Graph Cycle and Cut Spaces

Def: A tour in a graph is a sequence of vertices and edges $v_1 e_1 v_2 \dots e_k v_{k+1}$ where no two edges repeat.

Def: An Euler tour is a tour which covers all of the edges.

Theorem: A connected graph G has a closed Euler tour ⇔ G has all degrees even.

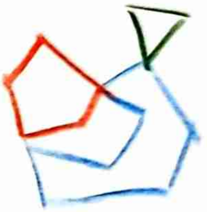
↳ closed tour := start node = end node

Pf:

⇒: The tour is closed, so every time it enters a vertex it also exits it and because it cannot reuse edges: 1 visit ⇒ degree + 2

⇐:

Def: $E' \subseteq E(G)$ is even $\equiv G' = (V, E')$ has all degrees even.



Lemma: Let $G = (V, E)$ where $E \neq \emptyset$ is even. Then G has a cycle.

Pf: Since $E \neq \emptyset$ there is a component of G with some edges

↳ lets focus on this component only

↳ if it didnt contain a cycle, then it would be a tree ⇒ E not even ↯

Corollary: Every graph with all degrees even is an edge-disjunct union of cycles.

↳ simply apply the previous lemma, find a cycle

↳ remove the cycle → all degrees still even

↳ repeat untill there are no edges left

→ it is easy to create the Euler tour from these cycles by induction by starting with no edges and adding one cycle at a time

↳ the next cycle always has to be conected to what we already have

↳ but G is connected so that is not an issue ■

This algorithm is polynomial $O(|V| \cdot (|V| + |E|))$

Fact: The following generalization of this problem can also be solved polynomially

$G = (V, E)$, $w : E \to \mathbb{Q}$, find even $E' \subseteq E$ s.t.

$w(E') := \sum_{e \in E'} w(e)$ is maximal

Fact: The problem of finding the shortest closed Euler tour (Hamiltonian cycle) cannot be solved polynomially → its NP complete

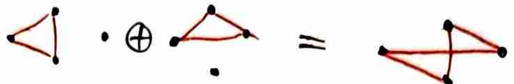
- Cycle space

Def: The cycle space of a graph $G=(V,E)$ is the set of all of its even-degree subgraphs over the finite field $\mathbb{F}_2 = \{0,1\}$.

- vectors $= \{(V,E') \mid \text{even } E' \subseteq E\}$
- zero vector $= (V,\emptyset)$

↳ symmetric difference $A \Delta B = (A \cup B) \setminus (A \cap B)$

- sum of two vectors : $(V,E_1) \oplus (V,E_2) := (V, E_1 \Delta E_2)$

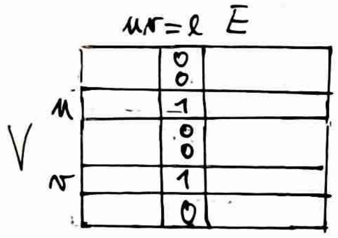


- opposite vector : $-(V,E) = (V,E) \because (V,E) \oplus (V,E) = (V,\emptyset)$
- scalar multiplication: $0 \cdot (V,E) = (V,\emptyset)$

$1 \cdot (V,E) = (V,E)$

Theorem: The kernel of the incidence matrix of G over $\mathbb{F}_2$ is the cycle space of G.

Def: The incidence matrix of $G=(V,E)$ is



$$(I_G)_{ij} := \begin{cases} 1, & v_i \in e_j \\ 0, & v_i \notin e_j \end{cases} \qquad I_G \in \mathbb{F}_2^{|V| \times |E|}$$

Pf: $\ker_{\mathbb{F}_2}(I_G) = \{x \in \mathbb{F}_2^{|E|} \mid I_G \cdot x = \sigma\}$

↳ x determines which edges will be used

$\Rightarrow x = \chi_{E'}$ = characteristical vector for some $E' \subseteq E$

👁 the i^{th} element of $I_G \cdot \chi_{E'}$ is $\deg(v_i)$ in (V,E')

$\Rightarrow$ we are in $\mathbb{F}_2 \Rightarrow I_G \cdot \chi_{E'} = \sigma \iff$ all degrees in (V,E') are even

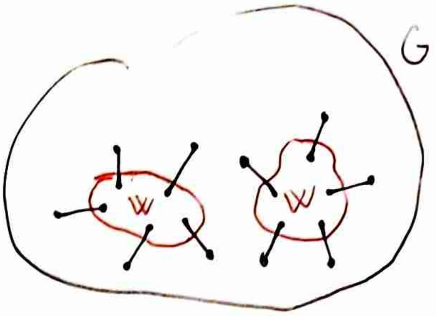
$\Rightarrow \ker_{\mathbb{F}_2}(I_G) = \{\chi_{E'} \mid E' \subseteq E \text{ is even}\}$ □

Corollary: Since $\ker I_G$ = cycle space and ker is a vector space, it follows that cycle space is really a vector space too.

• Cut Space

Def: An edge cut in $G=(V,E)$ is any $E' \subseteq E$ s.t.

$\exists W \subseteq V : E' = \{e \in E \mid |e \cap W| = 1\}$



Def: The cut space of $G=(V,E)$ is the set of all of its edge cuts over the field $\mathbb{F}_2$.

- vectors $= \{E' \subseteq E \mid E'$ is an edge cut in $G\}$
- zero vector $= \emptyset$
- vector addition: $E_1 \oplus E_2 := E_1 \Delta E_2$ ... symmetric difference
- opposite vector: $-E' = E'$ $\because E' \Delta E' = \emptyset$

Lemma: The symmetric difference between two edge cuts is an edge cut.

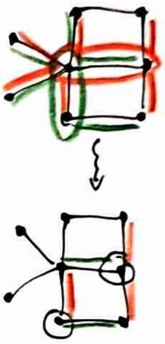
Pf: E_1 with W_1 ... $e \in E_1 \Leftrightarrow |e \cap W_1| = 1$

E_2 with W_2 ... $e \in E_2 \Leftrightarrow |e \cap W_2| = 1$

$\rightarrow E_1 \Delta E_2$ with $W_1 \Delta W_2$ ...

$\hookrightarrow e \in E_1 \Delta E_2 \Longleftrightarrow |e \cap (W_1 \Delta W_2)| = 1$

$e \in E_1 \cup E_2$ & $e \notin E_1 \cap E_2 \Longleftrightarrow \underbrace{|e \cap (W_1 \cup W_2)|}_{2 \text{ or } 1} - \underbrace{|e \cap (W_1 \cap W_2)|}_{1 \text{ or } 0} = 1$



■

Lemma: Each edge cut $E' \subseteq E$ with $W \subseteq V$ is the symmetric difference of elementary edge cuts $N(v)$ defined by $v \in W$. $N(v) = \{e \in E \mid v \in e\}$

Pf: Induction by adding vertices to W.

Corollary: $\{N(v) \mid v \in V\}$ generates $G=(V,E)$.

Theorem: The row space of I_G is the cut space of G.

Def: The row space of $A \in \mathbb{K}^{n \times m}$ is $R_{\mathbb{K}}(A) = \{A^T \tilde{x} \in \mathbb{K}^m \mid \tilde{x} \in \mathbb{K}^n\}$

$\hookrightarrow$ the space generated by the rows of A

Pf: Row i of I_G = characteristic edge vector of $N(v_i) \subseteq E$

$\hookrightarrow$ elementary edge cut

$\Rightarrow$ the rows of I_G form a basis for the cut space of G

$\Rightarrow R(I_G)$ = cut space of G

■

Fact: The problem of finding a max. cut in a graph is NP complete

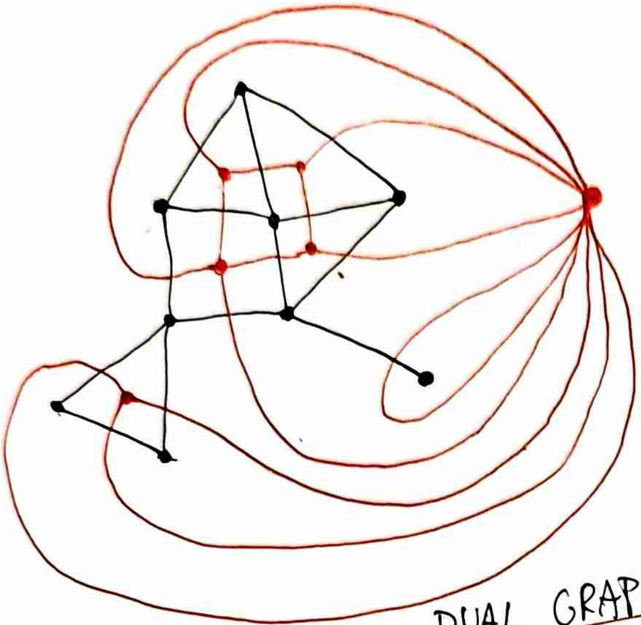
- **Geometric Duality**

↱ dual graph

Fact: For planar graphs: CutSpace(G) ~ Cycle Space(G^*).

→ it is very difficult to geometrically define what the dual of a planar graph is

⇒ we will use an abstract definition



DUAL GRAPH

→ faces of G (including the outer face) become the vertices of G^*

→ each edge from G touches 1 or 2 faces of G (vertices of G^*) and therefore defines an "edge" of G^*

Def: Let F be the faces of a planar graph $G=(V,E)$.

$$G^* := (F, f[E]) \quad , \quad f: E \to \binom{F}{2} \cup F$$

Where f(e) is the set of faces touching e.

Corollary: G planar ⇒ find max. cut is polynomial

↳ convert G to G^*, work in cycle space of G^* (polynomial), convert back

- **Ising Model**

→ some simple historical model of something in modern physics

~ named by the scientist Ising

$G = (V,E)$, $w: E \to \mathbb{Q}$

State: $s: V \to \{1,-1\}$

State energy: $e(s) := \sum_{uv \in E} s(u)s(v)w(uv)$

→ goal: find some state with minimum energy

👁 each state defines an edge cut

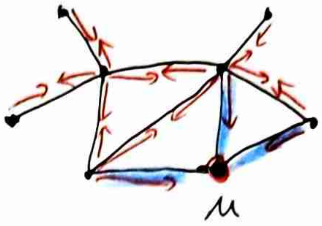
$\varphi(s) = \{uv \in E \mid s(u) \neq s(v)\}$ with $W = \{v \mid s(v) = 1\}$

👁 $w(E) - e(s) = 2 \cdot w(\varphi(s)) \Rightarrow e(s) = -2 \cdot$(weight of cut(s)) + constant

⇒ to find min. energy we need to find max. cut

• More on Cycle and Cut Spaces

① What is dim(Cut Space)? Find a basis.



→ Suppose G is connected

→ we know that $\{N(v) \mid v \in V\}$ generates G

👁 for any $u \in V$: $N(u) = \triangle_{\substack{v \in V \\ v \neq u}} N(v)$

👁 if we were to remove another vertex, we would not be able to generate it

⇒ $\{N(v) \mid u \neq v \in V\}$ is a minimal, lin-ind set → basis ... dim = $|V| - 1$

→ Suppose G is not connected ⇒ we handle each component separately

Theorem: Let G be a graph with k components and n vertices.

Then dim(CutSpace(G)) = n − k.

② What is dim(CycleSpace)? Find a basis.

→ Suppose $G = (V,E)$ is connected and consider a spanning tree $T = (V,F)$

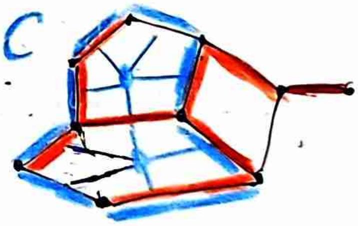
recall: $E' \in$ Cycle Space ≡ (V,E') has all degrees even

⇒ E' = edge-disjoint union of cycles

⇒ we need to generate all cycles in G

👁 Adding any edge $e \in E \setminus F$ to T creates a single cycle C_e

⇒ $\mathcal{B} := \{C_e \mid e \in E \setminus F\}$



claim: $\mathcal{B}$ generates ∀ cycle $C \subseteq E$ as $C = \triangle_{e \in C \setminus F} C_e$

↳ lets show $X := C \,\triangle \underbrace{\left(\triangle_{e \in C \setminus F} C_e\right)}_{} = \emptyset$

👁 $X \subseteq F$, & C even ∧ even ⇒ X even

→ x is a union of cycles & $x \subseteq F$ ⇒ $X = \emptyset$

claim: $\mathcal{B}$ is lin. ind. and minimal ⇒ basis

↳ C_e is the only cycle in $\mathcal{B}$ containing e

→ $|\mathcal{B}| = ?$ ... $|V| = n$ ⇒ $|F| = n - 1$ ⇒ $|E \setminus F| = m - n + 1$

→ for G non-connected we do each component separately and together $|F| = n - k$

Theorem: Let G be a graph with k components, $m := |E|$, $n := |V|$.

Then dim(CycleSpace(G)) = m − n + k.

③ Let $G=(V,E)$ be 2-connected planar and $G^*=(V^*,E^*)$ its geometric dual.
Prove $CutSpace(G) \cong CycleSpace(G^*)$

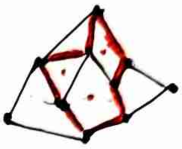
Def: Let U, V be vector spaces with bases B_U, B_V. We say that → field $\mathbb{K}$
U and V are isomorphic $U \cong V$ ≡ $\exists$ linear bijection $f: U \to V$

👁 Sufficient: $\exists$ bijection $B_U \leftrightarrow B_V$ ⇒ we can extend it to be a linear bijection $U \leftrightarrow V$
↳ $(\forall a \in \mathbb{K})(\forall u, v \in B_U)$: $f(a \cdot v) = a \cdot f(v)$, $f(u+v) = f(u) + f(v)$

Theorem: The inner faces of a 2-connected planar G form a basis of $CycleSpace(G)$.

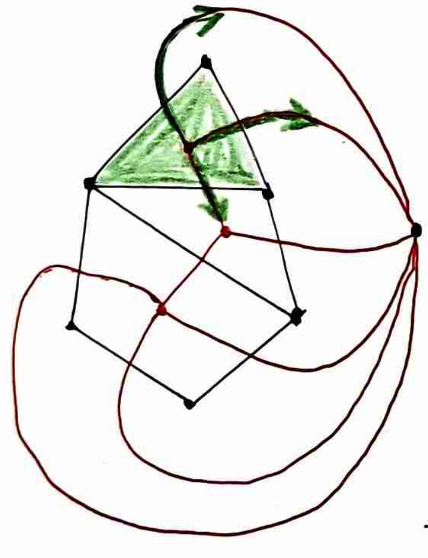
Pf: 1) they generate all cycles $C \subseteq E(G)$

👁 $C = \triangle \{F \mid F \text{ is a face inside } C\}$



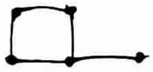
2) they are a minimal generating set
↳ a basis of G has size $m-n+1$ ... last theorem
↳ Eulers formula: $n - m + f = 2 \Rightarrow f = m - n + 2$
↳ we are using only inner faces ⇒ $|\mathcal{B}| = m - n + 1$ ■

Theorem: G 2-connected planar ⇒ $CutSpace(G) \cong CycleSpace(G^*)$



G 2-connected ⇒ we will not get things as

OR



$Basis(CycleSpace(G))$ = inner faces of G
$Basis(CutSpace(G^*)) = \{N_{G^*}(F) \mid F \text{ is inner face of } G\}$

→ there is a clear bijection between the basis

⇒ we can extend it to get a linear bijection between $CycleSpace(G)$ and $CutSpace(G^*)$ ■

Conclusion: The isomorphism maps an inner face of G to the edges of G^* which intersect its boundary
⇒ equivalently it maps an edge of G to the corresponding edge in G^*
↳ this is the general isomorphism which works for all planar graphs

Matroids

ground set

Def: $M=(X,\varphi)$, $\varphi \subseteq 2^X$ is a matroid $\equiv$

① $\emptyset \in \varphi$ ... nonempty } hereditary system

② $A \in \varphi$ & $A' \subseteq A \Rightarrow A' \in \varphi$ ... hereditary

③ $U, V \in \varphi$, $|U| > |V| \Rightarrow \exists x \in U \setminus V : (V \cup \{x\}) \in \varphi$ ... exchange axiom

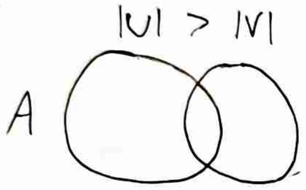
③' $Y \subseteq X \Rightarrow$ all $\max_{\subseteq} \{A \subseteq Y \mid A \in \varphi\}$ have the same cardinality

↳ bases of Y

→ The elements of φ are called independent sets

Lemma: ③ $\Longleftrightarrow$ ③'

Pf: $3' \Rightarrow 3$: We have $U, V \in \varphi$ s.t. $|U| > |V|$ → define $A := U \cup V$



↳ all $\max_{\subseteq}$ subsets of A have the same cardinality

↳ $|U| > |V|$ → we can add something from $A \setminus V$ to V

$3 \Rightarrow 3'$: for contradiction assume two maximal $|U| > |V|$ – different sizes

↳ by ③ $\exists x \in U \setminus V$: $(V \cup \{x\}) \in \varphi \Rightarrow V$ is not maximal ↯ ■

Def: $M=(X,\varphi)$ matroid, for $Y \subseteq X$ we define

$\text{rank}(Y) := |\max_{\subseteq} \{A \subseteq Y \mid A \in \varphi\}|$ ... this is OK because ③' guarantees the same size of all maxims

👁 $\text{rank}(Y) = \max\{|A| \mid A \subseteq Y \ \& \ A \in \varphi\}$

↳ this is a weaker definition which doesnt require ↗

Corollary: Matroids are exactly the hereditary set systems, where rank can be correctly defined.

→ max ind. subsets of $Y \subseteq X$ are called the bases of Y

Def: Max independent subsets of $M=(X,\varphi)$ are called the bases of M.

↳ i.e. elements of φ which are max with respect to inclusion $\subseteq$.

Def: The rank of $M=(X,\varphi)$ is the $\text{rank}(M) := \text{rank}(X) = \text{rank}(\text{base of } M)$

👁 The bases of $M=(X,\varphi)$ determine φ

↳ $\forall A \subseteq X$: $A \in \varphi \Longleftrightarrow \exists$ basis B of M s.t. $A \subseteq B$

👁 $A \in \varphi \Longleftrightarrow \text{rank}(A) = |A|$ $\Rightarrow$ rank determines φ

Ex: → also sometimes column matroid

- **Vectorial Matroid**

$N =$ matrix over a field $\mathbb{K}$

↳ $M = (X, \varphi)$ … $X =$ columns of N

$A \in \varphi \equiv A$ is lin. ind.

① $\emptyset \in \varphi$ ✓

② A lin. ind. & $B \subseteq A \Rightarrow B$ lin. ind. ✓

③ implied by the Steinitz exchange theorem

👁 A is a basis of $\mathrm{ColSpace}(N) \Leftrightarrow A$ is a basis of M

- **Cycle Matroid**

$$G = (V, E) \rightsquigarrow M_G := (E, \{\underbrace{E' \subseteq E \mid E' \text{ is acyclic}}_{E' \text{ is a forest}}\})$$

① $\emptyset$ is acyclic ✓

② E_1 acyclic & $E_2 \subseteq E_1 \Rightarrow E_2$ acyclic ✓

③' $E' \subseteq E \Rightarrow$ all $\max_{\subseteq} \{F \subseteq E' \mid F \text{ acyclic}\}$ have the same cardinality

👁 these $\max_{\subseteq}$ forrests are the spanning forrests of E' → same size

Ex: operations with matroids

- **deletion**: $M = (X, \varphi)$, $Y \subseteq X \rightsquigarrow M - Y := (X \setminus Y, \overbrace{\{A \setminus Y \mid A \in \varphi\}}^{\varphi'})$ & $'(A) = r(A)$

① $\emptyset \in M - Y$

② $A \in \{B \setminus Y \mid B \in \varphi\}$ & $A' \subseteq A \overset{?}{\Rightarrow} A' \in \{B \setminus Y \mid B \in \varphi\}$

↳ $(\exists \Delta \subseteq Y): A \dot\cup \Delta \in \varphi$ & $A' \subseteq (A \cup \Delta) \Rightarrow A' \in \varphi$

$A' \cap Y = \emptyset$ ✓

③' $Z \subseteq X \setminus Y$ … all $\max_{\subseteq} \{B \subseteq Z \mid B \in \varphi'\}$ have the same size

↳ we want to use ③' of M → need $(\forall B \subseteq Z): B \in \varphi' \Leftrightarrow B \in \varphi$

↳ $B \in \varphi' \Leftrightarrow \exists A \in \varphi: B = A \setminus Y \Rightarrow B \subseteq A \overset{②}{\Rightarrow} B \in \varphi$

↳ other direction: $B \in \varphi$ & $B \subseteq Z$ … $B \cap Y = \emptyset \Rightarrow B \in \varphi'$

- **direct sum**

↳ matroids $M_1 = (X_1, \varphi_1)$, $M_2 = (X_2, \varphi_2)$ s.t. $X_1 \cap X_2 = \emptyset$

$$M_1 + M_2 := (X_1 \dot\cup X_2, \{Y_1 \dot\cup Y_2 \mid Y_1 \in \varphi_1, Y_2 \in \varphi_2\})$$

Theorem: $r: 2^X \to \mathbb{N}$ is the rank function of a matroid $\iff \forall Y \subseteq X$:

(R1) $r(\emptyset) = 0$

(R2) $r(Y) \leq r(Y \cup \{y\}) \leq r(Y) + 1$

(R3) $r(Y \cup \{y\}) = r(Y \cup \{z\}) = r(Y) \implies r(Y) = r(Y \cup \{y, z\})$

Pf: $\Rightarrow$: Let $M = (X, \mathcal{I})$ and r its rank function. It satisfies:

(R1) $r(\emptyset) = 0$ ✓

(R2) $r(Y)$ = size of max. independent subset of Y

$r(Y \cup \{y\})$ ... adding 1 element can not increase it by more than 1

(R3) let max. ind. subset of Y be B $\Rightarrow r(Y) = |B|$

↳ since $r(Y \cup \{y\}) = r(Y \cup \{z\}) = r(Y)$, B can not be extended by y, z

$\Rightarrow Y \cup \{y\}$ and $Y \cup \{z\} \notin \mathcal{I}$

$\Rightarrow$ therefore $r(Y) = r(Y \cup \{y, z\})$ ... if $Y \cup \{y, z\} \in \mathcal{I}$ then by heredity $Y \cup \{y\}$ and $Y \cup \{z\} \in \mathcal{I}$ ↯

$\Leftarrow$: Define $\mathcal{I} := \{A \subseteq X \mid |A| = r(A)\}$

claim: $(X, \mathcal{I})$ is a matroid with rank function r

① $\emptyset \in \mathcal{I}$ ... because $r(\emptyset) = 0$ (R1)

② $|A| = r(A)$ & $A' \subseteq A \overset{?}{\Rightarrow} |A'| = r(A')$

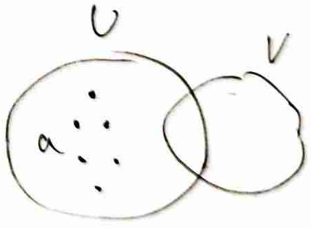
$r(A) = |A|$ → use (R2) repeatedly

👁 $r(A') \leq |A'|$ ∵ A' is created from $\emptyset$ by adding $|A'|$ elements

Note: A is created from A' by adding $k := |A \setminus A'|$ elements

$r(A) \leq r(A') + k \Rightarrow |A| \leq r(A') + |A \setminus A'| \Rightarrow r(A') \geq |A| - |A \setminus A'| = |A'|$

③ for contradiction assume $U, V \in \mathcal{I}$, $|U| > |V|$ ←
such that $\forall a \in U \setminus V$: $V \cup \{a\} \notin \mathcal{I}$



↳ $\forall a \in U \setminus V$: $r(V \cup \{a\}) \neq |V \cup \{a\}|$

↳ using (R2) we have $r(V \cup \{a\}) = r(V)$

→ by repeatedly using (R3) we can add every $a \in U \setminus V$ to V and the r never increases

$\Rightarrow$ we arrive at $r(V \cup (U \setminus V)) = r(V)$ $\Rightarrow r(U \cup V) = r(V)$ ↯

Note: $r(V) = |V|$, $U \subseteq U \cup V \Rightarrow r(U \cup V) \geq r(U) = |U| \Rightarrow (\text{something} \geq |U|) = |V|$

- Submodularity of rank

Def: $f: 2^X \to \mathbb{R}$ is submodular $\equiv$

$$\forall A, B \subseteq X \; : \; f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

Theorem: $r: 2^X \to \mathbb{N}_0$ is the rank function of a matroid $\iff \forall Y, Z \subseteq X$:

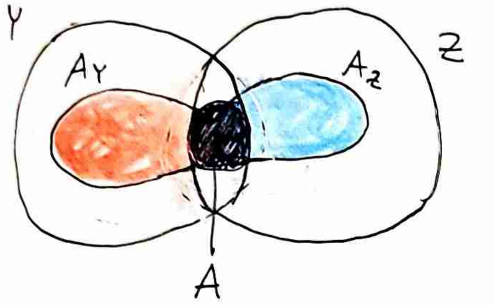
(R1') $0 \leq r(Y) \leq |Y|$

(R2') $Z \subseteq Y \Rightarrow r(Z) \leq r(Y)$ ... monotonicity

(R3') $r(Y) + r(Z) \geq r(Y \cup Z) + r(Y \cap Z)$ ... rank is submodular !

Pf: $\Rightarrow$: (R1') and (R2') trivially hold for all matroids

$\to$ for (R3') consider the following: let $Y, Z \subseteq X$ be arbitrary



$A :=$ max independent set of $Y \cap Z$

$A_Y :=$ max ind. in Y containing A

$A_Z :=$ max ind in Z containing A

$$\text{rank}(Y) + \text{rank}(Z) = |A_Y| + |A_Z| = |A_Y \cup A_Z| + |A_Y \cap A_Z| = |A_Y \cup A_Z| + \overbrace{\text{rank}(Y \cap Z)}^{A}$$

need: $\text{rank}(Y \cup Z) = |\text{max ind. in } Y \cup Z| \leq |A_Y \cup A_Z|$

↳ we want to find max ind in $Y \cup Z$

$\Rightarrow$ start with A_Y (max in Y) and add stuff to it from $Z \setminus Y$

claim: let $W \subseteq Z \setminus Y$ s.t. $A_Y \cup W$ is max ind. in $Y \cup Z$

↳ then $|W| \leq |A_Z \setminus A| \Rightarrow |A_Y \cup W| \leq |A_Y| + |A_Z \setminus A| = |A_Y \cup A_Z|$

$\to$ if (for contradiction) $|W| > |A_Z \setminus A|$, then $|W \cup A| > |A_Z|$

and since $A_Y \cup W$ is independent, $(W \cup A) \subseteq (A_Y \cup W)$ is also independent

$\Rightarrow A_Z$ is not max. independent in Z ↯

$\Leftarrow$: We will show (R1' & R2' & R3') $\Rightarrow$ (R1 & R2 & R3)

Ⓡ1 $r(\emptyset) = 0$ ... ∵ R1'

Ⓡ2 let $y \notin Y$ otherwise its trivial. $Y \subseteq Y \cup \{y\} \overset{R2'}{\Longrightarrow} r(Y) \leq r(Y \cup \{y\})$

submodularity: $r(Y \cup \{y\}) + \overbrace{r(\emptyset)}^{0} \overset{R3'}{\leq} r(Y) + r(\{y\}) \leq r(Y) + 1$

Ⓡ3 let $y, z \notin Y$ s.t. $r(Y \cup \{y\}) = r(Y \cup \{z\}) = r(Y)$. We have $r(Y \cup \{y, z\}) \overset{R2'}{\geq} r(Y)$

$r(Y \cup \{y\}) + 0 \overset{R3'}{\leq} r(Y) + r(\{y\})$ & $r(Y \cup \{y\}) = r(Y) \Rightarrow r(\{y\}) = 0$ ... similarly $r(\{z\}) = 0$

$\Rightarrow r(Y \cup \{y, z\}) + \underbrace{r(\emptyset)}_{0} \leq r(Y) + \underbrace{r(\{y, z\})}_{0}$, but $r(\{y, z\}) + 0 \leq 0 + 0 \Rightarrow r(\{y, z\}) = 0$ ▨

Corollary: Matroids are exactly the hereditary set systems where rank is monotone and submodular

Recall: $f: 2^X \to \mathbb{R}$ is submodular $\equiv f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$

Notation: For $x \in X$ define $\Delta f_x : 2^X \to \mathbb{R}$, $T \mapsto f(T \cup \{x\}) - f(T)$

Theorem: $f: 2^X \to \mathbb{R}$ is submodular $\iff (\forall x \in X): \Delta f_x$ is nonincreasing.

Corollary: Submodular functions model utility well.

↳ $\Delta f_x(T)$ = increase in utility after adding x to T

↳ we want $S \subseteq T \Rightarrow \Delta f_x(S) \geq \Delta f_x(T)$

↳ more significant utility increase when added to a smaller set

Proof:

👁 Δf_x is nonincreasing $\equiv S \subseteq T \Rightarrow \Delta f_x(S) \geq \Delta f_x(T)$

$\iff (\forall S \subseteq X)(\forall z \in X \setminus S): \underset{f(S+x)-f(S)}{\Delta f_x(S)} \geq \underset{f(S+z+x)-f(S+z)}{\Delta f_x(S \cup \{z\})}$

👁 $(\forall x \in X): \Delta f_x$ is nonincreasing $\iff$

(∗) $(\forall S \subseteq X)(\forall x, z \in X \setminus S): f(S+x) + f(S+z) \geq f(S) + f(S+x+z)$

$\Rightarrow$: f submodular $\Rightarrow$ (∗) ... pick $A = S+x$, $B = S+z$

$\Leftarrow$: (∗) $\Rightarrow f$ submodular ... want: $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$

By induction on $|A \Delta B| = (A \cup B) \setminus (A \cap B)$

👁 $A \subseteq B \Rightarrow$ want $f(A) + f(B) \geq f(A) + f(B)$ ... trivially holds

① $|A \Delta B| \leq 2$: one of the following two cases occur

a) $A \subseteq B \lor B \subseteq A$ ... use 👁

b)  $|A \Delta B| = 2$ & $A = (A \cap B) + a$ & $B = (A \cap B) + b$

$\Rightarrow$ use (∗) with $S = A \cap B$, $x = a$, $z = b$

② $|A \Delta B| \geq 3$: WLOG assume $|A \setminus B| \overset{\boxtimes}{\geq} 2$ ... $\exists a \in A \setminus B$

claim: $f(A \cup B) - f(A) \underset{(?)}{\leq} f(A \cup B - a) - f(A - a) \underset{(??)}{\leq} f(B) - f(A \cap B)$

(?) $Y = A$; $Z = A \cup B - a \Rightarrow Y \cup Z = A \cup B$, $Y \cap Z = A - a$

$|Y \Delta Z| = |(A \cup B) \setminus (A - a)| = |a + B \setminus (A \cap B)| \overset{\boxtimes}{<} |(A \setminus (A \cap B)) \cup (B \setminus (A \cap B))| = |A \Delta B|$

$\Rightarrow$ we can use induction hypothesis for Y and Z

(??) $Y = B$, $Z = A - a \Rightarrow Y \cup Z = A \cup B - a$, $Y \cap Z = A \cap B$

$|Y \Delta Z| = |(A \cup B - a) \setminus (A \cap B)| = |A \Delta B - a| < |A \Delta B|$

$\Rightarrow$ we can use induction hypothesis for Y and Z ▨

Simple Matroids

Def: $M=(X,\Psi)$ is simple $\equiv \forall A\subseteq X: \quad |A|\leq 2 \Rightarrow r(A)=|A|$

$\hookrightarrow$ meaning $\quad |A|\leq 2 \Rightarrow A\in\Psi$.

Def: Let $M=(X,\Psi)$. $A\subseteq X$ is closed $\equiv \forall x\in X\setminus A: r(A)<r(A\cup\{x\})=r(A)+1$

👁 Each simple matroid $M=(X,\Psi)$ of rank 3 $(r(X)=3)$ is determined by

$$L(M):=\{A\subseteq X \mid r(A)=2,\ |A|>2,\ A \text{ closed}\}.$$

This set determines the rank function and therefore Ψ. Let $A\subseteq X$.

$\downarrow$

$\Psi=\{A\subseteq X \mid r(A)=|A|\}$

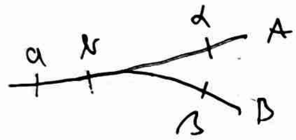
1) $|A|\leq 2 \Rightarrow r(A)=|A|$

(*) 2) $\exists B\in L(M): A\subseteq B \Rightarrow r(A)=2$

3) otherwise $r(A)=3$

Lemma: $L(M)$ is almost disjoint: $A,B\in L(M),\ A\neq B \Rightarrow |A\cap B|\leq 1$.

Pf: Suppose $|A\cap B|\geq 2$; $\alpha\in A, \alpha\notin B$ & $\beta\in B, \beta\notin A$



👁 α,β exist, else $A\subset B$ or $B\subset A$ ... ↯ with closed

👁 $\forall S\subseteq A, |S|\geq 2: r(S)=2$

$$2=r(A\cap B)=r(A\cap B+\alpha)=r(A\cap B+\beta)\ \underline{\overset{(R3)}{\ \ }}\ r(A\cap B+\alpha+\beta)$$

Now suppose that some additional $\gamma\in A\setminus\{a,b,\alpha\}$, $\gamma\notin B$.

By the same argument: $r(A\cap B+\gamma+\beta)=2$

$\Rightarrow$ using (R3) once more: $r(A\cap B+\alpha+\gamma+\beta)=2$

We can add all elements of $A\setminus B$ to this set and end up with

$$r((A\cap B)\cup(A\setminus B)+\beta)=r(A+\beta)=2$$

Recall $\beta\notin A$, and A is closed, therefore $r(A+\beta)=3$ ↯ ■

Def: $\mathcal{C}\subseteq 2^X$ is a configuration $\equiv$

① $A\in\mathcal{C} \Rightarrow |A|\geq 3$

② $A,B\in\mathcal{C} \Rightarrow |A\cap B|\leq 1$

Theorem: $\mathcal{C}\subseteq 2^X$ is a configuration $\Leftrightarrow \mathcal{C}=L(M)$ for some simple rank 3 matroid.

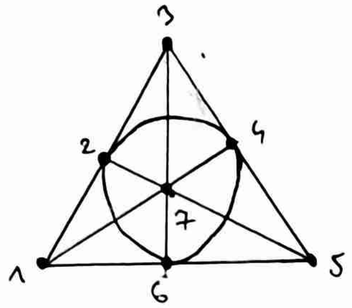
Pf: $\Leftarrow$: using previous lemma and by definition of $L(M)$

$\Rightarrow$: define rank

$$(*)\quad r_M(A):=\begin{cases}|A|, & |A|\leq 2\\ 2, & A\subseteq B\in\mathcal{C}\\ 3, & \text{otherwise}\end{cases} \quad \dots\ M=(X,\{A\subseteq X \mid r(A)=|A|\}),\ 👁: L(M)=\mathcal{C}$$

■

Ex: The Fano Matroid $F_7 = (X, \varphi)$



$X =$ points of the Fano projective plane

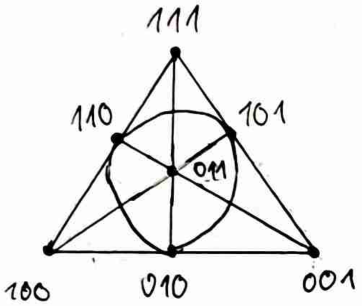
Bases of F_7 = lines

↳ $A \subseteq X \in \varphi \iff \exists$ basis B s.t. $A \subseteq B$

👁 F_7 is a simple matroid of rank 3 $\quad 246 = \bigcirc$

👁 Configurations $= \{1245,$ ~~1246~~$, 1267, 1346,$ ~~2346~~$, 2347, 2356\}$

👁 F_7 is isomorphic to a vectorial matroid of $\mathbb{F}_2^{3\times 7}$



$$V = \begin{pmatrix} 1&0&0&1&1&0&1 \\ 0&1&0&1&0&1&1 \\ 0&0&1&0&1&1&1 \end{pmatrix} \in \mathbb{F}_2^{3\times 7}$$

$M_V = (X, \varphi)$ ... $X =$ Columns of V

$A \subseteq X \in \varphi \iff A$ is lin. ind.

👁 bases of M_V = bases of Col-space of V

↳ every line = basis of V

Def: Let $M_1 = (X_1, \varphi_1)$, $M_2 = (X_2, \varphi_2)$ be matroids. The bijection $f: X_1 \to X_2$ is an isomorfism of M_1 and M_2 $\equiv$

$$\forall A \subseteq X_1 : \quad A \in \varphi_1 \iff f[A] \in \varphi_2$$

Note: Equivalent conditions (*)

- bases: $\forall B \subseteq X_1$: B is a basis of $M_1 \iff f[B]$ is a basis of M_2
- rank: $\forall A \subseteq X_1$: $\text{rank}_1(A) = \text{rank}_2(f[A])$

Def: The matroid M is representable over a field $\mathbb{F}$

$\equiv M$ is isomorphic to a vectorial matroid over $\mathbb{F}$

Def: The matroid M is binary $\equiv$ it is representable over $\mathbb{F}_2$

Def: The matroid M is graphic $\equiv \exists$ graph G s.t. $M \cong$ Cycle Matroid(G)

↳ isomorphic

↳ recall: CycleMatroid(G) = forests of G

Example: F_7 is binary.

(*) Also circuits - defined later

Operations with Matroids

① **Direct sum**: $M_1=(X_1,\mathcal{I}_1)$, $M_2=(X_2,\mathcal{I}_2)$ & $X_1\cap X_2=\emptyset$

$M_1+M_2 := (X_1\dot\cup X_2, \{Y_1\dot\cup Y_2 \mid Y_1\in\mathcal{I}_1, Y_2\in\mathcal{I}_2\})$, $\underline{r(A)=r_1(A)+r_2(A)}$

② **Uniform matroid**: $[n]:=\{1,2,\dots n\}$

$U(n,k):=\left([n], \binom{[n]}{\le k}\right)$ ... obviously matroid, bases $=\binom{[n]}{k}$

③ **Partition matroid**: $X=\{X_1,\dots X_n\}$ partition of E → $\forall X_i\neq\emptyset$, $X_i\cap X_j=\emptyset$, $\bigcup X=E$

$P(X):=(E,\{F\subseteq E\mid \forall i: |F\cap X_i|\le 1\})$

👁 $P(X_1,\dots X_n)=\sum_{i=1}^{n} U(|X_i|,1)$ ... direct sum of matroids is a matroid

④ **Restriction to T**: $M=(X,\mathcal{I})$, $T\subseteq X$

$M|T:=(T,\{A\in\mathcal{I}\mid A\subseteq T\})$, $\underline{r'(A)=r(A)}$ ... $A\subseteq T$

⑤ **Deletion of T**: $M=(X,\mathcal{I})$, $T\subseteq X$

$M-T:=(X-T,\{A-T\in\mathcal{I}\mid A\in\mathcal{I}\})$ → note: $A\in\mathcal{I}$ & $A-T\subseteq A\Rightarrow A-T\in\mathcal{I}$

👁 $M-T=M|(X-T)\Rightarrow \underline{r'(A)=r(A)}$ ... $A\subseteq X-T$

⑤ **Contraction of T**: $M=(X,\mathcal{I})$, $T\subseteq X$ ↱ 👁 $T\in\mathcal{I}\Rightarrow B=T$

→ let B be a basis of T ... (max ind. in T) ... $B\in\max_{\subseteq}\{J\subseteq T\mid J\in\mathcal{I}\}$

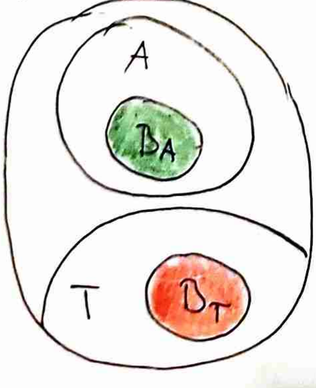
$M/T:=(X\setminus T,\{A\in\mathcal{I}\mid A\subseteq X-T\ \&\ A\dot\cup B\in\mathcal{I}\})$ → note: $A\dot\cup B\in\mathcal{I}$ & $A\subseteq A\dot\cup B$ $\Rightarrow A\in\mathcal{I}$

Theorem: M/T is a matroid with $r': X\setminus T\to\mathbb{N}_0$, $\underline{r'(A)=r(A\dot\cup T)-r(T)}$.

Corollary: Even though the definition of contraction has two parameters (T,B), the rank of M/T depends only on T

$\Rightarrow M/T$ is uniquely determined only based on T

Pf:



Let $A\in X\setminus T$ and denote

- B_T ... basis of T in M (used to construct M/T)
- B_A ... basis of A in M/T

👁 $B_A\cup B_T$ is a basis of $A\cup T$ in M
↳ can not extend B_* because then $*$ is not maximal

$r'(A)=|B_A|=|B_A\dot\cup B_T|-|B_T|=r(A\cup T)-r(T)$ ∎

Examples:

① $U(3,2) = \left([3], \binom{[3]}{\leq 2}\right)$ is binary = representable over $\mathbb{F}_2$

→ bases = subsets of size 2, 3 elements in total

$V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. $\{1,2\} \Leftrightarrow \left\{\binom{1}{0}, \binom{0}{1}\right\}$, $\{1,3\} \Leftrightarrow \left\{\binom{1}{0}, \binom{1}{1}\right\}$, $\{2,3\} \Leftrightarrow \left\{\binom{0}{1}, \binom{1}{1}\right\}$

② $U(4,2) = \left(\{1,2,3,4\}, \binom{[4]}{\leq 2}\right)$ is not binary

→ bases = subsets of size 2, 4 elements in total

$V = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}$ need: every pair of vectors is lin. ind.
every triplet of vectors is lin. dep.

(*) equivalently need: every pair of vectors is a basis of ColSpace(V)

⇒ $\{v_1, v_2\}$ basis and v_3, v_4 are generated from $\{v_1, v_2\}$

→ we are over $\mathbb{F}_2$ so the only possible lin. comb. of $\{v_1, v_2\}$ are

$\mathrm{Span}(\{v_1, v_2\}) = \{0, v_1, v_2, v_1 + v_2\}$

$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

⇒ 0 can not be in V ... (*) would be broken

⇒ $\{v_3, v_4\} \subseteq \{v_1, v_2, v_1 + v_2\}$ ... for example: $v_3 = v_1 + v_2$, $v_4 = v_2$

⇒ either v_1 or v_2 (or both) is repeated in V

⇒ the pair of these two identical vectors is not a basis ⇒ (*) ↯

Proposition: All graphic matroids are binary.

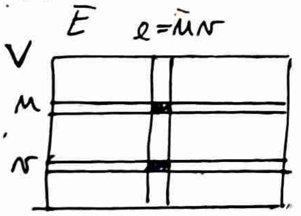
Pf: Let $G = (V, E)$ be a graph and consider its cycle matroid

$M_G = (E, \{E' \subseteq E \mid E' \text{ is acyclic}\})$ .... $|\text{Ground set}| = |E|$

want: $V \in \mathbb{F}_2^{\& \times |E|}$ s.t $M_V \cong M_G$ with isomorphism $f: 2^E \to$ Colums of V

↳ $E' \subseteq E$: E' is acyclic $\Leftrightarrow f[E']$ is lin. ind.
E' has a cycle $\Leftrightarrow f[E']$ is lin. dep.

The incidence matrix of G meets our criteria



Assume E' has a cycle $C \subseteq E$

↳ $C \in$ Cycle Space $(G) \cong \ker(I_G)$

⇒ char. vector of C, $\chi_C \in \ker(I_G)$

⇒ the columns corresponding to C are lin. dep. ▨

Duality of Matroids

Motivation: $G=(V,E) \rightsquigarrow M_G = (E, \{E' \subseteq E \mid E' \text{ is a forrest}\})$

↳ $M_G^* := (E, \{E' \subseteq E \mid G-E' \text{ has the same \# components as } G\})$ ↗ show $M_G^* =$ matroid

👁 $\text{rank}_M(E) =$ size of a spanning forrest of G

$E' \in M_G^* \iff$ removing edges E' doesnt create new components

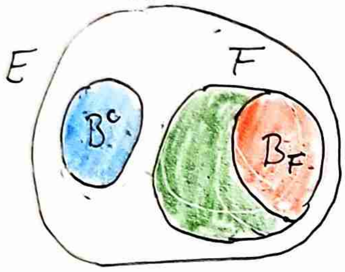
$\iff$ the size of a spanning tree of G doesnt change by removing E'

$\iff \text{rank}(E-E') = \text{rank}(E)$

Theorem: M_G^* is a matroid with rank $r^*(F) = |F| - r(E) + r(E-F)$

Pf: ① $\emptyset \in M_G^*$ ② $E' \in M_G^*$ & $F \subseteq E' \Rightarrow F \in M_G^*$

③' we will show that rank is well defined



→ let $F \subseteq E$ and let

- B_F be a basis of F in M^* ... largest removable part
- B^c be a basis of $E-F$ in M ... some forrest

→ now expand B^c with edges from $E-B_F$ to form a spanning forrest of G

↳ we can ∵ excluding B_F does not change the # of components

→ formally we made $B \subseteq E - B_F$ s.t. $B^c \subseteq B$ and B is a basis of M and we have used $B_F \in \varphi^* \Rightarrow (E-B_F) = r(E)$

👁 $B_F = F - B \iff$ all edges from $F - B_F$ have been used

↳ if $\exists e \in F-B-B_F$ ... unused edge which isnt forbidden by B_F then this e can be added to B_F and B is still a spanning forrest

$\Rightarrow B_F$ would not be maximal independent subset of F

👁 $B = \underline{B^c} \,\dot\cup\, \underline{(F-B_F)}$ ... else B^c wouldnt be max. ind. in $E-F$

👁 $|B_F| = |F| - |F \cap B| = |F| - \underline{|F-B_F|} = |F| - (|B|-|B^c|) = |F| - r(E) + r(E-F)$ ■

Def: The dual of a matroid $M=(X,\varphi)$ is the matroid

$M^* = (X, \{A \subseteq X \mid r(X-A) = r(X)\})$ with $\underline{r^*(A) = |A| - r(X) + r(X-A)}$

👁 B is a basis of $M \iff X-B$ is a basis of M^* ← complementary bases

Corollary: $\underline{(M^*)^* = M}$, $\underline{\text{rank}(M) + \text{rank}(M^*) = |X|}$

Deletion and Contraction are dual operations

Theorem: $M=(X,\mathcal{I})$, $T\subseteq X$. Then $\underline{(M-T)^* = M^*/T}$.

Pf: Denote rank of $M, M^*, (M-T)^*, M^*/T$ as r, r^*, r_D, r_C respectively. Let $A\subseteq X-T$. We will show $r_D(A)=r_C(A)$

- **LHS:** $M-T \dots \text{rank}_{M-T}(A) = r(A)$

  $\Rightarrow r_D(A) = r^*_{M-T}(A) = |A| - r(X-T) + r(X-T-A)$

- **RHS:** $M^* \dots r^*(A) = |A| - r(X) + r(X-A)$

  recall: M with rank r $\Rightarrow$ M/T has rank $r'(A) = r(A\dot\cup T) - r(T)$

  $\Rightarrow r_C(A) = r'_{M^*}(A) = r^*(A\dot\cup T) - r^*(T)$

  $= |A\dot\cup T| - \underset{\sim}{r(X)} + r(X-(A\dot\cup T)) - |T| + \underset{\sim}{r(X)} - r(X-T)$

  $= |A| + r(X-A-T) - r(X-T)$ ■

Theorem: $M=(X,\mathcal{I})$, $T\subseteq X$. Then $\underline{(M/T)^* = M^* - T}$.

Pf: Let $A\subseteq X-T$. We will show $r_C(T) = r_D(T)$

- **RHS:** $M^* \dots r^*(A) = |A| - r(X) + r(X-A) = r_D(A)$
- **LHS:** $M/T \dots r'(A) = r(A\dot\cup T) - r(T)$

  $\Rightarrow r_C(A) = |A| - r'(X-T) + r'(X-T-A)$

  $= |A| - r(X) + r(T) + r(X-A) - r(T) = |A| - r(X) + r(X-A)$ ■

Duality of Graphic Matroids

Def: The circuits of $M=(X,\mathcal{I})$ are its minimum dependent subsets. That is $C\subseteq X$ s.t. $C\notin\mathcal{I}$ & $A\subset C \Rightarrow A\in\mathcal{I}$.

👁 Circuits of a graphic matroid are cycles in the corresponding graph.

Terminology: When talking about a matroid property and prepend "co" $\Rightarrow$ dual
- ↳ cocircuits, cobases = circuits, bases of the dual matroid
- ↳ M is cographic = M^* is graphic

👁 Cocircuits of a graphic matroid are minimum edge cuts in the corresp. graph.
- ↳ F is a cocircuit of M_G^* $\Leftrightarrow$ $F\notin\mathcal{I}^*$ & $A\subset F \Rightarrow A\in\mathcal{I}^*$
- ↳ $F\notin\mathcal{I}^*$ $\Leftrightarrow$ deleting F reduces $r(M)$ $\Leftrightarrow$ creates new components $\Leftrightarrow$ is edge cut
- ↳ $A\subset F \Rightarrow A\in\mathcal{I}^*$ $\Leftrightarrow$ A is not edge cut $\Leftrightarrow$ F is minimal

Theorem: $M=(X,\mathcal{I})$ is completely characterized by any of the following.

① rank function r : $A\in\mathcal{I} \iff r(A)=|A|$

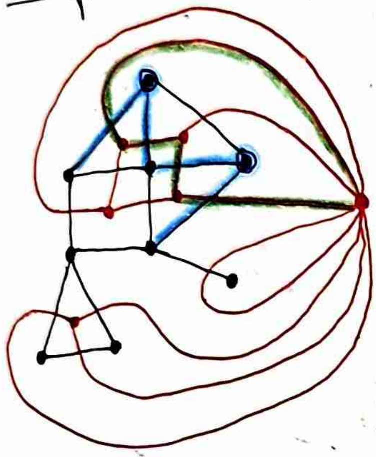
② dependent sets $\emptyset$: $A\in\mathcal{I} \iff A\notin\emptyset$.

③ bases $\mathcal{B}$: $A\in\mathcal{I} \iff \exists B\in\mathcal{B}: A\subseteq B$ ... subset of a basis

④ circuits $\mathcal{C}$: $A\in\mathcal{I} \iff \nexists C\in\mathcal{C}: C\subseteq A$ ... doesn't contain a circuit

Theorem: Let $G=(V,E)$ be a planar graph. Then $M^*(G)\cong M(G^*)$ G^* = geom. dual

Proof: We will show that $M^*(G)$ and $M(G^*)$ have the same circuits.



- Circuits of $M^*(G)$ = min. edge cuts of G
- Circuits of $M(G^*)$ = cycles of G^*

👁 there is a very clear bijection between the edges of G and the edges of G^* ... call it f

know (*): f isomorphism of CutSpace(G) and CycleSpace(G^*)

want: f isomorphism of $M^*(G)$ and $M(G^*)$

$A\subseteq E(G)$ edge cut in G $\overset{(*)}{\iff}$ $f[A]\subseteq E(G^*)$ even subgraph of G^*.

→ isomorphisms preserve relative properties → $\nexists$ even $F\subset f[A]$

$\Rightarrow$ A min. edge cut in G $\iff$ $f[A]$ min. even subgraph of G^*

$\iff$ $f[A]$ cycle in G^* ■

Planar Graphs Recap

Def: The graph H is a minor of the graph G, $H\leq G$

$\equiv$ H can be obtained from G using a sequence of operations

① vertex deletion ② edge deletion ③ edge contraction

👁 G planar & $H\leq G \Rightarrow H$ planar

👁 K_5 and $K_{3,3}$ are not planar

Theorem (Kuratowski–Wagner): G planar $\iff K_5\nleq G$ & $K_{3,3}\nleq G$

→ we will define a matroid minor and show that there is a strong connection between the minors of graphic matroids and graph minors

Matroid Minors

Def: The matroid N is a minor of the matroid M, $N \leq M$

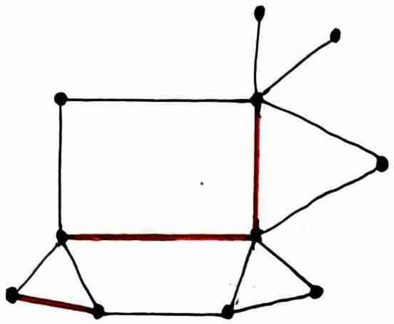
$\equiv$ N can be obtained from M using a sequence of operations

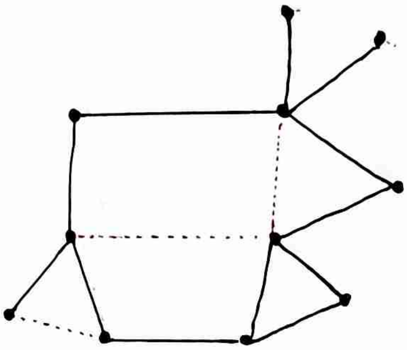
① deletion of T ①' restriction to T ② contraction of T

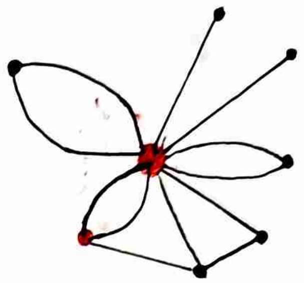
Note: Recall that $M - T = M|(X-T)$.

(multigraph)

If M is graphic, then ① ~ deletion of edges and ② ~ contraction of edges







Theorem: $G=(V,E)$, $F \subseteq E$. Then $M(G-F) \cong M(G)-F$ and $M(G.F) = M(G)/F$.

$\hookrightarrow M(G) = (E, \{E' \subseteq E \mid E' \text{ acyclic in } G\})$

① deletion of $F \subseteq E$

$M(G-F) = (E-F, \{E' \subseteq E-F \mid E' \text{ is acyclic}\})$

$M(G)-F \doteq M(G)|(E-F) = (E-F, \underbrace{\{E' \subseteq E-F \mid E' \in \mathcal{I}(M_G)\}}^{E' \text{ acyclic}})$

② contraction of $F \subseteq E$

$M(G.F) = (E-F, \{E' \subseteq E-F \mid E' \text{ is acyclic in } G.F\})$

→ let K be a spanning forest of F

$M(G)/F = (E-F, \{E' \subseteq E-F \mid E' \text{ acyclic in } G \ \& \ E' \dot\cup K \text{ acyclic in } G\})$

→ need to show $M(G.F) \cong M(G)/F$

- $E' \subseteq E-F$ acyclic in $G.F$ → want $E' \dot\cup K$ acyclic in G
  - ↳ replacing a contracted vertex by its spanning tree doesnt introduce cycles
  - → if there was a cycle in $E' \dot\cup K$ in G then there must have been a cycle in E' in $G.F$ because trees are acyclic
- $E' \subseteq E-K$ acyclic in G & $E' \dot\cup K$ acyclic in G → want E' acyclic in $G.F$
  - → if there was a cycle in E' in $G.F$, then there must have been a cycle in $E' \dot\cup K$ in G (we replace contracted vertices by their spaning trees) because trees are connected. ■

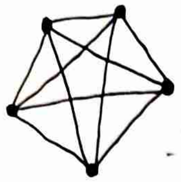
← other direction tricky because of the multiedges

Corollary: $H \leq G \Rightarrow M(H) \leq M(G)$.

① $G-e \Rightarrow M(G)-e$ ② $G-v \Rightarrow M(G) - \{e \in E \mid v \in e\}$ ③ $G.e \Rightarrow M(G)/e$ & reduce multiedges to simple edges

Lemma: $M(K_5)$ and $M(K_{3,3})$ are **not cographic**. → $M^*(K_5)$ and $M^*(K_{3,3})$ are not graphic

Pf: Suppose $M(K_5)$ is cographic i.e. $\exists H$ s.t. $M^*(K_5) \cong M(H)$ i.e. $M^*(H) \cong M(K_5)$



recall: cocircuits of $M(H)$ = circuits of $M^*(H) \cong M(K_5)$ = cycles of K_5

= min. edge cuts of H (*)

- The ground sets of $M(K_5)$ and $M^*(K_5) \cong M(H)$ are edges of K_5.

$\Rightarrow$ 10 elements in total, rank of $M(K_5) = 4$ (size of spanning tree)

$\Rightarrow$ rank of $M(H) = 10 - 4 = 6$

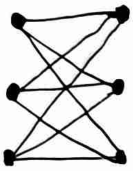
👁 rank of $M(H)$ = size of spanning forest of $H = |V(H)| - k$ ← # components

$\Rightarrow |V(H)| = \text{rank} + k = 6 + k \geq 7$

$\Rightarrow$ average degree of $H = \frac{2 \cdot |E(H)|}{|V(H)|} \leq \frac{2 \cdot 10}{7} < 3$

$\Rightarrow \exists v \in V(H)$ with $\deg(v) \leq 2 \Rightarrow$ min edge cut of H with size ≤ 2

(*) $\Rightarrow \exists$ cycle in K_5 with size ≤ 2 which is obviously nonsense ↯



Using the same approach for $K_{3,3}$ we have

rank$(M(K_{3,3})) = 5 \Rightarrow$ rank$(M(H)) = |E(K_{3,3})| - 5 = 9 - 5 = 4$

$\Rightarrow |V(H)| = \text{rank} + k = 4 + k \geq 5$

① $k = 1$: $|V(H)| = 5$ & $|E(H)| = 9 \Rightarrow H \cong K_5$ without one edge

circuits of $M(H)$ = cycles in H = min edge cuts in $K_{3,3}$

👁 $K_{3,3}$ has 6 edge cuts of size 3 ↯

👁 H has 7 cycles of size 3 ... $\binom{5}{3} = 10$ in K_5 -3

② $k \geq 2$: $|V(H)| \geq 6$ & $|E(H)| = 9 \Rightarrow$ avg degree $\leq \frac{2 \cdot 9}{6} = 3$

$\Rightarrow \exists v \in H$ with degree $\leq 3 \Rightarrow \exists$ min edge cut of H of size ≤ 3

$\Rightarrow \exists$ cycle in $K_{3,3}$ of size ≤ 3 ↯ ... smallest cycle has size 4 ■

👁 M graphic & $N \leq M \Rightarrow N$ graphic

Lemma: $N \leq M \iff N^* \leq M^*$.

Pf: We will use the duality of the deletion and contraction operations. We will show "$\Rightarrow$", the other direction is similar. From the sequence of operations $M \to N$, we will construct a series of operations $M^* \to N$.

$$M = M_1 \geq M_2 \geq \dots \geq M_n = N \rightsquigarrow M^* = M'_1 \geq M'_2 \geq \dots \geq M'_n = N^*$$

recall: $(M - T)^* = M^*/T$ & $(M/T)^* = M^* - T$

induction: $M'_1 = M^*$, now assume $M_k = M_{k-1} - T$. (contraction similar)

$\Rightarrow M'_k := M'_{k-1}/T = M^*_{k-1}/T = (M_{k-1} - T)^* = M^*_k$ ■

Whitney's Planarity Criterion

Theorem (Whitney): G planar $\Longleftrightarrow$ $M(G)$ cographic. $\equiv M^*(G)$ graphic

Pf: $\Longrightarrow$: G planar $\Rightarrow \exists$ geometric dual G^* and $M(G^*) \cong M^*(G)$

↳ $M(G^*)$ is graphic (by definition) $\Rightarrow M^*(G)$ is graphic

$\Longleftarrow$: Suppose G is nonplanar. Then by the K-W theorem: $K_5 \leq G$ or $K_{3,3} \leq G$.

recall: $H \leq G \Rightarrow M(H) \leq M(G) \ldots \Rightarrow M(K_5) \leq M(G)$ or $M(K_{3,3}) \leq M(G)$

duality: $M^*(K_5) \leq M^*(G)$ or $M^*(K_{3,3}) \leq M^*(G)$

recall: $M^*(K_5)$ and $M^*(K_{3,3})$ are not graphic

$\Rightarrow M^*(G)$ is graphic and minors of graphic matroids are also graphic ↯ ■

Lemma: Let G, H be graphs. If H is simple (no loops or multiedges), then

$$H \leq G \Longleftrightarrow M(H) \leq M(G)$$

Pf: We know "$\Rightarrow$". Now lets assume $M(H) \leq M(G)$ and H is simple.

$$M(G) = M(G_1) \geq M(G_2) \geq \cdots \geq M(G_m) = M(H)$$

We want to create a sequence of operations to turn G into H.

Deletion is straightforward: $M(G_i) - T \rightsquigarrow$ delete edges of T from G_i

Matroid contraction is problematic because it creates multiedges.

$(*)$ { Each contraction is followed by a deletion which turns all newly created multiedges into simple edges

$(**)$ { The graph corresponding to the final matroid $M(H)$ has no multiedges

☼ $(*) \Rightarrow H \leq G$ ... simply contract the corresponding edges

claim: $M(H) \leq M(G)$ & $(**)$ $\Rightarrow \exists$ construction of $M(H)$ from $M(G)$ satisfying $(*)$

↳ suppose a contraction $M(G)/T$ created an multiedge

→ H doesnt have the multiedge, so it must have been deleted / reduced

1) one of the edges in the multiedge contracted later

☼ we can contract it now - edges outside this multiedge cont care

2) the multiedge was deleted / reduced to a simple edge

☼ we can do that now ▨

Forbidden Minor Characterization

Theorem (Tutte): M is graphic $\iff$ $M^*(K_5) \not\leq M$ & $M^*(K_{3,3}) \not\leq M$.

Pf: M graphic $\iff$ M^* cographic $\iff$ $M^* = M(G)$ for some planar G

G planar $\iff$ $K_5 \not\leq G$ & $K_{3,3} \not\leq G$ ← Kuratowski - Wagner

$\iff$ $M(K_5) \not\leq M(G) = M^*$ & $M(K_{3,3}) \not\leq M(G) = M^*$ ← previous lemma

$\iff$ $M^*(K_5) \not\leq M$ & $M^*(K_{3,3}) \not\leq M$ ← taking duals ■

Def: M is binary $\equiv$ it is representable over $\mathbb{F}_2$. ← $M \cong$ Vectorial Matroid $(V \in \mathbb{F}_2^{m \times n})$

Recall: All graphic matroids are binary. $U(4,2)$ is not binary

Def: M is regular $\equiv$ it is representable over every field.

Fact: M regular $\iff$ M^* regular. Minors of a regular matroid are regular.

Theorem (Tutte): M is regular $\iff$ it is representable with an totally unimodular matrix.

Corollary: M graphic $\Rightarrow$ M regular.

Pf: $M(G)$ can be represented by the directed incidence matrix of G.

Recall: Incidence matrices for directed graphs are allways totally unimodular. ■

Theorem (Tutte): We can characterize certain minor-closed matroid classes using a finite set of forbiden minors.

① binary $\iff$ not contains $U(4,2)$ ← Fano matroid

② regular $\iff$ not contains $U(4,2)$, F_7, F_7^*

③ graphic $\iff$ not contains $M^*(K_5)$, $M^*(K_{3,3})$

Note: This is not as strong as the graph minor theorem.
↳ matroid minors are not a well-quasi ordering.