ ~ Cycle Matroid(G).

$M_2 := (E, \{F\subseteq E \mid \forall v\in V: \text{indegree}_F(v)\le 1\})$

👁 M_2 is a partition matroid ... let $N(v)=\{e\in E \mid e = uv,\ u \xrightarrow{e} v\}$, then

$M_2 = P(N(v_1),\dots N(v_n))$

👁 Branchings of G are exactly the common independent subsets of M_1 and M_2

Simple upper bound

Let $(X,\mathcal{I}_1)$, $(X,\mathcal{I}_2)$ be matroids, $J\in \mathcal{I}_1\cap\mathcal{I}_2$, $A\subseteq X$, $\bar{A} := X - A$.

- $J\in\mathcal{I}_1 \Rightarrow J\cap A\in \mathcal{I}_1$, $J\in\mathcal{I}_2 \Rightarrow J\cap\bar{A}\in\mathcal{I}_2$

$\Rightarrow |J| = \underbrace{|J\cap A|}_{\in\mathcal{I}_1} + \underbrace{|J\cap\bar{A}|}_{\in\mathcal{I}_2} \le r_1(A) + r_2(\bar{A})$ ... for $\forall J, \forall A$

Theorem (Edmonds): For matroids $M_1=(X,\mathcal{I}_1)$, $M_2=(X,\mathcal{I}_2)$ holds

$$\max\{|J| \mid J\in\mathcal{I}_1\cap\mathcal{I}_2\} = \min\{r_1(A)+r_2(\bar{A}) \mid A\subseteq X\}, \quad \bar{A}=X-A$$

Proof: We already observed that max $\le$ min, now we need to show $\ge$.
Induction on $|X|$. Let $k=\min\{r_1(A)+r_2(\bar{A}) \mid A\subseteq X\}$

① $\nexists x\in X: \{x\}\in\mathcal{I}_1\cap\mathcal{I}_2$

$\to$ for $\forall x\in X: \{x\}\notin\mathcal{I}_1 \vee \{x\}\notin\mathcal{I}_2$

$\Rightarrow$ let $A:=\{x\in X \mid r_1(\{x\})=0\}$

$\Rightarrow \bar{A}=\{x\in X \mid r_1(\{x\})\neq 0\} = \{x\in X \mid r_2(\{x\})=0\}$ $\Rightarrow k=0$ ✓

② $\exists x\in X: \{x\}\in\mathcal{I}_1\cap\mathcal{I}_2 \Rightarrow X':=X-\{x\}$ < deletion: $M_1^-:=M_1-\{x\}$, $M_2^-:=M_2-\{x\}$; contraction: $M_1':=M_1/\{x\}$, $M_2':=M_2/\{x\}$

induction: $\max\{|J^-| \mid J^-\in\mathcal{I}_1^-\cap\mathcal{I}_1^-\} = \min\limits_{A\subseteq X'}\{r_1(A)+r_2(X'-A)\} =: k^-$

$\max\{|J'| \mid J'\in\mathcal{I}_1'\cap\mathcal{I}_2'\} = \min\limits_{A\subseteq X'}\{r_1'(A)+r_2'(X'-A)\} =: k'$

a) $k^-\ge k$: since $\mathcal{I}_1^-\subseteq\mathcal{I}_1$ and $\mathcal{I}_2^-\subseteq\mathcal{I}_2$ we have $\max\ge\max^-=k^-\ge k$ ✓

b) $k'\ge k-1$: $\max'$ gives $J'\in\mathcal{I}_1'\cap\mathcal{I}_2'$ with size $\ge k-1$

$\Rightarrow$ add back x $\Rightarrow$ $J:=J'\cup\{x\}\in\mathcal{I}_1\cap\mathcal{I}_2$ has size $\ge k$ $\Rightarrow \max\ge k$ ✓

definition of contraction

c) otherwise

- $k^-\le k-1 \Rightarrow \exists A\subseteq X': r_1(A)+r_2(X'-A)=k^-\le k-1$
- $k'\le k-2 \Rightarrow \exists B\subseteq X': r_1'(B)+r_2'(X'-B)=k'\le k-2$

$\hookrightarrow$ since M_i' are contraction matroids: $r_i'(Y)=r_i(Y\cup\{x\})-\overbrace{r_i(\{x\})}^{1}$

$\Rightarrow r_1(B+x)-1+r_2(X'-B+x)-1\le k-2$

$r_1(B+x)+r_2(X-B)\le k$

$\to$ add both equations and use submodularity

$$r_1(A)+r_1(B+x)+r_2(X'-A)+r_2(X-B)\le 2k-1$$

$$\qquad\qquad \text{VI} \qquad\qquad\qquad\qquad \text{VI}$$

$$\underbrace{r_1(A\cup B+x)}_{a}+\underbrace{r_1(A\cap B)}_{b}+\underbrace{r_2(X-(A\cap B))}_{c}+\underbrace{r_2(X-(A\cup B+x))}_{d}\le 2k-1$$

$\to$ we have $a+b+c+d\le 2k-1 \Rightarrow b+c\le k-1 \vee a+d\le k-1$

$\hookrightarrow$ both of these imply $k=\min\limits_{Y\subseteq X}\{r_1(Y)+r_2(X-Y)\}\le k-1$ ↯ ▨

Fact: For most purposes there $\exists$ polynomial max matroid intersection algorithm.

Perfect and Maximum matchings

Def: The vertex v is exposed w.r.t. matching $M \equiv \nexists e \in M: v \in e$

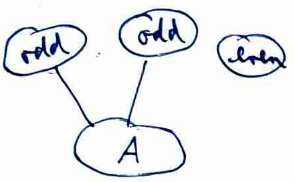
Def: The path P is an augmenting alternating path AAP $\equiv$

$P =$  , endpoints exposed

Lemma: M is a max matching in $G \iff G$ doesnt have an AAP.

Def: $G=(V,E) \Rightarrow ODD(G) :=$ # components of G with odd size

if $\exists A \subseteq V$ s.t $ODD(G-A) > |A|$, then G doesnt have a perfect matching.



if we want PM, we need at least 1 edge from A to each odd compont of $G-A$

Theorem (Tutte): G has perfect matching $\iff \forall A \subseteq V: ODD(G-A) \leq |A|$.

Theorem (Tutte, Berge): The size of a max. matching of G is

$$\frac{1}{2}(|V| - \text{defect}(G)), \quad \text{defect}(G) := \max_{A \subseteq V} \{ODD(G-A) - |A|\}$$

Intuition: If G has PM, then defect $= 0$ and the size of the PM is $|V|/2$.
Otherwise, this theorem says that the odd components are the only obstacle of PM

Algorithm (Edmons blossom alg): Finds max matching in $O(n^2(n+m))$

↑

FOR PROOFS & DETAILS SEE THE START OF NOTES FROM COMBINATORICS & GRAPHS II.

Def: A graph G is <u>hypomatchable</u> $\equiv \forall v \in V(G)$: $G-v$ has PM

Theorem (Edmons-Gallai decomposition): Let $G=(V,E)$ be a graph and divide the vertices of G into two groups

- essential vertices ... every max. matching covers these vertices
- inessential vertices ... $D(G) := \{v \in V \mid \exists$ max matching not covering $v\}$

Now split the essential vertices into 2 groups

$A(G) :=$ vertices adjaicent to at least one vertex from $D(G)$

$C(G) :=$ the rest = essential, not adjacent

The following holds:

① $C(G)$ has a perfect matching

② components of $D(G)$ are hypomatchable

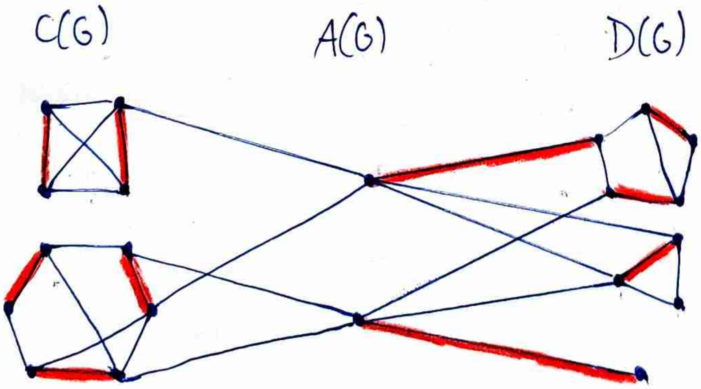
③ every $\emptyset \neq X \subseteq A(G)$ has neighbours in at least $|X|+1$ components of $D(G)$

every max matching has the following structure

- perfect matching in $C(G)$
- near perfect matching in $D(G)$
- edges from vertices in $A(G)$ to distinct components in $D(G)$

if $D(G)$ has k components, then the size of a max. matching is

$$\frac{1}{2}\left(|V(G)| - k + |A(G)|\right) \Rightarrow \text{defect}(G) = k - |A(G)|$$



only 1 uncovered vertex

defect = 3-2 = 1

How to find decomposition?

→ run blossom algorithm and stop at the very end, when we have Edmons forrest F fully build and contract all blossoms ... smaller graph G' with forrest F'

- $D(G') = v \in F'$, even dist. from root $\Rightarrow D(G) = D(G')$ + vertices contracted into blossoms
- $A(G') = v \in F'$, odd dist. from root $\Rightarrow A(G) = A(G')$
- $C(G') = v \in G' - F'$ $\Rightarrow C(G) = C(G')$

$\Big\}$ vertices never contracted

ARC ROUTING PROBLEMS

$G=(V,E)$ graph (road network), $l: E \to \mathbb{Q}^+$ lenghts of edges

- Chinese Postman Problem (CPP) [polynomial]
  - → find shortest closed route containing all edges at least once
- Traveling Salesman Problem (TSP) [NP-complete]
  - → find shortest closed route containing all vertices at least once
    - ↳ shortest Hamiltonian cycle

Chinese Postman Problem ... G connected

a) All degrees are even

solution to CPP is a closed euler tour, which can be found polynomially

Recall: G has closed euler tour $\Leftrightarrow$ all degrees even (G connected)

b) Some degrees are odd

for any solution to CPP there is at least one edge traversed twice

↳ otherwise all degrees even

$T := \{v \in V \mid \deg(v) \text{ is odd}\}$ ... $|T|$ is even

Def: $G=(V,E)$ graph, $T \subseteq V$. Then $E' \subseteq E$ is called a T-join $\equiv$

graph $G_T := (V, E')$ satisfies: $v \in T \Leftrightarrow \deg_{G_T}(v)$ is odd

→ E is a T-join

Lemma: Let $E' \subseteq E$ be the set of edges of a min. CP route which are traversed at least twice. Then each edge of E' is traversed exactly twice and E' is a min T-join for $T = \{v \in V \mid \deg_G(v) \text{ odd}\}$

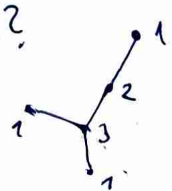
Pf: Observe that E' is the set of edges of min total lenght s.t. when we add E' to G (create multiedges), $E \dot{\cup} E'$ has all degrees even and so we can use euler tour

$\Rightarrow$ Hence each edge in E' is traversed exactly twice in the CP route

↳ we dont need more for even degrees

$\Rightarrow$ we know that E' is minimal ... is it T-join?

$v \in T \Leftrightarrow \deg_{E'}(v)$ is odd



Conclusion: In order to solve CPP, it suffices to find a min T-join.
→ we then simply add the edges in the T-join and get a degree even graph

Algorithm for min T-join in G, $T \subseteq V(G)$, $|T|$ even

① Construct graph $H = (T, \binom{T}{2})$ with edge weight w

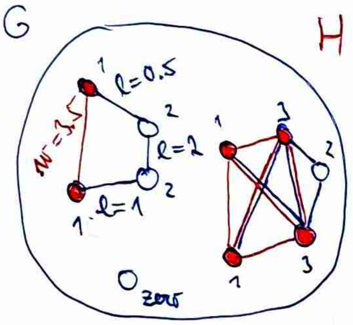
↳ for $uv \in E(H)$, let $w(uv) :=$ lenght of shortest path $u \leftrightarrow v$ in G

↳ recall, G has edge lenghts l

calculating w is polynomial... shortest paths via Dijkstra

② Find min. weight perfect matching of H – fact: this is polynomial

↳ H is complete & $|V(H)|$ is even ⇒ PM exists



→ weight of the PM uses weights w

→ the PM pairs up vertices and then sums up the lenghts of the shortest path between paired vertices

⇒ Let M be the min perfect matching, for $e \in M$ define

$P_e :=$ a shortest path between the endpoints of e in G

③ Join the shortest paths defined by M together – claim: this is min T-join

→ let $E' := \dot{\bigcup}_{e \in M} P_e$

E' has no multiedges – we could create a smaller perfect matching

E' is a T-join

- $v \in T \Leftrightarrow v$ covered by some $e \in M \Leftrightarrow v$ endpoint of P_e ⇒ $\deg(v)$ += 1
- any $v \in V(G)$ can used by one of the shortest paths, but it can be an end point only if $v \in T$ (and it can be an endpoint only once)
  → all subsequent pass-throughs add $\deg(v)$ += v

⇒ $v \in T \Leftrightarrow \deg_{E'}(v)$ is odd

claim: E' is a min T-join ... $l(E') := \sum_{e \in E'} l(e)$ is minimal

every T-join F can be partitioned into paths between the vertices of T

⇒ hence F defines a perfect matching M_F of H s.t. $w(M_F) \leq l(F)$

because of the way how E' was constructed we have $w(M) = l(E')$

Since M is a min PM we have $l(E') = w(M) \leq w(M_F) \leq l(F)$ ▨

Algorithm: Chinese Postman Problem ($G=(V,E)$)

1. if $\exists v \in V$ with odd degree:
2. $\quad T := \{v \in V \mid \deg(v) \text{ is odd}\}$
3. $\quad E' \leftarrow \text{MinTJoin}(G,T)$ ← polynomial
4. $\quad G \leftarrow G + E'$ ← G is a multigraph

   # now all degrees of G are even
5. return ClosedEulerTour(G) ← polynomial

Traveling Salesman Problem ... G connected

→ NP complete ⇒ we use an approximation algorithm

APS2: 2-approx

now: 1.5-approx

Assumptions

(A) G complete – if not, add edges with $l(e) := +\infty$

(B) $l(e) \geq 0$ for $\forall e \in E$

(C) triangle inequality holds

} dist(u,v) is a metric on V(G)

Algorithm (Christofides)

1. $T \leftarrow$ min spanning tree of G
2. $W \leftarrow \{v \in V \mid \deg_T(v) \text{ is odd}\}$
3. $M \leftarrow$ min. lenght perfect matching of vertices in W ... PM of $G[W]$
4. $T' \leftarrow T + M$ ← T' is a multigraph

   # all degrees of T' are even

   because of T
5. if all degrees in T' are 2:

   → covers all vertices, is connected and all degrees are 2

   # T' is a Hamiltonian cycle

   return T'
6. while $\exists v \in V$ with $\deg_{T'}(v) \geq 4$:

   reduce $\deg_{T'}(v)$ by 2 using shortcut

   # now all degrees are 2
7. return T'

(*) $uw \in E$ ∵ G is complete

Δ-ineq: $l(uv) + l(vw) \geq l(uw)$

The alg. returns a H. cycle, but we want the shortest H. cycle

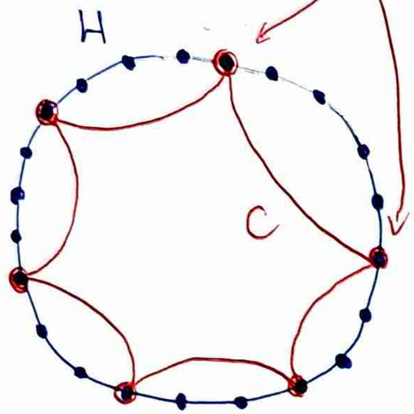
Theorem: Assume Ⓐ, Ⓑ, Ⓒ. Let H be a shortest TS tour and H^c the output of the Christofides algorithm. Then $\ell(H^c) \leq \frac{3}{2}\ell(H)$.

Proof: We will follow the alg. step-by-step

① Take any $e \in H$, then $H-e$ is a spanning tree

$\Rightarrow \ell(T) \leq \ell(H)$ $\because$ T is min. spanning tree & $\ell(e) \geq 0$

② Construct W



→ order vertices of G based on how they are visited by H

→ create C = cycle on W following order of H

$\ell(C) \leq \ell(H)$ using Δ-inequality

$|W|$ is even $\Rightarrow$ C defines 2 perfect matchings of W

↳ call them M_1, M_2

③ Since $\ell(M_1) + \ell(M_2) = \ell(C) \leq \ell(H)$, at least one M_i has $\ell(M_i) \leq \frac{1}{2}\ell(H)$

$\Rightarrow \ell(M) \leq \frac{1}{2}\ell(H)$ $\because$ M is min. PM of W $\Rightarrow$ $\ell(M) \leq \ell(M_i)$

④ Hence $\ell(T') = \ell(T) + \ell(M) \leq \ell(H) + \frac{1}{2}\ell(H) = \frac{3}{2}\ell(H)$

⑥ As we observed in (*), doing this doesn't increase lenght

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