RAMSEY THEORY

Def: For cardinals κ, λ, μ and $k \in \omega$ the expression

$$\kappa \to (\lambda)^k_\mu$$

means that $\forall X: |X| = \kappa$ and $\forall$ μ-coloring of k-element subsets of X $f: [X]^k \to \mu$, there $\exists A \subseteq X$ s.t. $|A| = \lambda$ & $f \restriction [A]^k$ is constant.

k-element subsets of X

$\Rightarrow$ there is always a large subset on which the coloring is monochromatic

Examples:

- $6 \to (3)^2_2$ ... 6 element set, coloring pairs using 2 colors $\Rightarrow$ find Δ
- $(\forall n)(\forall r)(\exists N): N \to (n)^1_r$ ... pigeonhole for r colors ... take $N \geq (n-1)r+1$
- Finite Ramsey Thm for graphs: $(\forall n)(\forall r)(\exists N): N \to (n)^2_r$
  hypergraphs: $(\forall n)(\forall r)(\forall k)(\exists N): N \to (n)^k_r$
- Infinite RT: $(\forall r)(\forall k): \omega \to (\omega)^k_r$

? Why in the definition $k \in \omega$ and not $k \in Cn$?

Proposition: $\forall$ cardinal κ: $\kappa \not\to (\omega)^\omega_2$.

Intuition: Ramsey-style theorems only make sense for coloring finite tuples

Pf: We will construct a bad coloring of $[\kappa]^\omega$ = countable subsets of κ

- let $\sqsubseteq$ be a well-ordering of $[\kappa]^\omega$ given by AC
- define coloring $\chi: [\kappa]^\omega \to \{\text{red}, \text{blue}\}$ as

  $\chi(A) := \text{red} \equiv (\exists B \subseteq A): B \sqsubseteq A$ ... A not minimal

  $\chi(A) := \text{blue} \equiv$ otherwise ... A minimal

- now assume that some $X \in [\kappa]^\omega$ is monochromatic (for contradiction)

  1) if $\chi(X) = R$, we want $A \in [X]^\omega$ s.t. $\chi(A) = B$

     $\to$ take $A := \min\{Y \in [X]^\omega \mid Y \subseteq X\}$ ... A is blue ↯

  2) if $\chi(X) = B$, we want $A \in [X]^\omega$ s.t. $\chi(A) = R$

     $\to$ let $A' = \min\{Y \in [X]^\omega \mid X \setminus Y \text{ is infinite}\}$

     $\to$ choose $a \in X \setminus A'$ and take $A := A' \cup \{a\}$ ... A is red $\blacksquare$

Pf #2: Define an equivalence relation on $[\kappa]^\omega$ as

$$X \sim Y \;\equiv\; X \text{ and } Y \text{ differ only in finitely many elements}$$

- using AC, choose from each equivalence class $[X]_\sim$ a representative $\overline{X}$
- now define coloring $\chi : [\kappa]^\omega \to \{\text{red}, \text{blue}\}$ as

$$\chi(A) = \text{red} \quad \equiv \quad |X \setminus \overline{X}| \text{ is odd}$$
$$\chi(A) = \text{blue} \quad \equiv \quad |X \setminus \overline{X}| \text{ is even}$$

- assume that some $X \in [\kappa]^\omega$ is monochromatic
  - → choose $x \in X$ and let $Y = X \setminus \{x\}$, then $X \sim Y$ and $\overline{X} = \overline{Y}$, so $|X \setminus \overline{X}|$ and $|Y \setminus \overline{Y}|$ have different parities ⇒ X, Y different colors ↯ ▨

? What about coloring all finite subsets?

Def: $\kappa \to (\lambda)^{<\omega}_\mu$ means that for ∀ μ-coloring of finite subsets of κ $f : [\kappa]^{<\omega} \to \mu$, there $\exists A \subseteq \kappa$ s.t. $|A| = \lambda$ & $\forall n < \omega : f \restriction [A]^n$ is constant, although the value of f on $[A]^n$ may differ based on n.

→ λ = infinite cardinal

Def: A cardinal κ is λ-Erdős $\equiv \kappa \to (\lambda)^{<\omega}_2$
A cardinal κ is Ramsey $\equiv \kappa \to (\kappa)^{<\omega}_2$

Fact: these are large cardinals
↳ their existence cannot be proved in ZFC

? $\omega \not\to (\omega)^\omega_2$ says that there ∃ bad colorings, but which colorings are good?.

Def: A set $X \subseteq [\omega]^\omega$ (countable subsets of ω) is Ramsey ≡

$$\exists A \in [\omega]^\omega \text{ either } [A]^\omega \subseteq X \text{ or } [A]^\omega \subseteq \overline{X}$$

In other words: ∃ infinite $A \subseteq \omega$ such that

- $A \in X \Rightarrow \forall$ infinite $B \subseteq A : B \in X$
- $A \notin X \Rightarrow \forall$ infinite $B \subseteq A : B \notin X$

set $X := \{Y \in [\omega]^\omega \mid \chi(Y) = \text{red}\}$ is not Ramsey

$X \subseteq [\omega]^\omega$ is Ramsey ⟺ the coloring $\chi : [\omega]^\omega \to \{R, B\}$, $A \mapsto \begin{cases} R \text{ if } A \in X \\ B \text{ if } A \notin X \end{cases}$ is good (∃ ∞ monochromatic subset)

Theorem (Galvin–Prikry, 1972): All subsets $X \subseteq [\omega]^\omega$ which can be constructed without AC are Ramsey.

Corollary: It is impossible to prove $\omega \not\to (\omega)^\omega_2$ in ZF.

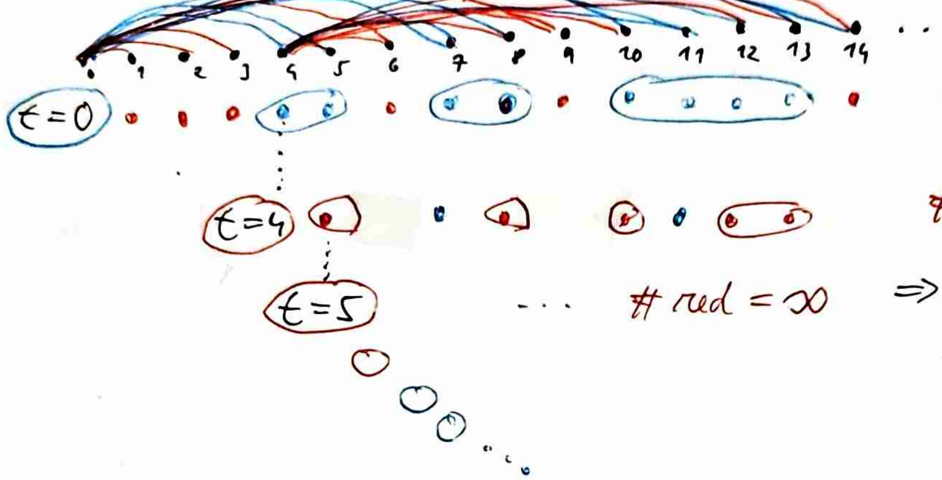
INFINITE RAMSEY THEOREM - 1930

Theorem: $(\forall k)(\forall r)$: $\omega \to (\omega)^k_r$

Proof: Induction on k (dimension)   We are given a coloring $\chi : [\omega]^k \to r$

- $k=1$: is pigeonhole principle
- $k=2$, $r=2$: we will prune the vertex set to make the coloring behave nicely

  ⇒ for $t \in \omega$ we define a coloring of $\omega \setminus \{t\}$ as $\chi_t(n) := \chi(\{t,n\})$



blue = ∞ ⇒ remove red,

red = ∞ ⇒ remove blue

... # red = ∞ ⇒ remove blue

- at each step we are left with ∞ many elements
- after repeating this for ∞ many values of t, we have a sequence of blue and red ts ⇒ pick those whose color repeats ∞ many times, this is a monochromatic subset

- $k-1 \to k$ ... we formalize the picture above
  - let χ be a r-coloring of k-element subsets of ω
  - denote $t_0 := 0$ and consider all subsets of ω containing $t_0 = 0$

  → for $t \in \omega$ define a coloring of $[\omega \setminus \{t\}]^{k-1}$ as $\chi_t(A) := \chi(A \cup \{t\})$
  - use induction hypothesis for $k-1$ and coloring χ_{t_0} to get an infinite $A_0 \subseteq \omega$ where χ_{t_0} is homogeneous

  ⇒ all k-element subsets of $A_0 \cup \{t_0\}$ containing t_0 have the same color w.r.t. χ
  - now define $t_1 := \min A_0$ and find infinite $A_1 \subseteq A_0$ where χ_{t_1} is homogeneous

  ⇒ all k-element subsets of $A_1 \cup \{t_1\}$ containing t_1 have the same color w.r.t χ
  - repeat this to construct the set $\{t_n \mid n < \omega\}$ where the color of a k-element subset is determined by its smallest element

⇒ by pigeonhole, there ∃ a subsequence $t_{i_1}, t_{i_2}, \dots$ where all t_{i_j} correspond to the same color (as there are only finitely many colors) ■

⇒ the set $\{t_{i_j} \mid j < \omega\}$ is homogeneous w.r.t. χ.

FINITE RAMSEY THEOREM - 1930

Theorem: $(\forall m)(\forall k)(\forall r)(\exists N): N \to (m)^k_r$

Proof: The same idea as for IRT but we dont have ∞ many numbers, so for each step t_i, A_i we need to invoke the induction hypothesis for a sufficiently large $n_i = |A_i|$ so that A_i is large enough and we can keep going – the final set has to have size ≥ 1 to pick the t_i.

→ to use pigeonhole at the very end we need to do ⊛ $(m-1)r+1$ steps to get enough $t_1, t_2, \dots$ so that there is a homogenous subset of size m ∎

¿ How large are Ramsey numbers?

Def: $R_{k,r}(m) := \min\{N \mid N \to (m)^k_r\}$, $R_k(m) := R_{k,2}(m)$ ... makes sense for $m \geq k$

→ 2 colors

👁 $R_1(m) = 2m-1 \leq 2m$, $R_{1,r}(m) = (m-1)r+1 \leq r \cdot m$ ... pigeonhole

Notation: $f(m): \omega \to \omega$, then $f^k(m) := \overbrace{f(f(\cdots f(m)\cdots))}^{k}$ repeated k times.

Ex: $f(m) = m+1 \Rightarrow f^4(m) = m+4$

Proposition: $R_k(m) \leq R^{2m-1}_{k-1}(k)$, $R_{k,r}(m) \leq R^{r \cdot m-1}_{k-1,r}(k)$.

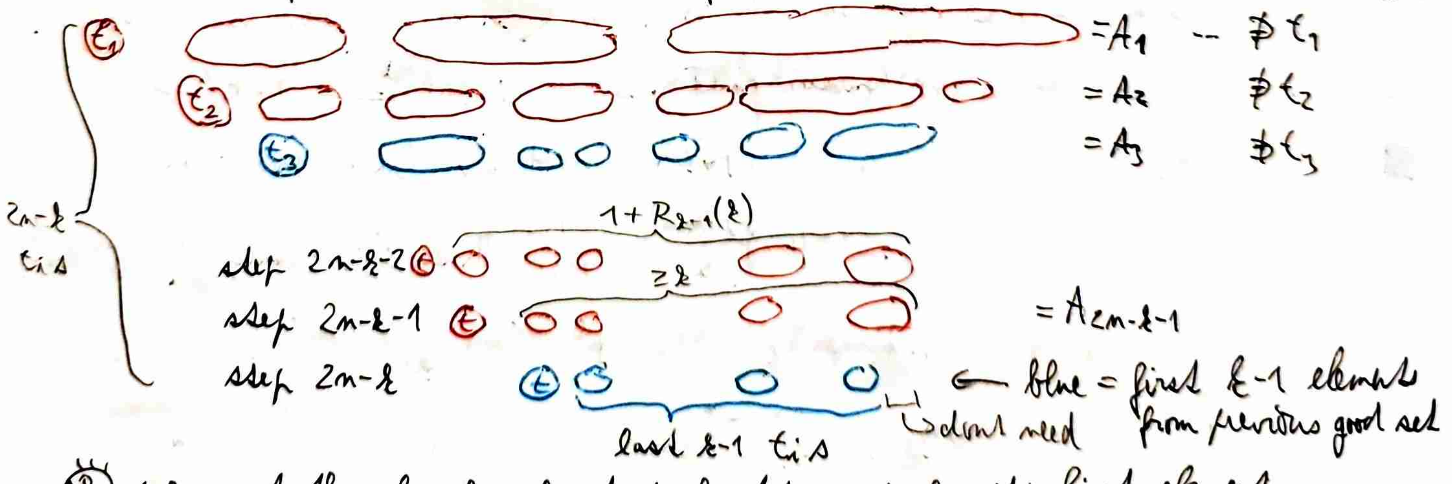
Corollary: Upper bounds for graph FRT ... $k=2$

- **2 colors:** $R_2(m) \leq R_1^{2m-1}(2) \leq \overbrace{2 \cdot 2 \cdot 2 \cdots 2}^{2m-1} \cdot \boxed{2} = 2^{2m} = \underline{4^m}$
- **r colors:** $R_{2,r}(m) \leq R^{rm-1}_{1,r}(2) \leq \overbrace{r \cdot r \cdots r}^{rm-1} \cdot \boxed{2} \leq r^{rm}$

} Ramsey numbers for graphs are exponentially bounded

Proof: We will examin more closely what happens in the proof of FRT, specifically ⊛

We shall focus on $r=2$, similar for more colors $\Rightarrow r=2$ needs $2m-1$ t_i's



👁 we want the color of a k-set to be determined by its first element

$\Rightarrow$ we can choose the last $k-1$ t_i's arbitrarily, they will never be first

$\Rightarrow$ at step $2m-k$ we take the first k elements from A_{2m-k-1}

$\Rightarrow A_{2m-k-1}$ has to have size $\geq k$ $\Rightarrow |A_{2m-k-2}| \geq 1 + R_{k-1}(k)$

$\Rightarrow |A_{2m-k-3}| \geq 1 + R_{k-1}(1 + R_{k-1}(k))$ ... $|A_{i-1}| \geq 1 + R_{k-1}(|A_i|)$

→ if we continue from $|A_{i-1}| \geq 1+R_{k-1}(|A_i|)$ and $|A_{2n-k-1}| \geq k$
we get that $|A_1| \geq 1+R_{k-1}(1+R_{k-1}(1+R_{k-1}(\cdots 1+R_{k-1}(k)\cdots)))$
where R_{k-1} is repeated $2n-k-2$ times

⇒ the original set has to have size $\geq N = 1+R_{k-1}(|A_1|)$, so R_{k-1} repeated $2n-k-1$ times

→ the +1 s are annoying, but they can be easily disregarded

<u>Lemma</u>: Let $f:\omega\to\omega$ satisfy $\forall k$: $f(k)\geq 2k-1$, $f(k+1)-f(k)\geq 2$. Let $g(k)=1+f(k)$.
Then $\forall k\geq 2$ we have $g^n(k)\leq f^{n+1}(k)$.

<u>Pf</u>: By induction we show that $g^n(k)\leq f^{n+1}(k)-1$, skip here ■

→ since $R_1(x)=2x-1$ and for $k>1$, $R_k(x)\geq R_1(x)$ we have $\forall k\geq 2$:

$R_k(n)\leq N = R_{k-1}^{2n-k}(k)$, the proposition only claims $R_k(n)\leq R_{k-1}^{2n-1}(k)$ ■

↳ such N is enough for $N\to(n)^k_2$

<u>Def</u>: The <u>fast growing hierarchy</u> functions $f_\alpha:\omega\to\omega$ are defined for $\alpha\in On$ as

- $f_0(n)=n+1$ $\qquad \alpha[n]$ = fundamental sequence of α
- $f_{\alpha+1}(n)=f_\alpha^n(n) = \underbrace{f_\alpha(f_\alpha(\cdots f_\alpha(n)\cdots))}^{n \text{ times}}$
- $f_\alpha(n)=f_{\alpha[n]}(n)$ for limit α, where $\sup\{\alpha[0],\alpha[1],\ldots\alpha[n],\ldots\}=\alpha$

<u>Ex</u>: $f_1(n)=2n$, $f_2(n)=2\cdot 2\cdots 2n = 2^n\cdot n$

<u>Proposition</u>: <u>$R_k(n)\leq f_k(4n)$</u>, <u>$R_{k,r}(n)\leq f_k(2r\cdot n) < f_{k+1}(n)$</u> $\quad\forall n$ large enough

<u>Pf</u>: $R_1(n)=2n-1\leq 2n = f_1(n)$

$R_2(n)\leq R_1^{2n}(k)\leq f_1^{2n}(k)\leq f_1^{2n}(2n)=f_2(2n) \qquad < f_2^2(n)$

$R_3(n)\leq R_2^{2n}(k)\leq f_2^{4n}(k)\leq f_2^{4n}(4n)=f_3(4n) \qquad < f_3^2(n)$

$R_4(n)\leq R_3^{2n}(k)\leq f_3^{4n}(k)\leq f_3^{4n}(4n)=f_4(4n) \qquad < f_4^2(n)$

→ for r colors:
$\qquad$ n large enough ↓

$R_1(n)\leq r\cdot n = 2^{\log_2 r}\cdot n = f_1^{\log_2 r}(n)$

$R_2(n)\leq R_1^{rn}(k)\leq f_1^{rn\cdot\log r}(k)\leq f_1^{rn\log r}(rn\log r) = f_2(n\cdot r\log r) \qquad < f_2^2(n)$

$R_3(n)\leq R_2^{rn}(k)\leq f_2^{2r\cdot n}(k)\leq f_2^{2rn}(2rn) = f_3(2rn) \qquad < f_3^2(n)$ ■

LOWER BOUND ON RAMSEY NUMBERS

→ famous Erdős probabilistic argument

very quickly goes to 0

Theorem: $R_{2,r}(m) > \frac{1}{e\sqrt{r}} \cdot m\sqrt{r}^{m} \cdot (1-o(1))$, .. $R_{k,r}(m) > \frac{1}{e} \cdot m \sqrt[k!]{r}^{(m-1)^{k-1}} \cdot (1-o(1))$.

Proof: We are coloring $[N]^k$ randomly using r colors

$\Rightarrow$ for $\forall S \subseteq N$ s.t. $|S| = m$ define random variable $X_S := \begin{cases} 1, & S \text{ is monochr.} \\ 0, & \text{otherwise} \end{cases}$

Define $X_m := (\#\text{ monochr. subsets } S \subseteq N \text{ with } |S| = m) = \sum_{|S|=m} X_S$

👁 $\mathbb{E}[X_S] = P[S \text{ colored monochr}] = \frac{\#\text{ monochr. colorings}}{\#\text{ all colorings}} = \frac{r}{r^{\binom{m}{k}}} = r^{1-\binom{m}{k}}$

$\Longrightarrow \mathbb{E}[X_m] = \sum_{|S|=m} \mathbb{E}[X_S] = \binom{N}{m} \cdot r^{1-\binom{m}{k}}$

If we pick N small enough so that $\mathbb{E}[X_m] < 1$, then there must exist a coloring of N which has 0 monochr. subsets of size $m \Rightarrow R_{k,r} > N$

Fact: $\binom{N}{m} \leq \left(\frac{e \cdot N}{m}\right)^m$, we pick N s.t. $\left(\frac{eN}{m}\right)^m r^{1-\binom{m}{k}} < 1$

$\Longleftrightarrow N^m < \left(\frac{m}{e}\right)^m \cdot r^{\binom{m}{k}-1} \quad \Longleftrightarrow N < \frac{m}{e} \cdot r^{\frac{1}{m}\binom{m}{k} - \frac{1}{m}}$ $\quad \Big| \binom{m}{k} = \frac{m^k}{k!}$

$N < \frac{m}{e} \cdot r^{(m-1)^{k-1}/k!} \cdot \boxed{\frac{1}{\sqrt[m]{r}}} \longrightarrow \sqrt[m]{r} \to 1$, so $\frac{1}{\sqrt[m]{r}} = 1 + o(1)$

$N < \frac{m}{e} \cdot \sqrt[k!]{r}^{(m-1)^{k-1}}$

For $k=2$ we have $N < \frac{m}{e} \cdot \sqrt{r}^{m-1} = \frac{m}{e} \cdot \frac{1}{\sqrt{r}} \cdot \sqrt{r}^m$ ■

Theorem: If $m \geq \frac{r^2}{2\pi}$, then we can change $(1-o(1))$ to $(1+o(1))$.

Proof: Instead of $\binom{N}{m} \leq \left(\frac{eN}{m}\right)^m$ we use $\binom{N}{m} \leq \frac{N^m}{m!}$, so $\frac{N^m}{m!} r^{1-\binom{m}{2}} < 1$

$N < (m!)^{\frac{1}{m}} \cdot r^{\frac{1}{m}\binom{m}{2}} \cdot r^{-\frac{1}{m}}$, use $m! > \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$

$\Rightarrow (m!)^{\frac{1}{m}} > \sqrt{2\pi m}^{\frac{1}{m}} \cdot \frac{m}{e}$ ... we will make less demands on N

$$N < \frac{m}{e} \cdot r^{\frac{1}{m} \cdot \binom{m}{2}} \cdot \underbrace{\sqrt{2\pi m}^{\frac{1}{m}} \cdot r^{-\frac{1}{m}}}_{\left(\frac{\sqrt{2\pi m}}{r}\right)^{\frac{1}{m}} = \sqrt{\frac{2\pi m}{r^2}}^{\frac{1}{m}} = 1 + o(1) \text{ if } m \geq \frac{r^2}{2\pi}}$$

■

COMPACTNESS PRINCIPLE

finite edges

Def: $H=(V,E)$ is a hypergraph $\equiv$ V is a set and $E \subseteq [V]^{<\omega}$.

Def: $f: V \to r$ is an r-coloration of H $\equiv$ no $X \in E$ is monochromatic

Def: The chromatic number $\chi(H)$ is the min. r s.t. H can be r-colored.

☼ Finite Ramsey theory results can be expressed using hypergraphs

$N \to (m)^k_r \iff H=(V,E) \to V=[N]^k$

$\hookrightarrow E = \{[A]^k \mid |A| = m\}$ has $\chi(H) > r$

⇓

for r colors ∃ monochromatic edge

Def: For a hypergraph $H=(V,E)$ and $W \subseteq V$ define

$H_W = (W, E_W)$ as $E_W = \{X \in E \mid X \subseteq W\} = E \cap [W]^{<\omega}$.

Theorem (Compactness principle): Let $H=(V,E)$ be a hypergraph: V arbitrary, $E \subseteq [V]^{<\omega}$.

$$(\forall \text{finite } W \subseteq V): \chi(H_W) \le r \implies \chi(H) \le r.$$

Intuition: There are other formulations of the compactness principle, this one is the most useful for Ramsey theory. It will allow us to prove finite Ramsey theorems from infinite ones, using the following contraposition:

Corollary: $\chi(H) > r \implies (\exists \text{finite } W \subseteq V): \chi(H_W) > r$

Example: IRT ⇒ FRT

infinite ① ⇒ finite ②

→ we want to show that $\exists N: N \to (m)^k_r$

$V = [\omega]^k$ ← what we are coloring

$E = \{[A]^k \mid A \in [\omega]^m\}$ ← what we want to find

① $\chi(H) > r$: $(\forall r\text{-coloring } f: [\omega]^k \to r)(\exists X = [A]^k \in E)$: X is f-monochromatic

IRT ⇒ ①: IRT in fact gives us infinite A s.t. $X = [A]^k$ is monochromatic

⇒ take any n-element subset of this A

② $(\exists \text{finite } W \subseteq V): \chi(H_W) > r$: ∃ finite $W \subseteq [\omega]^k$ s.t. ∀ r-coloring $f: W \to r$

$\exists X = [A]^k \in E_W$ s.t. X is f-monochromatic

② ⇒ FRT: W contains finitely many k-sets of ω ⇒ collect the elements into

$A = \bigcup W$, now $A \subseteq \omega$ is finite.

⇒ for FRT take $N = \max A$

Proof of compactness principle: $H=(V,E)$

① V is countable → Königs lemma, requires AC_ω

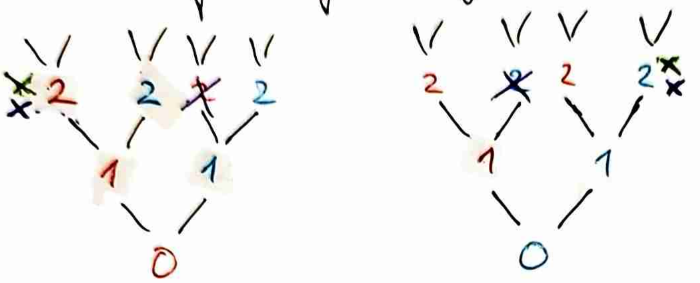
② V arbitrary → Zorns lemma, requires AC

We will show only ①. Assume $V=\omega$ and that for $\forall n\in\omega$ $\exists$ coloring

$\chi_n : n \to r$ s.t. none $A\in E$, $A\subseteq n$ is monochromatic

↳ good coloring

consider the forest of colorings: example $r=2$



→ 8 {R, B}-colorings of {0, 1, 2}

- we want only good colorings

→ suppose $\{0,2\}\in E$ and $\{0,1,2\}\in E$

if $\chi_n : n\to r$ is good then $\forall m<n$: $\chi_n\restriction m$ is good

- # colors is finite ⇒ finite braching factor
- each level has at least 1 good coloring (χ_n) ⇒ infinite forest

} Königs lemma: $\exists\,\infty$ branch

→ the ∞ branch corresponds to a coloring $\chi^*: \omega\to r$

χ^* is good, otherwise one of its finite restrictions would not be good ∎

Intuition: We might sometimes need different versions of the compactness principle, but they can be proven in a similar fashion – just use König's lemma.

PROVABILITY OF RAMSEY THEOREMS

≈ theory of finite sets

- Infinite RT: cannot even be formulated in Peano Arithmetic
- Finite RT: if we used IRT $\xrightarrow{\text{comp.}}$ FRT then we would not know if FRT can be proved in PA, but we proved FRT directly, without no infinity, so ✓
- Large - FRT: natural extension of FRT, unprovable in PA

Def: $A\subseteq\omega$ is large $\equiv |A|\geq \min(A)$

↳ if the sets starts late (min A is large), then it has to be big

Def: $N\xrightarrow{*}(m)^k_r$ means $(\forall f: [N]^k\to r)(\exists A\subseteq N)$: $|A|\geq m$, A large, and f is homogeneous on A

↳ like $N\to(m)^k_r$, but the monochromatic subset has to be large.

Theorem (Paris – Harrington, 1977):

① $(\forall m)(\forall k)(\forall r)(\exists N)$: $N \xrightarrow{*} (m)^k_r$

② PA cannot prove this theorem

→ or draw a tree of colorings when assuming it does not hold

Proof: ① By compactness principle

$V = [\omega]^k$, $E = \{[A]^k \mid A \in [\omega]^{<\omega}, |A| \geq m, A \text{ large}\} \rightsquigarrow H = (V, E)$

- IRT $\Rightarrow \chi(H) > r$ ...($\forall$ r-coloring of V): $\exists$ monochromatic $[A]^k \in E$

  ↳ IRT gives us infinite A^* s.t. $[A^*]^k$ is monochromatic

  $\Rightarrow$ take the first $\max(m, \min A^*)$ elements of A^* to form A

  ↳ $|A| \geq m$, A is large as $|A| \geq \min A$ and $[A]^k$ is monochromatic

- Compactness: $\exists$ finite $W \subseteq V$ s.t. $\chi(H_W) > r$ ∴ ① ■

② Fact: If $F: \omega \to \omega$ grows like f_{ε_0} or faster, then PA cannot prove its totality ... that it gives an output for $\forall$ input

Ketonen and Solovay gave in 1981 a purely combinatorial proof that the numbers N grow roughly like f_{ε_0}, hence PA cannot find them all ■

Idea: We used IRT to prove ①, but ② claims that it cannot be proved without using some kind of infinite argument (at least as strong as $TI(\varepsilon_0)$)

CONSEQUENCES OF RAMSEY THEOREMS

Def: $\alpha(G)$:= size of largest independent set in G
$\omega(G)$:= size of largest clique in G

Theorem: If $\mathcal{C}$ is an infinite class of graphs and $\exists k$ s.t.

a) $\forall G \in \mathcal{C}: \alpha(G) \leq k \Rightarrow \sup\{\omega(G) \mid G \in \mathcal{C}\} = \infty$

b) $\forall G \in \mathcal{C}: \omega(G) \leq k \Rightarrow \sup\{\alpha(G) \mid G \in \mathcal{C}\} = \infty$

} bounded α / ω ⇓ unbounded ω / α

Proof: edges = red, non-edges = blue

→ the sizes (# vertices) of graphs in $\mathcal{C}$ are unbounded (otherwise $\mathcal{C}$ cannot be ∞)

→ suppose α is bounded, then

FRT: $(\forall m)(\exists N)$: $|G| \geq N \Rightarrow \exists$ monochromatic $A \subseteq V(G)$ of size $\geq m$

↳ α (blue) is small $\Rightarrow$ for large m we have large blue (ω) ■

Theorem: Every infinite ordered set X has an ∞-chain or ∞-antichain.

Pf: Define coloring of $[X]^2$ as $\chi(a,b)$ = red if a, b comparable, blue otherwise. Use IRT ■

WHERE RAMSEY BRAKES DOWN

? what instead of coloring $[n]^k$ we colored $[n]^{\le k} = \{A \subseteq n \mid |A| \le k\}$?

Theorem: $(\forall n)(\forall r)(\forall k)(\exists N)$: $N \longrightarrow (n)^{\le k}_r$

Recall: This means that ($\forall r$-coloring $f: [N]^{\le k} \to r$)($\exists A \subseteq N$): $|A| = n$ & $\forall l \le k$: $f \restriction [A]^l$ is constant
↳ monochromatic in each dimension, but colors may differ between dimensions

Proof: Define a sequence $N_1, N_2, \dots N_k$ and take $N := N_k$

$N_1 \to (n)^1_r$

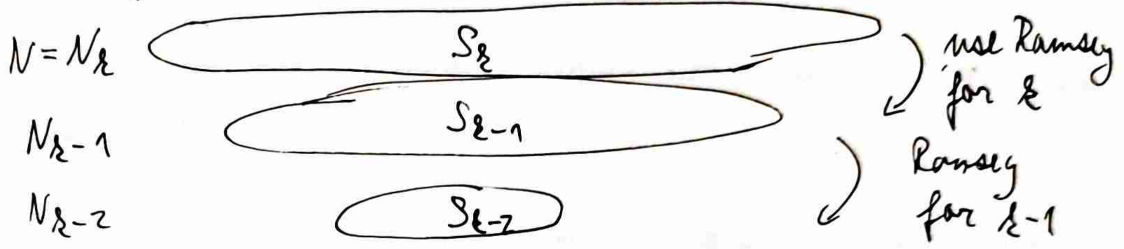
$N_2 \to (N_1)^2_r$

$N_3 \to (N_2)^3_r$

$\vdots$

$N_k \to (N_{k-1})^k_r$

We are given $\chi: [N]^{\le k} \to r$



We take $S = S_k$ of size N with coloring χ.

- FRT for k: find $S_{k-1} \subseteq S_k$, $|S_{k-1}| = N_{k-1}$ and $[S_{k-1}]^k$ is monochromatic
- FRT for $k1$: find $S_{k-2} \subseteq S_{k-1}$, $|S_{k-2}| = N_{k-2}$ and $[S_{k-2}]^{k-1}$ is monochromatic

At the end: $S_k, S_{k-1}, \dots S_0$ where $|S_0| = n$ & $\forall l \le k$: $[S_0]^l$ is monochromatic
↳ since $S_0 \subseteq S_{l-1}$, so $[S_0]^l \subseteq [S_{k-1}]^l$, which is monochromatic ■

Theorem: $(\forall r)(\forall k)$: $\omega \longrightarrow (\omega)^{\le k}_r$.

Proof: Essentially the same, but we use $\forall l \le k$: $\omega \to (\omega)^l_r$ k-times, to always find an infinite subset where all l-sets are monochromatic ■

Theorem: $\omega \not\to (\omega)^{<\omega}_2$. coloring all finite subsets fails

Proof: We define a bad coloring $\chi: [\omega]^{<\omega} \to \{red, blue\}$. Let $A \in [\omega]^k$, then

$$\chi(A) := \begin{cases} red, & k \in A \\ blue, & k \notin A \end{cases}$$

Now let $S \subseteq \omega$ be an infinite subset.

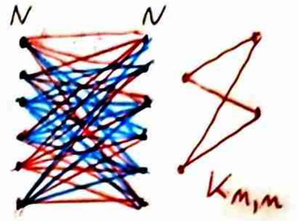
Take $k = \min(S)$. Then $[S]^k$ contains both $\left\langle \begin{array}{l} k\text{-sets with } k \to red \\ k\text{-sets without } k \to blue \end{array} \right.$ ↯ ■

Fact: ZFC cannot prove the existence of a cardinal κ satisfying $\kappa \longrightarrow (\omega)^{<\omega}_2$. Such a large cardinal is said to be ω-Erdős.

Recall (AC): $\forall$ cardinal κ: $\kappa \not\to (\omega)^{\omega}_2$. ← start of this notebook

PRODUCT RAMSEY THEOREM

Def: $N_1, N_2 \longrightarrow (m_1, m_2)_r^{k_1, k_2}$ means that $\forall r$-coloring $f : [N_1]^{k_1} \times [N_2]^{k_2} \longrightarrow r$ of tuples of k_1-sets of N_1 and k_2-sets of N_2, there exists $A_1 \subseteq N_1$ and $A_2 \subseteq N_2$ such that $|A_1| = m_1$, $|A_2| = m_2$ and $[A_1]^{k_1} \times [A_2]^{k_2}$ is monochromatic



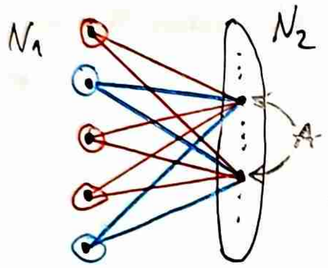
Example: $N, N \longrightarrow (m, m)_2^{1,1}$ means that if we 2-color the edges of $K_{N,N}$, then we will find monochromatic $K_{m,m}$

↳ this is essentially 2-D pigeonhole ... standard: $N \longrightarrow (m)_2^1$

Theorem: $(\forall r)(\forall m_1, m_2)(\forall k_1, k_2)(\exists N_1, N_2)$: $N_1, N_2 \longrightarrow (m_1, m_2)_r^{k_1, k_2}$.

Proof: First focus on $N_1, N_2 \longrightarrow (m_1, m_2)_r^{1,1}$ ... bipartite graphs, r colors

It would be nice of the coloring had the following property:



Want: $\exists A \subseteq N_2$ s.t. $\forall v \in N_1$: all edges from v to A are monochromatic ⊛
$|A| = m_2$

That is, the signature of colors  is the same $\forall u \in A$

Now if $N_1 \longrightarrow (m_1)_r^1$ then $\exists m_1$ elements from N_1 associated with the same color $\Rightarrow$ take them and A ✓

How to do this?

→ important trick! we let $N_2 \longrightarrow (m_2)_{\boxed{r^{N_1}} = \#\text{colors}}^1$ and $N_1 \longrightarrow (m_1)_r^1$

For a given $\chi : N_1 \times N_2 \rightarrow r$ define coloring χ_2 of N_2 using r^{N_1} labels as follows. Treat the N_1-tuples $(c_1, c_2, \dots c_{N_1}) \in r^{N_1}$ (vector of colors) as labels and notice that there is exactly r^{N_1} of them. For $v \in N_2$ define

$$\chi_2(v) := \left(\chi(0, v), \chi(1, v), \dots \chi(N_1, v)\right) \in r^{N_1}$$

Since $N_2 \longrightarrow (m_2)_{r^{N_1}}^1$, we find a monochromatic $A \subseteq N_2$ of size m_2

→ the elements of A have the same color signature ⊛, and we are done

- To show $N_1, N_2 \longrightarrow (m_1, m_2)_r^{k_1, k_2}$ use

$$N_1 \longrightarrow (m_1)_r^{k_1} \quad \text{and} \quad N_2 \longrightarrow (m_2)_{r^T}^{k_2} \quad \text{where } T = \binom{N_1}{k_1}$$

The same argument works, just now on the right are not points, but k_2-sets and their signature is a list of $\binom{N_1}{k_1}$ colors, as the elements being colored on the left-hand side are not points but k_1 sets of N_1

Def: $N_1, N_2, \dots, N_d \longrightarrow (m_1, m_2, \dots, m_d)_r^{k_1, k_2, \dots, k_d}$ means that

∀ r-coloring $f: [N_1]^{k_1} \times [N_2]^{k_2} \times \dots \times [N_d]^{k_d} \to r$ of d-tuples of k_i-sets of N_i, there exist subsets $A_1 \subseteq N_1$, $A_2 \subseteq N_2, \dots A_d \subseteq N_d$ s.t. $|A_i| = m_i$, ∀i and $[A_1]^{k_1} \times [A_2]^{k_2} \times \dots \times [A_d]^{k_d}$ is monochromatic

Theorem: $(\forall r)(\forall d)(\forall m_1, \dots, m_d)(\forall k_1, \dots, k_d)(\exists N_1, \dots N_d): N_1, \dots, N_d \to (m_1, \dots, m_d)_r^{k_1, \dots, k_d}$

Example: if $\forall k_i = 1$, then this says that ∀ r-coloring of a complete d-partite graph with partites $N_1, \dots, N_d$ contains a monochromatic $K_{m_1, \dots, m_d}$.

Proof: By induction on d. Very similar to the proof of d=2, just take

$N_1 \longrightarrow (m_1)_r^{k_1}$ and $N_2, \dots, N_d \to (m_2, \dots, m_d)_{r^T}^{k_2, \dots, k_d}$, where $T = \binom{N_1}{k_1}$

This allows us to pick a subset $A \subseteq [N_2]^{k_2} \times \dots \times [N_d]^{k_d}$ with the proper m_i sizes such that the color-signature of each $X \in A$ is the same. ■

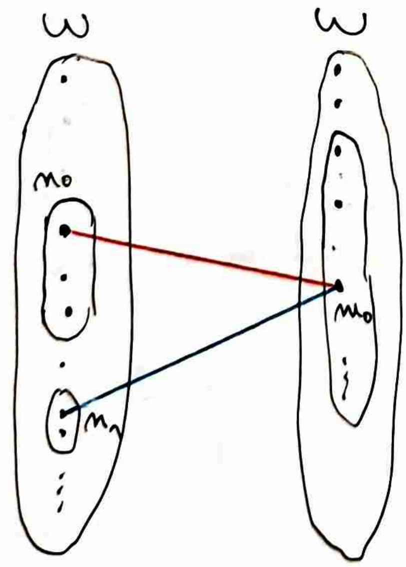
? Infinite 1-D pigeonhole $\omega \longrightarrow (\omega)_2^1$

What about 2-D pigeonhole $(\omega, \omega) \longrightarrow (\omega, \omega)_2^{1,1}$?

Theorem: $\omega, \omega \not\longrightarrow (\omega, \omega)_2^{1,1}$

Proof: We define a bad coloring $\chi: \omega \times \omega \to \{red, blue\}$ as

$$\chi(n, m) := \begin{cases} \text{red} & \text{if } n < m \\ \text{blue} & \text{if } n \geq m \end{cases}$$

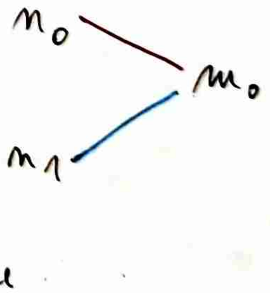


Given infinite $N \subseteq \omega$, $M \subseteq \omega$, find

$n_0 := \min N$

$m_0 := \min \{m \in M \mid m > n_0\}$

$n_1 := \min \{n \in N \mid n \geq m_0\}$



All these numbers must exist since the infinite sets N, M are unbounded ■

TREES AND STRONG SUBTREES

→ we will be dealing only with countable trees.

Def: A (model-theoretic) forest is a partial order $(T, <_T)$ where $|T| \leq \omega$ and

- $\forall x \in T$ is the set $\{y \in T \mid y <_T x\}$ finite and linearly ordered

↳ predecessors of $X = (\leftarrow, x)$

Every forest T has a minimal element ... pick any $x \in T$ and take $\min(\leftarrow, x)$

Def: The roots of a forest T are its minimal elements.

Def: If a forest has a single root $r = \min(T)$, then it is called a tree.

Def: The branching factor of a node $x \in T$ is # of its immediate successors.
The forest T is finitely branching ≡ $\forall x \in T$ has finite branching factor.

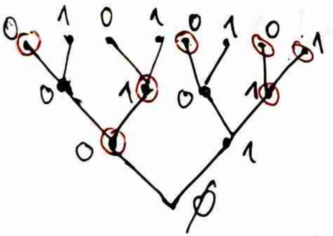
Def: A subtree of a tree $(T, <_T)$ is a subset $S \subseteq T$ with inherited order $<_S$.
subforest forest

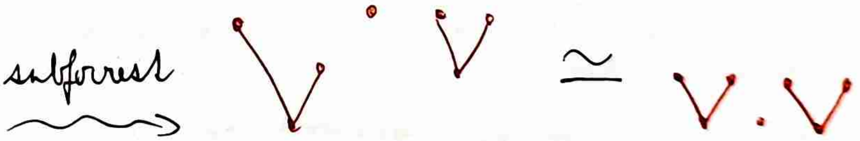
Examples:

- motivation from graph trees:

$a < b$ ≡ a lies on the unique path from r to b

- Cantor's tree: $T = 2^{<\omega}$ = all finite words in the alphabet $\{0,1\}$
  $a \sqsubseteq b$ ≡ a is a prefix (initial segment) of b

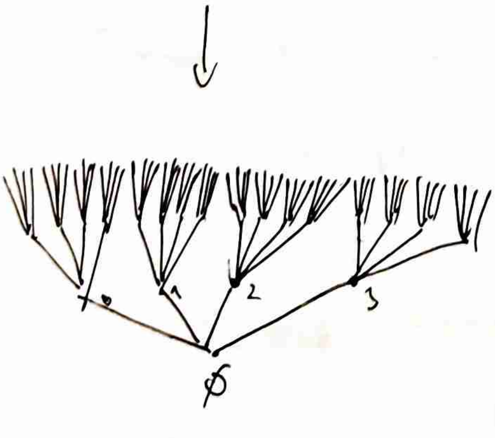


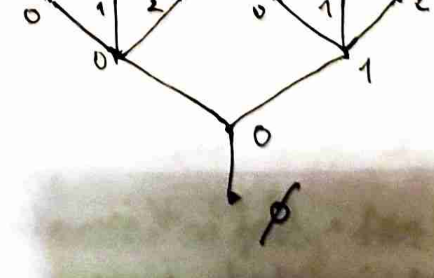


- more wildly branching trees:

$T = 4^{<\omega}$

characters in alphabet increases on each level

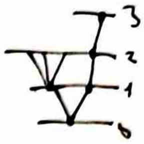




$A = \{0,1,2,3\}$
$A = \{0,1,2\}$
$A = \{0,1\}$
$A = \{0\}$
$A = \emptyset$

Tree vocabulary: Let $(T, <_T)$ be a tree

- $x \in T$ is a leaf if it has no immediate succesor
- the level of x in T is $|x|_T := |\{y \in T \mid y <_T x\}$ ... $|root|_T = 0$
- the level $n \in \omega$ of T is $T(n) := \{x \in T \mid |x|_T = n\}$
- the height of T is the least $\alpha \leq \omega$ s.t. $\forall x \in T: \alpha > |x|_T$



height = 4

- a branch of T is a $\subseteq$-maximal chain $A \subseteq T$

👁 infinite branch ⇒ infinite height

König's lemma: ∞ height & finitely branching ⇒ ∞ branch

! from now on we only consider finitely branching trees = FB tree

👁 For $\forall$ FB tree T of height ω, and $\forall$ 2-coloring of nodes of T there $\exists$FB subtree $S \subseteq T$ of heigt ω, with all nodes monochromatic.

↳ suppose we color T with red and blue, and assume the root is red

⇒ try to make a red subtree starting from the root, that is taking r and adding everything red above it ... that is all red nodes.

→ if # red nodes $= \infty$, then this is a red subtree of height ω.

→ if # red nodes is finite, then $\exists$ blue vertex x such that nothing above x is red (find an infinite branch and go until there are no red vertices). Then the entire subtree starting at x is blue ■

? Can we strenghten this so that the subtree retains some of the structure of the original tree?

Def: A tree S of height α is a strong subtree of a tree T if

① S is a subtree of T, namely S has a root $r(S) \in T$

② S is level preserving, that is $\exists$ function $f: \alpha \to \omega$ called a level function s.t.

$$(\forall n < \alpha): x \in S(n) \Rightarrow x \in T(f(n)) \quad \ldots \quad S(n) \subseteq T(f(n))$$

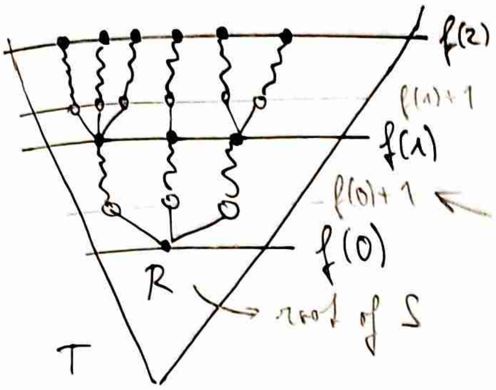
③ $\forall x \in S$ which is not a leaf in S satisfies

$$BF_S(x) = \ell \iff BF_T(x)$$ where $BF_S(x)$ = branching factor of x in S

👁 ③ ⇒ $(\forall x \in S)$ not a leaf, $\forall$ immediate successor y of x in T $\exists$ unique immediate successor y' of x in S s.t. $x <_T y \leq_T y'$

Def: Given a tree T and $1 \le \alpha \le \omega$, denote by $STR_\alpha(T)$ the set of all strong subtrees of T of height α.

Example: S strong subtree of T



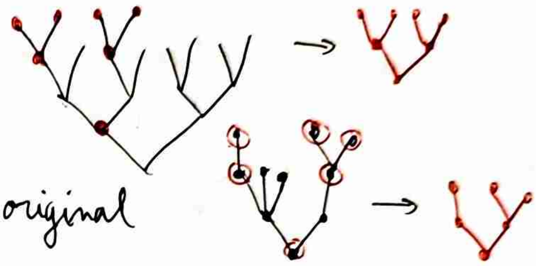
← strong subtree of height $\alpha=3$

levels are preserved

← level in T but not in S

$\alpha \le \beta$, $S_\beta \in STR_\beta(T)$, $S_\alpha \in STR_\alpha(S_\beta) \Rightarrow S_\alpha \in STR_\alpha(T)$

a strong subtree of a binary tree is a binary tree, but in general, a strong subtree is no isomorphic to the original



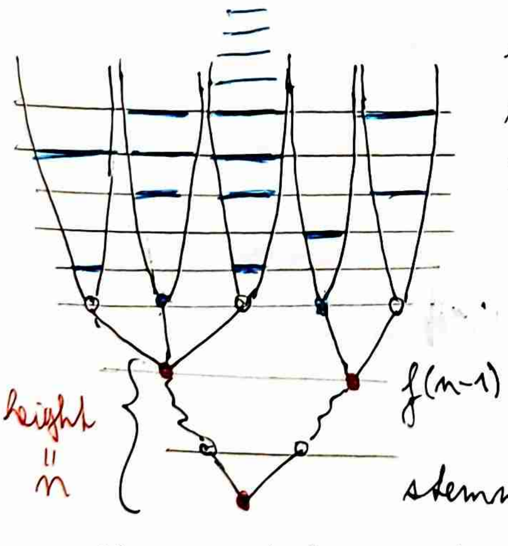
T is a forrest $\Leftrightarrow$ it is a union of disjoint trees

$\Rightarrow$ we will use the same terminology as for trees when talking about forrests

Proposition: Let T be a FB tree of height ω with no leaves, and let $r \in \omega$.
Then for $\forall$ r-coloring of nodes of T there
$\exists$ monochromatic strong subtree $S \subseteq T$ of height ω.

Proof: Consider $r=2$, red and blue; general idea is the same.

Assume that the root is red and attempt to build a red strong subtree S starting from the root. If this fails and the resulting subtree has height $n < \omega$. Then the final level of S is $n-1$, which corresponds to the level $f(n-1)$ in T. The strong subtree cannot be extended



$\Rightarrow$ if $x_1, x_2, \dots x_k$ are the immediate successors of $S(n-1)$ and $T_1, \dots T_k$ are the subtrees of T stemming from $x_1, \dots x_k$, then for $\forall$ level $N \ge 0$ in $T_1, \dots T_k$ there must be a subtree T_i such that everything in $T_i(N)$ is blue

→ as such a tree exists for $\forall$ level, there are ∞ many of these blue segments

→ there are finitely many T_is $\Rightarrow \exists j: T_j$ contains ∞ many blue levels

$\Rightarrow$ we can create ∞ blue strong subtree in T_j ∎

HALPERN-LAÜCHLI and MILLIKEN TREE THEOREMS

Def: A product tree of dimension d is a sequence of d trees $\pi = (T_1, \dots T_d)$.

- the product of a product tree π is

$$\prod_\pi = \prod_i T_i := \{(x_1, \dots, x_d) \mid x_1 \in T_1, \dots, x_d \in T_d\} = T_1 \times \dots \times T_d$$

- the level product is

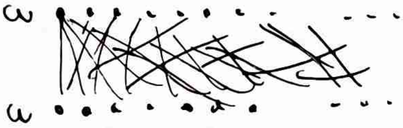
$$\prod_\pi^{lev} = \prod_i^{lev} T_i := \left\{(x_1, \dots, x_d) \;\middle|\; \begin{array}{l} x_1 \in T_1, \dots, x_d \in T_d \\ |x_1|_{T_1} = |x_2|_{T_2} = \dots = |x_d|_{T_d} \end{array}\right\}$$

that is, all x_i have to be on the same level in their respective trees

Intuition: Forest ~ unordered collection of trees, product tree ~ ordered forest

Example:

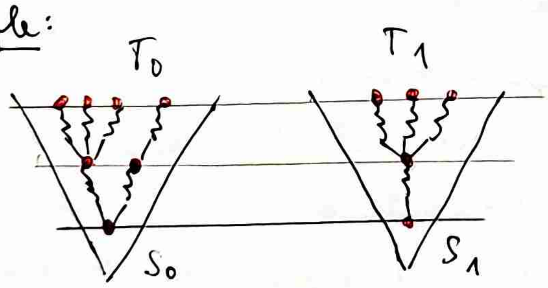
$\prod_{(\omega,\omega)}$



$\prod^{lev}_{(\omega,\omega)}$

Def: Given a product tree $\Pi = (T_1, \dots, T_d)$, a strong product subtree of height $\alpha \leq \omega$ is a product tree $S = (S_1, \dots, S_d)$ s.t. $\forall i$: $S_i \in STR_\alpha(T_i)$, and all these strong subtrees S_i share the same level function $f: \alpha \to \omega$.

Example:



same height & share levels

Def: For a product tree $\Pi = (T_1, \dots, T_d)$ and $\alpha \leq \omega$ we denote by $STR_\alpha(\Pi)$ the set of all strong product subtrees of Π of height α.

Theorem (Halpern–Laüchli, 1966): For ∀d, ∀r, ∀ product tree $\Pi = (T_1, \dots, T_d)$ of FB trees of height ω with no leaves, and ∀ r-coloring of Π^{lev}_{Π}, the level product of Π, there ∃ homogeneous $\mathcal{S} = (S_1, \dots, S_d) \in STR_\omega(\Pi)$.

↳ $\Pi^{\text{lev}}_{\mathcal{S}}$ monochromatic

👁 If $\Pi = (\omega)$ then this is pigeonhole principle

Theorem (Milliken's tree theorem, 1979): For ∀n, ∀r, ∀ FB tree T of height ω with no leaves, and ∀ r-coloring of $STR_n(T)$, the strong subtrees of T of height n, there ∃ homogeneous $S \in STR_\omega(T)$. → $S_n(S)$ monochromatic.

👁 Milliken ⟹ Infinite Ramsey theorem

↳ take the tree $T = 2^{<\omega}$ of finite $\{0,1\}$ strings and prefix order

→ given a coloring $f : [\omega]^k \to r$, define a coloring $g : STR_n(T) \to r$ as:

- for $S \in STR_n(T)$ define $\vec{X}_S := \{\#\text{bits in } x \mid x \in S\}$ … $|\vec{X}_S| = n$
- set $g(S) := f(\vec{X}_S)$ ↳ bits of levels of S … $|0011| = |1001| = 4$

→ let $\mathcal{S} \in STR_\omega(T)$ be given by Milliken's theorem for this g.
then $A = \{\#\text{bits in } x \mid x \in \mathcal{S}\}$ is homogeneous for f ■

Theorem (Product version of Milliken): For ∀n, ∀d, ∀r, ∀ product tree $\Pi = (T_1, \dots, T_d)$ of FB trees of height ω with no leaves, and ∀ r-coloring of $STR_n(\Pi)$, the strong product subtrees of Π of height n, there ∃ homogeneous $\mathcal{S} = (S_1, \dots, S_d) \in STR_\omega(\Pi)$. → $STR_n(\mathcal{S})$ is monochromatic

👁 Halpern–Laüchli is product Milliken for $n = 1$

PROOF OF HALPERN-LAÜCHLI

Def: Given a tree T, we say that $y \in T$ extends $x \in T$ if $x \leq_T y$. When we talk about nodes "above" x, we mean the set of its extensions. Similarly, when we talk about the nodes above a set of nodes $X \subseteq T$.

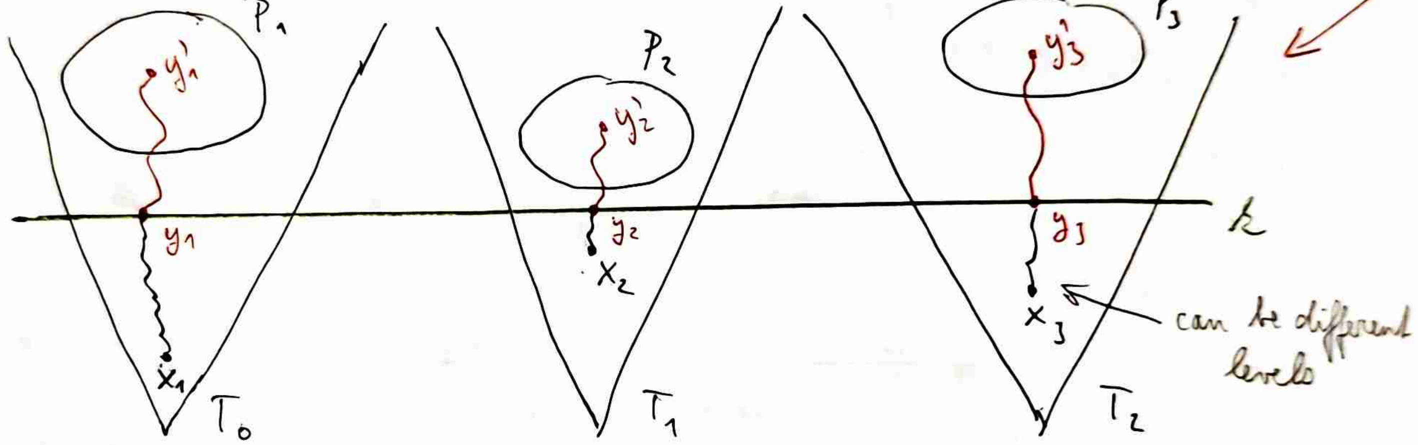
Def: Given a tree T, node $x \in T$ and $k \in \omega$ s.t. $k > |x|_T$, a subset $P \subseteq T$ is k-x-dense $\equiv$ $\forall y \in T(k)$ above x has an extension $y' \in P$.

→ for all successors of x at level k, there $\exists$ a path to P

P strictly above k

Def: Given a product tree $\Pi = (T_1, \dots, T_d)$, nodes $\overline{x} = (x_1, \dots, x_d) \in \prod_\Pi$ ($\to x_i \in T_i$) and $k \in \omega$ s.t. $k > |x_i|_{T_i}$ for $\forall i$, a subset $P = P_1 \times P_2 \times \dots \times P_d \subseteq \prod_\Pi$ is a k-$\overline{x}$-dense matrix $\equiv$ $\forall P_i$ is a k-x_i-dense subset of T_i.

→ dense matrix = product of dense subsets for each component



→ for $\forall \overline{y} \in \prod_\Pi^{lev}$ in level k extending $\overline{x}$, there $\exists \overline{y}' \in \prod_\Pi$ extending $\overline{y}$ such that $\forall i : y'_i \in P_i$

Note: $\Pi = (T_1, \dots, T_d)$ will always be a product tree of FB trees of height ω with no leaves

→ Halpern-Laüchli = SS

→ to prove will use 4 versions of the theorem and an induction based on them

$\boxed{SS_d}$ $\forall$ finite coloring of $\prod_\Pi^{lev}$: $\exists$ homogeneous $S \in STR_\omega(\Pi)$

$\boxed{SD_d}$ $\forall$ finite coloring of $\prod_\Pi$: $\exists \overline{x} \in \prod_\Pi$, $\exists k \in \omega$ s.t. $\exists$ homogeneous k-$\overline{x}$-dense matrix

$\boxed{DS_d}$ $\forall$ finite coloring of $\prod_\Pi$: $\exists \overline{x} \in \prod_\Pi$ s.t. $\forall k$ large enough $\exists$ homogeneous k-$\overline{x}$-dense matrix, and all of those matrices have the same color.

👁 clearly $DS_d \Rightarrow SD_d$

$\boxed{DS_d^{lev}}$ $\forall$ finite coloring of $\prod_\pi^{lev}$: $\exists \bar{x} \in \prod_\pi$ s.t. $\forall k$ large enough $\exists$ homogeneous k-$\bar{x}$-dense matrix $\mathbb{P}_k = P_1 \times P_2 \times \dots \times P_d$ such that all sets $P_1, \dots, P_d$ are contained in a single shared level ... $\exists n : \forall P_i \subseteq T_i(n)$. Furthermore, all of the matrices $\mathbb{P}_k$ share the same color.

$\boxed{SD_d^{lev}}$ $\forall$ finite coloring of $\prod_\pi^{lev}$: $\exists \bar{x} \in \prod_\pi$, $\exists k \in \omega$ s.t. $\exists$ homogeneous k-$\bar{x}$-dense matrix $\mathbb{P} = P_1 \times P_2 \times \dots \times P_d$ s.t. all P_i are contained in a single shared level ... $\exists n : \forall P_i \subseteq T_i(n)$

Strategy:

① prove SD_1

② prove $\forall d$: $SD_d \Rightarrow DS_d$

③ prove $\forall d$: $SD_d \Rightarrow SD_d^{lev}$

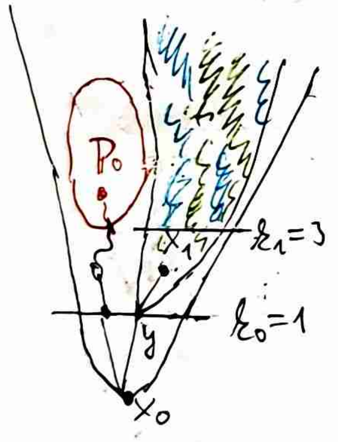
④ prove $\forall d$: $SD_d^{lev} \Rightarrow DS_d^{lev}$

⑤ prove $\forall d$: $DS_d^{lev} \Rightarrow SS_d$

⑥ prove $\forall d$: SD_d & DS_d & DS_d^{lev} & $SS_d \Rightarrow SD_{d+1}$

} $\forall d$: all above statements :

Proving SD_1 - homogenous $\bar{x}$-k-dense matrix in a tree = x-k-dense subset



Suppose there are 3 colors: red, blue and green

→ try x_0 = root, $k_0 = |x_0|_T + 1 = 1$ and P_0 = red vertices above level k_0

• if P_0 is a x_0-k_0-dense subset, we are done

otherwise $\exists$ some $y \in T(1)$ in level $k_0 = 1$ such that all nodes above y are either blue or green

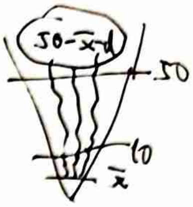
→ pick any x_1 immediate successor of y and set $k_1 = |x_1|_T + 1 = 3$

→ red has been eliminated, try blue next : P_1 = blue vertices above k_1

→ if blue fails as well, select X_3 and green above X_3 will work, as everything above X_3 must be green

$SD_d \Rightarrow DS_d$ assume DS_d fails for a given coloring of Π_{Π}

$\Rightarrow \forall \bar{x} \in \Pi_{\Pi}\ \exists k_{\bar{x}}$ s.t. there is no $k_{\bar{x}}$-$\bar{x}$-dense matrix

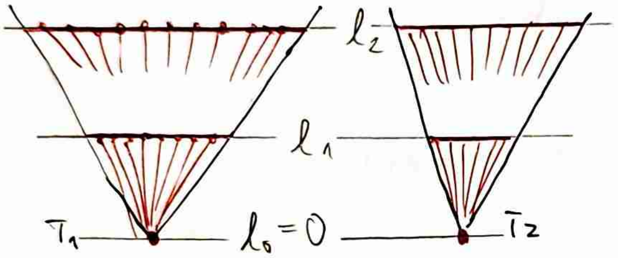


⊛ 👁 if $k > l$, then k-$\bar{x}$-dense is also l-$\bar{x}$-dense
50-$\bar{x}$-dense is 10-$\bar{x}$-dense

→ define levels (integers) $l_0, l_1, l_2, \ldots$ as $l_0 := 0$ and

- to get l_{i+1} look at all $\bar{x} = (x_1, \ldots, x_d)$ s.t. $\forall i: |x_i|_{T_i} \leq l_i$ at most on level l_i, and set l_{i+1} to be grater than all of their $k_{\bar{x}}$ values. So if $\bar{x}$ is at most at level l_i with value $k_{\bar{x}}$, then since $l_{i+1} > k_{\bar{x}}$, there is no l_{i+1}-$\bar{x}$-dense matrix

→ the compressed subtree $S \subseteq \Pi$ is the product subtree of Π with exactly those elements of Π which lie on one of the l_i levels



👁 S has no k-$\bar{x}$-dense matrix, ↯ SD_d

→ suppose $k = l_{i+1}$, so $\bar{x}$ is strictly bellow l_{i+1}, that is, at most l_i

→ if there was a k-$\bar{x}$-dense matrix, then it would also be a $k_{\bar{x}}$-$\bar{x}$-dense in T ↯

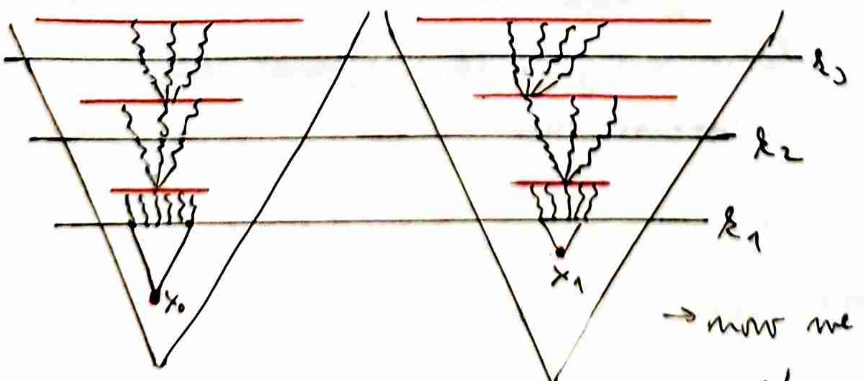
Hence $\exists \bar{x}\ \forall k$ large enough there $\exists\, k$-$\bar{x}$-dense matric of color C_k

→ we want all matrices to have the same color

- finitely many colors $\Rightarrow \exists C \in r$ s.t. there are ∞ many k s.t. $C_k = C$
  - ↳ specifically, matrices of this color are unbounded
  - → combining with ⊛ we have matrix of color C for $\forall k$ large enough ■

$SD_d^{lev} \Rightarrow DS_d^{lev}$: the same proof works

$DS_d^{lev} \Rightarrow SS_d$: DS_d^{lev} gives us $\bar{x}$ s.t. $\forall k$ large enough $\exists$ homogenous pancake above k



- take k_1 as the first such k
- to get k_{i+1}, find a dense pancake above k_i and set k_{i+1} to any level above this pancake

→ now we can easily find a homogeneous ∞ strong product subtree using these pancakes

Proving $SD_d \Rightarrow SD_d^{lev}$

👁 if ∃ homogeneous k-$\bar{x}$-dense matrix $P = P_1 \times P_2 \times \dots \times P_d$, then ∃ homogenous $P' = P_1' \times P_2' \times \dots \times P_d'$ contained below some finite level N

↳ the trees have finite branching factor, so there are only finitely $y \in T_i(k)$ above x_i for each T_i. We need only one $y' \in P_i$ above y, so we can take $P_i' \subseteq P_i$ by keeping only these finitely many y'.

⇒ hence $P_1', P_2', \dots, P_d'$ are finite, so contained below some finite level

claim: (∀ num colors r)(∃ finitel level N) s.t. ∀ r-coloring of $\prod_\pi$ $\exists \bar{x} \in \prod_\pi$, $\exists k$ s.t. ∃ homogenous k-$\bar{x}$-dense matrix below level N.

Proof: We use a compactness argument. Suppose that $\forall N$ existed a bad r-coloring χ_N of $\prod_{\pi \restriction N}$ (where $\pi \restriction N$ is the restriction of π to the first N levels). That is, there is no χ_N-homogenous k-$\bar{x}$-dense m. in $\pi \restriction N$.

→ Imagine a forrests whose elements are:

- level 0: colorings χ of $\prod_{\pi \restriction 0}$
- level 1: colorings χ of $\prod_{\pi \restriction 1}$
- level n: colorings χ of $\prod_{\pi \restriction n}$

node χ extends node χ' if $\chi' \subset \chi$

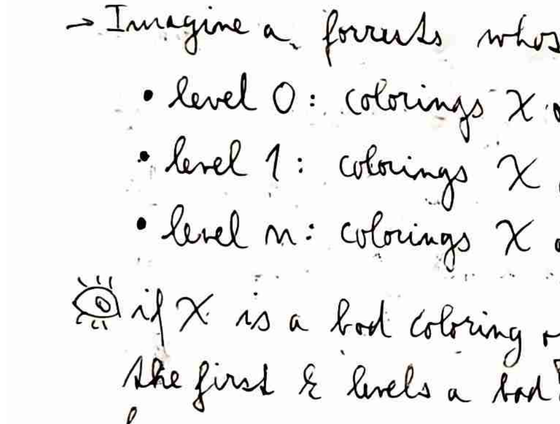
👁 finite colors ⇒ it has finite branching factor

👁 if χ is a bad coloring of $\prod_{\pi \restriction n}$, then $\forall k < n$ is the restriction of χ to the first k levels a bad coloring of $\prod_{\pi \restriction k}$. So if we remove all good colorings from our forrest, the levels do not change.

⇒ since $(\forall N)(\exists \text{ bad } \chi_N)$, the forrest is infinite with finite branching factor

→ König's lemma: ∃ infinite branch = bad coloring of the entire $\prod_\pi$ ↯ ■

To show $SD_d \Rightarrow SD_d^{lev}$ let N be the level from the claim, and $\chi : \prod_\pi \to r$.



→ define a coloring $\chi' : \prod_{\pi \restriction N} \to r$ as $\chi'(x_1, x_2, \dots, x_d) := \chi(x_1', x_2', \dots, x_d')$, where x_i' is the left-most successor of x_i in level N

→ find a k-$\bar{x}$-dense matrix P homogenous for χ' using SD_d

→ notice that $P' := \{\text{leftmost successors of } \bar{y} \text{ in level } N \mid \bar{y} \in P\}$ is a k-$\bar{x}$-dense matrix homogeneous for χ contained entirely in level N ■

Note: The compactness claim follows from the hypergraph formulation

V = what we are coloring = $\prod_{\pi}$

E = what we are looking for = $\{P \subseteq \prod_{\pi} \mid \exists \bar{x}, \exists k : P \text{ is a } k\text{-}\bar{x}\text{-dense matrix}\}$

$\chi(H) > r \Rightarrow (\exists \text{ finite } W \subseteq V) : \chi(H) > r$

SD_d: ($\forall$ r-coloring)($\exists$ homogenous k-$\bar{x}$-dense matrix)

$\Rightarrow \exists$ monochromatic edge $\Rightarrow \chi(H) > r$

$\Rightarrow (\exists \text{ finite } W \subseteq \prod_{\pi})(\forall r\text{-coloring of } W)\overbrace{(\exists \text{ monochr edge})}^{\circledast}$

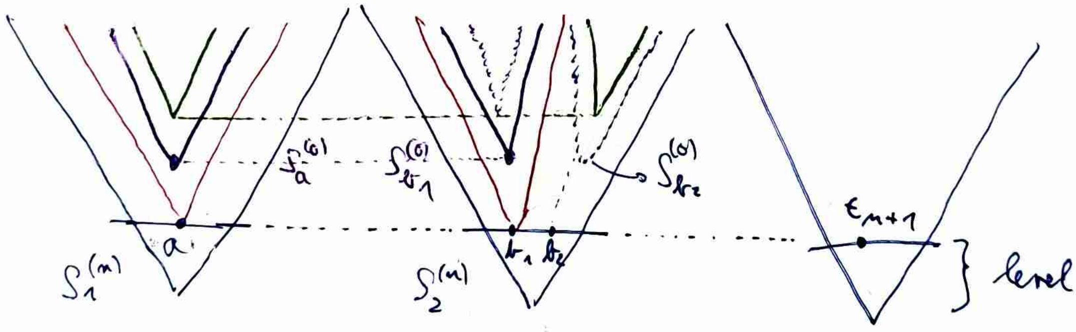
$\circledast$ = $\exists$ homogenous k-$\bar{x}$-dense matrix inside W

→ we let N be the first level above everything in W

We will now fix the roots of $S_1^{(0)}, \dots, S_d^{(0)}$ and construct new strong subtrees with those roots that work for t_1 as well, and then t_2, and t_3, ...

- improving the subtrees for t_{m+1} ... suppose it already works for $t_0, t_1, \dots, t_m$

→ we have $S_1^{(m)}, \dots, S_d^{(m)}$, want $S_1^{(m+1)}, \dots, S_d^{(m+1)}$



$\circledast$ (a, b_1)
(a, b_2)
in the example

→ suppose t_{m+1} is on level k and consider all $\underbrace{(x_1, \dots, x_d) \in \prod^{lev}_{S_1 \dots S_d}}_{\circledast}$ on level k

proof by example: consider (a, b_1) and their subtrees S_a, S_{b_1} in red V V

→ SS_d: we find $X_{t_{m+1}}$-homogenous strong subtree of $S_a\, S_{b_1}$ ⇝ $S_a^{(0)}, S_{b_1}^{(0)}$

→ we take $S_{b_2}^{(0)}$ ... a strong subtree of S_{b_2} that shares levels with $S_a^{(0)}, S_{b_1}^{(0)}$

→ now consider (a, b_2) and $S_a^{(0)}, b_{b_2}^{(0)}$ ... SS_d: for $X_{t_{m+1}}$, find homogenous strong subtree of $(S_a^{(0)}, S_{b_2}^{(0)})$ ⇝ $S_a^{(1)}, S_{b_2}^{(1)}$ in green V V

→ take $S_{b_1}^{(1)}$ ... a strong subtree of $S_{b_1}^{(0)}$ sharing levels with $S_a^{(1)}, S_{b_2}^{(1)}$

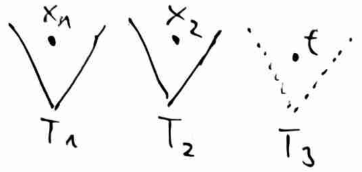
→ these trees work for all $t_0, \dots, t_{m+1}$

→ after we make them work for $\forall t$ on level k, then the roots of $S_a^{(*)}, S_{b_1}^{(*)}, S_{b_2}^{(*)}$ will become level k of the final subtrees $S_1, \dots, S_d$ ∎

SS_d & $DS_d^{lev} \Rightarrow SD_{d+1}$

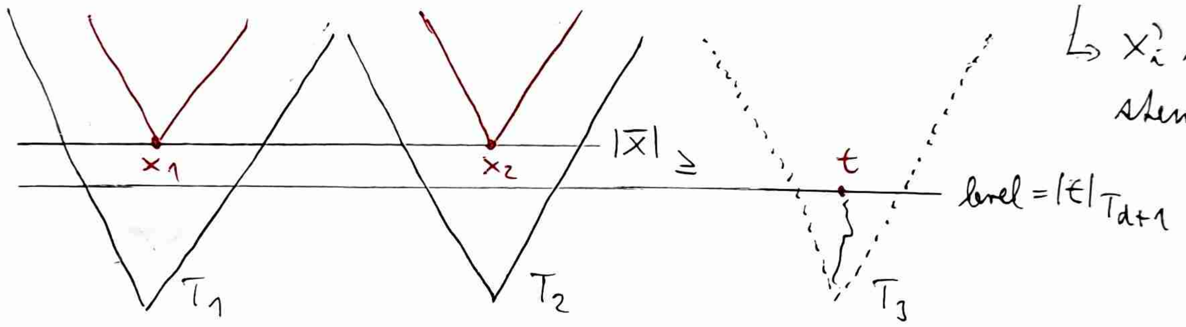
→ suppose we can do everything for d trees $T_1, T_2, \dots, T_d$ and want to find a k-$\overline{x}$-dense matrix in $d+1$ trees $T_1, T_2, \dots, T_d, T_{d+1}$

→ we have a coloring χ of $\Pi_{(T_1,\dots T_{d+1})}$, to apply induction step, we define for $\forall t \in T_{d+1}$ a coloring χ_t of $\Pi_{(T_1,\dots,T_d)}$ as $\chi_t(\overline{x}) = \chi_t(\overline{x}{}^\frown t)$



$\chi_t(x_1, x_2) := \chi(x_1, x_2, t)$ similar to Ramsey thm

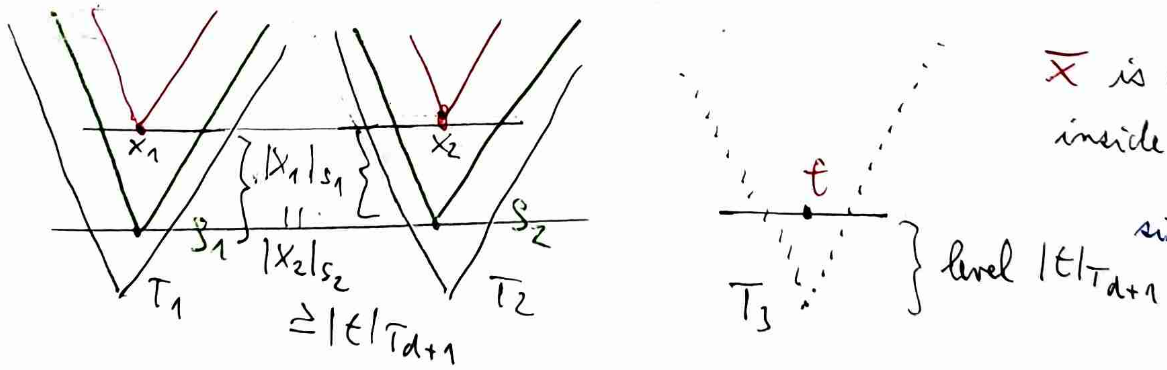
Def: A coloring χ of $\Pi_{(T_1,\dots,T_{d+1})}$ is good $\equiv (\forall t \in T_{d+1})(\forall \overline{x} \in \Pi^{lev}_{(T_1,\dots,T_d)})$ s.t. $\overline{x}$ is on level $|\overline{x}| \geq |t|_{T_{d+1}}$, the coloring χ_t is monochromatic on all $\overline{x}' \in \Pi^{lev}_{(T_1,\dots,T_d)}$ on level $|\overline{x}'| \geq |\overline{x}|$ s.t. x_i' extends x_i for $\forall i$



↳ x_i' s form a subtree stemming from x_i

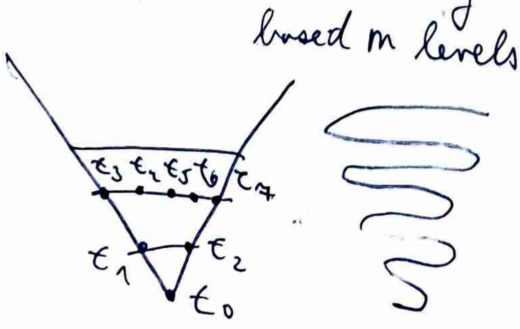
? the color can differ for distinct $t \in T_{d+1}$

Lemma $\boxed{1}$: There $\exists$ strong product subtree $(S_1,\dots,S_d) \in STR_\omega(T_1,\dots,T_d)$ s.t. the coloring χ is good on $\Pi' := (S_1,\dots,S_d,T_{d+1})$



$\overline{x}$ is on the same level inside $(S_1,\dots,S_d)$ and $(T_1,\dots,T_d)$ since $S_1,\dots,S_d$ share levels

Proof: enumerate vertices of T_{d+1} based on levels



- use SS_d to find a strong product subtree in $T_1,\dots,T_d$ for the coloring χ_{t_0}

⇝ $S_1^{(0)}, S_2^{(0)}, \dots, S_d^{(0)}$ works for t_0

We have found $(S_1, \dots, S_d) \in STR_\omega(T_1, \dots, T_d)$ s.t. the given r-coloring χ is good for $(S_1, \dots, S_d, T_{d+1})$

Lemma $\boxed{2}$: Given a tree T denote $[T] := \{b \mid b \text{ is a } \underline{\text{branch}} \text{ of } T\}$ $\subseteq$-max chain (infinite) ↳ no leaves

$\forall$ finitely branching tree T of height ω with no leaves, $\forall \chi : [T] \to \omega$ $\exists t^* \in T$, and $\exists$ color $C^* \in \omega$, and $\exists$ set of nodes D s.t.

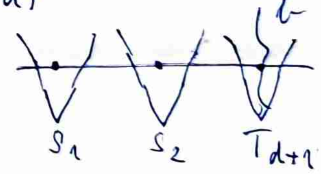
① $(\forall s \in D)(\exists b \in [T])$ s.t. $\chi(b) = C^*$ & $s \in b$
↳ there is a color C^* s.t. every $s \in D$ has a branch with color C^* going through s

② D is dense above t … $(\forall t \geq_T t^*)(\exists s \geq_T t)\ s \in D$
↳ for $\forall t$ above t^* there is some $s \in D$ above t $\Rightarrow$ D has to be "infinitely tall"

Proof: Try $t_0 = $ root, $C_0 = 0$, $D = $ all nodes s above t_0 s.t. $\exists$ branch $b \ni s$ with color 0
→ if it fails, then $\exists t$ above t_0 s.t. no s above t has a branch of color 0 going through
⇒ color 0 has "died out" in the subtree above t
- take $t_1 := t$ and try $C_1 = 1$, $D = $ all nodes above t_1 s.t. $\exists$ branch going throug of color 1

→ if fails, then $\exists t_2$ above t_1 s.t. no s above t_2 has a branch of color 0 or 1 going through
→ repeat this to construct a sequence $t_0, t_1, \dots t_n, \dots$
→ if at some point success ✓, if no, then this is an infinite branch of no color, since t_n has no branch of color $0, 1, \dots, n$ going through ↯ ▨

Def: For a branch b of T_{d+1} define a coloring $\chi^{(b)} : \prod^{lev}_{(S_1, \dots, S_d)} \to r$ as $\chi^{(b)}(x_1, \dots, x_d) := \chi(x_1, \dots, x_d, x_{d+1})$ where x_{d+1} is the unique node $\in b$ at height $|x_1|_{S_1} = \dots = |x_d|_{S_d}$



→ We now for each branch b of T_{d+1} use DS^{lev}_d on $S_1, \dots, S_d$ for $\chi^{(b)}$
⇒ we get $(x_1, \dots, x_d) \in \prod_{(S_1, \dots, S_d)}$ s.t. $\forall k$ large enough $\exists k$-$\bar{x}$-dense pancake, homogenous for $\chi^{(b)}$ with color C_b
👁 WLOG $(x_1, \dots, x_d) \in \prod^{lev}_{(S_1, \dots, S_d)}$ … the x_i which are low can be moved more up
⇒ $\forall b$ we have a pair $(\bar{x}, c)$ … basically a coloring $b \mapsto (\bar{x}, c)$
👁 only countably many "colors": $C \in r < \omega$
$x_i \in S_i \simeq \omega$ … finite levels ⇒ AC_ω: S_i is contable
$\Rightarrow \#\text{colors} \leq |r \cdot \underbrace{\omega \times \dots \times \omega}_{d}| = r \cdot \omega^d = \omega$ (basic cardinal arithmetic)

⇒ Lemma $\boxed{2}$: $\exists$ winning "color" $(\bar{x}^*, C^*)$ and t^* and dense $D \subseteq T_{d+1}$ s.t. $\forall t$ above t^* in T_{d+1} there is s above t that has a branch $b \ni s$ going through it s.t. $\forall k$ large enough $\exists k$-$\bar{x}^*$-dense pancake in $S_1, \dots, S_d$ with $\chi^{(b)}$-color C^*

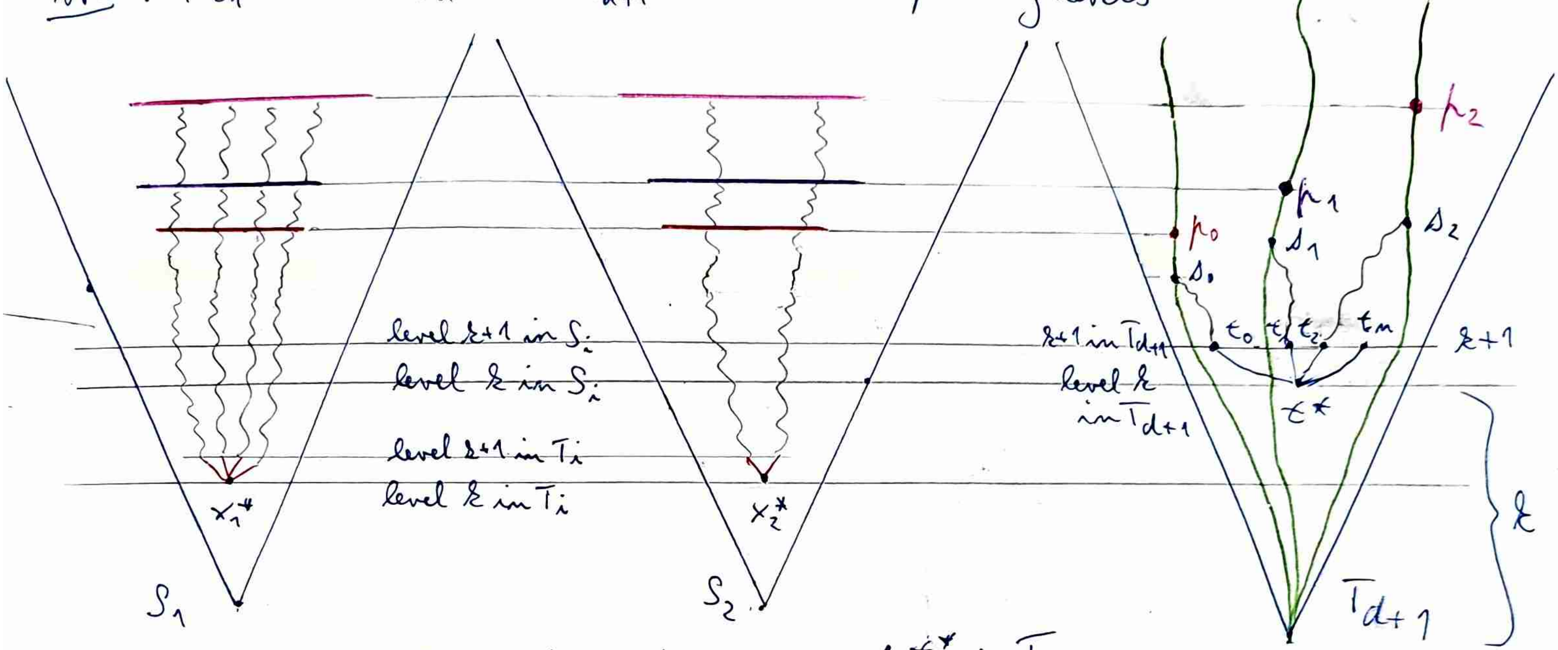
Goal: SD_{d+1} ... we want to find a k-$\bar{x}$-dense matrix in $(T_1, \ldots, T_d, T_{d+1})$

👁 we can move x_i^* up (through S_i) and t^* up (through T_{d+1})

$\Rightarrow$ we move them up to the same level: $|x_1^*|_{T_1} = \cdots = |x_d^*|_{T_d} = |t^*|_{T_{d+1}} =: k$, but note that still $\bar{x}^* \in \Pi^{lvr}_{(S_1, \ldots, S_d)}$

→ we will construct a $(k+1)$-$(x_1^*, \ldots, x_d^*, t^*)$-dense matrix in $T_1, \ldots, T_d, T_{d+1}$

Note: $|x_1^*|_{S_1} = \cdots = |x_d^*|_{S_d} \ll |t^*|_{T_{d+1}} = k$ ... S_i skip many levels



→ let $t_0, t_1, \ldots, t_m$ be the immediate successors of t^* in T_{d+1}

→ there $\exists s_0$ above t_0 s.t. there is a branch b_0 going through s_0, and $\forall l$ large enough (level in S_i) there $\exists$ l-$\bar{x}^*$-dense pancake in $S_1, \ldots, S_d$ with $\chi^{(b_0)}$-color C^*

$\Rightarrow$ take a pancake for $l > |s_0|_{T_{d+1}}$, so we have $P_1^{(0)}, \ldots, P_d^{(0)}$, pancake strictly above level $|s_0|_{T_{d+1}}$ in S_i, such that $\forall \bar{y} \in P_1^{(0)} \times \cdots \times P_d^{(0)}$ we have $\chi^{(b_0)}(\bar{y}) = C^*$

→ the color of $\bar{y}$ is determined by the unique $p_0 \in b_0$ on the same height (in T_{d+1}) as is the height of $P^{(0)}$ in S. Note that p_0 is strictly above s_0, so p_0 extends t_0

Note: $\forall \bar{y} \in P_1^{(0)} \times \cdots \times P_d^{(0)}$ we have $\chi^{(b_0)}(\bar{y}) = \chi_{p_0}(\bar{y}) = \chi(y_1, \ldots, y_d, p_0) = C^*$

We have: $\forall i \leq d$: ⊛-x_i^*-dense set $P_i^{(0)}$, and for t_0 we have p_0 strictly above t_0 s.t. $P_1^{(0)} \times \cdots \times P_d^{(0)} \times \{p_0\}$ is monochromatic w.r.t. χ

⊛: $P_i^{(0)}$ is $(k+1)$-x_i^*-dense in T_i since it is (something big)-x_i^*-dense in S_i and $x_i^* \in T_i(\underset{\substack{\uparrow \\ \text{level } k}}{k})$

Extending to $t_1, t_2, \ldots, t_m$:

→ there $\exists s_1$ above t_1 s.t. there is a branch b_1 going through s_1 and ... $\forall l$ large enough...

→ take a pancake $P_1^{(1)}, \ldots, P_d^{(1)}$ for $\underset{☒}{l > \text{level of } P_i^{(0)}}$ in S, and $l > |s_1|_{T_{d+1}}$

- $\forall \bar{y} \in P_1^{(1)} \times \cdots \times P_d^{(1)}$ we have $\chi^{(b_1)}(\bar{y}) = \chi_{p_1}(\bar{y}) = \chi(y_1, \ldots, y_d, p_1) = C^*$, where p_1 is above s_1

→ problem: we want $\chi(y_1, \ldots, y_d, p_0) = C^*$, but why should that hold?

$\Rightarrow$ ☒ every $\bar{y} \in P_1^{(0)} \times \cdots \times P_d^{(0)}$ has an extension $\bar{y}' \in P_1^{(1)} \times \cdots P_d^{(1)}$ and because χ is good on S_i, we have $\chi_{p_0}(\bar{y}) = \chi_{p_0}(\bar{y}') = C^*$ since the level of $\bar{y}$ in S is the same as $|p_0|_{T_{d+1}}$

→ $P_i^{(0)}$ is already dense for X_i^*, we do not need the full $P_i^{(1)}$

⇒ for each $y \in P_i^0$, take some extension $y' \in P_i^{(1)}$

→ and remove from $P_i^{(1)}$ everything else ... so $P_i^{(1)}$ is reduced to extensions of $P_i^{(0)}$

<u>Now</u>: $P_i^{(1)}$ is $(k+1)$-X_i^*-dense in T_i, and $\{p_0, p_1\}$ is strictly above t_0, t_1,
and $P_1^{(1)} \times \dots \times P_d^{(1)} \times \{p_0, p_1\}$ is monochromatic w.r.t. χ

→ iteratively, we construct $(k+1)$-t^*-dense subset $P = \{p_1, \dots, p_m\}$ in T_{d+1},
such that p_j is strictly above t_j, and $P_1^{(m)} \times \dots \times P_d^{(m)} \times P$ is monochromatic ▨

FILTERS AND ULTRAFILTERS

Def: Let $X \neq \emptyset$. A *non-empty* family of subsets $\mathcal{F} \subsetneq P(X)$ is a *filter* on $X \equiv$
... so $\mathcal{F} \neq P(X)$

① $\emptyset \notin \mathcal{F}$

② $A \in \mathcal{F}$ & $A \subseteq B \Rightarrow B \in \mathcal{F}$ ... upward closed

③ $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

Observations: For every filter $\mathcal{F}$ on X:

Notation: $\overline{A} := X \setminus A$

- $X \in \mathcal{F}$ ... since $\mathcal{F} \neq \emptyset$ and ②
- $\forall A, B \in \mathcal{F} : A \cap B \neq \emptyset$ ... from ③ and ①
- $\mathcal{F}$ is *closed under finite intersections* ... ③
- $\nexists A \subseteq X$ s.t. $A, \overline{A} \in \mathcal{F}$ ... from ③ and ①

Def: A filter $\mathcal{F}$ on X is an *ultrafilter* if in addition

④ $\nexists A \subseteq X$ s.t. $A, \overline{A} \notin \mathcal{F}$. Hence $\forall A \subseteq X$ *exactly one of $A, \overline{A}$ is in $\mathcal{F}$.*

Proposition: If $\mathcal{F}$ is an ultrafilter on X and $A = A_1 \dot{\cup} \cdots \dot{\cup} A_k \in \mathcal{F}$, then there is a *unique index* j s.t. $A_j \in \mathcal{F}$.

Proof: From ③ and ① there can be at most one such index. Suppose that $A_j \notin \mathcal{F}$ for $\forall j$. Then from ④ $\overline{A_j} \in \mathcal{F}$ for $\forall j$ and ③ gives $\bigcap \overline{A_j} \in \mathcal{F}$. But $\bigcap \overline{A_j} = \overline{A}$. So both $A, \overline{A} \in \mathcal{F}$ ↯ ■

Example: For every set X and every $\emptyset \neq Y \subseteq X$ is the set $F_Y := \{A \subseteq X \mid Y \subseteq A\}$ a filter on X.

Def: Filters of this form are called *principal filters.*

Proposition: A *principal filter* F_Y on X is an *ultrafilter* $\Leftrightarrow$ $|Y| = 1$ ... $Y = \{x\}$.

Proof: $\Rightarrow$: suppose $x, y \in Y$, $x \neq y$. Then $Y = (Y \setminus \{x\}) \dot{\cup} \{x\} \in \mathcal{F}$, but neither $Y \setminus \{x\}$ or $\{x\}$ is in $\mathcal{F}$ ↯

$\Leftarrow$: to show ④ let $A \subseteq X$ and $Y = \{x\}$. Either $x \in A$, then $A \in \mathcal{F}$, or $x \notin A$, then $x \in \overline{A}$ so $\overline{A} \in \mathcal{F}$ ■

Def: *Ultrafilters* of the form $F_x = \{A \subseteq X \mid x \in A\}$ for some $x \in X$ are called *principal*, and the other ultrafilters are called *non-principal*.

Def: A filter $\mathcal{F}$ is maximal ≡ there is no other filter $\mathcal{F}'$ s.t. $\mathcal{F} \subsetneq \mathcal{F}'$?

every principal ultrafilter is maximal

↳ suppose $\mathcal{F}_x = \{A \mid x \in A\}$ and $B \notin \mathcal{F}_x$, so $x \notin B$. Then $x \in \overline{B}$, so $\overline{B} \in \mathcal{F}_x \Rightarrow B \notin \mathcal{F}_x$

Theorem: Let $\mathcal{F}$ be a filter on X. The following statements are equivalent:

i) $\mathcal{F}$ is an ultrafilter

ii) $\mathcal{F}$ is a maximal filter

iii) $A \cup B \in \mathcal{F} \Rightarrow A \in \mathcal{F} \vee B \in \mathcal{F}$

Proof: (i)⇒(iii): suppose $A, B \notin \mathcal{F}$, so $\overline{A}, \overline{B} \in \mathcal{F}$ and $\overline{A} \cap \overline{B} = \overline{A \cup B} \in \mathcal{F}$ ↯

(iii) ⇒ (ii): suppose $\mathcal{F} \subsetneq \mathcal{F}'$ for some filter $\mathcal{F}'$. Hence there $\exists A \in \mathcal{F}'$ such that $A \notin \mathcal{F}$, so $\overline{A} \in \mathcal{F}$. Because $\mathcal{F} \subseteq \mathcal{F}'$, also $\overline{A} \in \mathcal{F}'$, so $A, \overline{A} \in \mathcal{F}'$ ↯

(ii) ⇒ (i): suppose that $\mathcal{F}$ is not an ultrafilter. Hence there $\exists A \subseteq X$ such that $A, \overline{A} \notin \mathcal{F}$. Note that $\forall B \in \mathcal{F}: A \cap B \neq \emptyset$, otherwise $B \subseteq \overline{A}$, so from ② we would have $\overline{A} \in \mathcal{F}$ ↯. Define a filter

$$\mathcal{F}' := \{B' \subseteq X \mid (A \cap B) \subseteq B' \text{ for some } B \in \mathcal{F}\}$$

It is indeed a filter: $\emptyset \notin \mathcal{F}$ because always $A \cap B \neq \emptyset$. It is upward closed from definition. To check ③ suppose $B_1', B_2' \in \mathcal{F}'$, so for some $B_1, B_2 \in \mathcal{F}$: $(A \cap B_1) \subseteq B_1'$ and $(A \cap B_2) \subseteq B_2'$. Notice that $(A \cap B_1 \cap B_2) \subseteq B_1' \cap B_2'$. Since $B_1 \cap B_2 \in \mathcal{F}$, we have $B_1' \cap B_2' \in \mathcal{F}'$ ... definition of $\mathcal{F}'$.

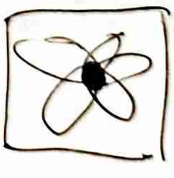
⇒ so $\mathcal{F}'$ is a filter such that $\mathcal{F} \subseteq \mathcal{F}'$ and $A \in \mathcal{F}' \setminus \mathcal{F} \Rightarrow \mathcal{F}$ not max ↯

Theorem: Let $\mathcal{F}$ be an ultrafilter on X. The following statements are equivalent:

i) $\mathcal{F}$ is a principle ultrafilter $\mathcal{F}_x = \{A \subseteq X \mid x \in A\}$

ii) $\mathcal{F}$ contains a finite set

iii) $\bigcap \mathcal{F} = \bigcap_{A \in \mathcal{F}} A \neq \emptyset$



Proof: (i) ⇒ (ii) since $\{x\} \in \mathcal{F}_x$.

(iii) ⇒ (ii) Define $A_0 := \bigcap \mathcal{F}$. Note that $\overline{A_0} \notin \mathcal{F}$ (since $A_0 \subseteq A$ for $\forall A \in \mathcal{F}$), so because $\mathcal{F}$ is an ultrafilter, we have $A_0 \in \mathcal{F}$.

- claim: $|A_0| = 1$. Suppose $|A_0| > 1$, so A_0 has a proper nonempty subset $B \subset A_0$.

  $B, \overline{B} \notin \mathcal{F}$, contradicting that $\mathcal{F}$ is an ultrafilter

  ↳ neither B or $\overline{B}$ contains A_0, but A_0 is contained in every $A \in \mathcal{F}$.

(ii) ⇒ (i): Let $A_0 \in \mathcal{F}$ be a set of the smallest (finite) size. We want to show that $\mathcal{F}$ is a principal ultrafilter.

A) if $A_0 = \{x\}$, then $\forall B \in \mathcal{F} \underset{⊛}{:} x \in B$, otherwise $B \cap A_0 = \emptyset$ ↯ $(\forall A, B \in \mathcal{F}: A \cap B \neq \emptyset)$

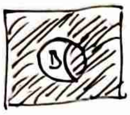
⇒ so $\mathcal{F} \subseteq \mathcal{F}_x = \{A \mid x \in A\}$, we want to show $\mathcal{F} = \mathcal{F}_x$.

claim: $\forall A$ s.t. $x \in A$ we have $A \in \mathcal{F}$.

↳ $\mathcal{F}$ is an ultrafilter so if $A \notin \mathcal{F}$, then $\bar{A} \in \mathcal{F}$. But $x \notin \bar{A}$ ↯ ⊛

B) if $|A_0| > 1$, then A_0 contains a proper nonempty subset $B \subset A_0$.

→ $\mathcal{F}$ is an ultrafilter and $B \notin \mathcal{F}$ (A_0 is the smallest), we have $\bar{B} \in \mathcal{F}$



Hence $\bar{B} \cap A_0 \in \mathcal{F}$, but $|\bar{B} \cap A_0| < |A_0|$ ↯ ■

Corollary: If X is finite, then all ultrafilters on X are principal.

? Are there non-principal ultrafilters on infinite sets?

Example: If X is an infinite set, then $\mathcal{F}_{co} := \{A \subseteq X \mid \bar{A} \text{ is finite}\}$ is a filter on X called the co-finite filter, or Fréchet filter.

If $\mathcal{F}$ is a non-principal ultrafilter on an infinite set X, then $\mathcal{F}_{co} \subseteq \mathcal{F}$

↳ for $\forall A$ s.t. $\bar{A}$ is finite, $\bar{A} \notin \mathcal{F}$ (last theorem), so $A \in \mathcal{F}$

Theorem (The Ultrafilter lemma, Tarski, 1930): Every filter $\mathcal{F}$ on a set X can be extended to an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ on X.

Corollary: Every infinite set has a nonprincipal ultrafilter.

Proof: First recall Zorn's lemma: Every partially ordered set A where chains have upper bounds, contains a maximal element. b above every $a \in A$.

→ let $\mathbb{F} \subseteq P(P(X))$ be the set of all ultrafilters on an infinite set X

→ $(\mathbb{F}, \subseteq)$ is a partially ordered set, notice that

if $\mathcal{C} \subseteq \mathbb{F}$ is a chain of filters $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$, then $\bigcup \mathcal{C} = \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$

is also a filter and an upper bound of $\mathcal{C}$

→ hence we can use Zorn's lemma to obtain a maximal filter (and thus an ultrafilter) $\mathcal{U} \in \mathbb{F}$ above the given filter $\mathcal{F}$ (so $\mathcal{F} \subseteq \mathcal{U}$) ■

Proof of corollary: Chose $\mathcal{F}$ as the co-finite filter $\mathcal{F}_{co}$ on X and get $\mathcal{U} \supseteq \mathcal{F}_{co}$

→ since $\mathcal{U}$ extends $\mathcal{F}_{co}$, it contains no finite sets ... otherwise $\underset{\text{finite}}{A}, \bar{A} \in \mathcal{U}$ ↯

↳ hence $\mathcal{F}$ is not a principal ultrafilter ■

Remark: We have used Zorn's lemma to prove the existence of nonprincipal ultrafilters, but we do not actually need the full strength of AC.

→ In fact, the Ultrafilter lemma is equivalent in ZF to the Boolean prime ideal theorem, which is strictly weaker than AC.

→ however, we need something stronger than ZF to prove the existence of nonprincipal ultrafilters even for $X = \omega$. AC_ω or DC are not enough.

Fact: There are models of ZF where ω has no nonprincipal ultrafilters.

ULTRAFILTERS AND RAMSEY THEORY

→ we will now use ultrafilters to prove the Infinite Ramsey Theorem

→ however, note that this proof (or any other Ramsey theory ultrafilter proofs) does not work in ZF or even in ZF + DC

Theorem: The ultrafilter lemma $\Rightarrow$ $(\forall r): \omega \to (\omega)^2_r$

Proof: Suppose $\chi: [\omega]^2 \to \{\mathrm{red}, \mathrm{blue}\}$ is a 2-coloring - the general case is similar

- let $\mathcal{U}$ be a non-principal ultrafilter on ω
- let $m \in \omega$ and notice that we have a partition of ω:

$$\omega = \{m\} \mathbin{\dot\cup} \{a \in \omega \mid \chi(a,m) = \mathrm{red}\} \mathbin{\dot\cup} \{a \in \omega \mid \chi(a,m) = \mathrm{blue}\}$$

→ since $\mathcal{U}$ is an ultrafilter, exactly one of these pieces belongs to $\mathcal{U}$

→ since $\mathcal{U}$ is non-principal, $\{m\} \notin \mathcal{U}$, so either

- $\{a \in \omega \mid \chi(a,m) = \mathrm{red}\} \in \mathcal{U}$ $\rightsquigarrow$ m is $\mathcal{U}$-red, or
- $\{a \in \omega \mid \chi(a,m) = \mathrm{blue}\} \in \mathcal{U}$ $\rightsquigarrow$ m is $\mathcal{U}$-blue

→ this induces another partition: $\omega = \{m \in \omega \mid m \text{ is } \mathcal{U}\text{-red}\} \mathbin{\dot\cup} \{m \in \omega \mid m \text{ is } \mathcal{U}\text{-blue}\}$

→ again, exactly one of these belongs to $\mathcal{U}$, WLOG assume that

$$A := \{m \in \omega \mid m \text{ is } \mathcal{U}\text{-red}\} \in \mathcal{U}$$

→ $\forall m \in A$: since m is $\mathcal{U}$-red, $\mathcal{U}$ contains the set $A_m = \{a \in \omega \mid \chi(a,m) = \mathrm{red}\}$

→ to construct an infinite homogeneous subset of ω, we create a sequence

- $m_0 \in A$, $m_1 \in A \cap A_{m_0}$, $m_2 \in A \cap A_{m_0} \cap A_{m_1}$, $m_3 \in A \cap A_{m_0} \cap A_{m_1} \cap A_{m_2}, \dots$

→ since filters are closed on finite intersections, all of these sets are in $\mathcal{U}$ and they are nonempty (in fact infinite), so the sequence is correctly defined

→ also notice that $\{m_i \mid i \in \omega\}$ is a homogeneous subset of ω

- $m_i \in \bigcap_{j<i} A_{m_j}$ $\Rightarrow$ m_i is connected to all m_j for $j<i$ with a red edge ▨

→ this argument can be generalized for $k>2$ but it just mimics the standard IRT proof

Def: Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on X, Y. Their tensor product (sometimes also Fubini product) is the ultrafilter $\mathcal{U}\otimes\mathcal{V}$ on $X\times Y$ defined as

$$\forall A\subseteq X\times Y: \quad A\in\mathcal{U}\otimes\mathcal{V} \equiv \{x\in A_X \mid A_Y(x)\in\mathcal{V}\}\in\mathcal{U},$$

where $A_X := \{x\in X \mid (x,y)\in A\}$ and $A_Y(x)=\{y\in Y \mid (x,y)\in A\}$ for $x\in X$.

Intuition: Ultrafilters are a way of defining "large" sets. Given an ultrafilter $\mathcal{U}$ on X, we say that $A\subseteq X$ is $\mathcal{U}$-large $\equiv A\in\mathcal{U}$.

👁 If A is $\mathcal{U}$-large and $B\supseteq A$, then B is also $\mathcal{U}$-large

⇒ $A\in\mathcal{U}\otimes\mathcal{V} \iff$ there are $\mathcal{U}$-many $x\in A_X$ for which $A_Y(x)$ is $\mathcal{V}$-large

→ if $\mathcal{U}$ and $\mathcal{V}$ are non-principal ultrafilters (so no finite elements), then $A\in\mathcal{U}\otimes\mathcal{V} \Rightarrow$ there are ∞-many $x\in A_X$ for which $A_Y(x)$ is infinite

👁 $\mathcal{U}\otimes\mathcal{V}$ is an ultrafilter on $X\times Y$

- $\emptyset\notin\mathcal{U}\otimes\mathcal{V}$ as empty sets are not large ... $\emptyset\notin\mathcal{U},\mathcal{V}$
- if $A\in\mathcal{U}\otimes\mathcal{V}$ and $B\supseteq A$, then $B\in\mathcal{U}\otimes\mathcal{V}$, since adding more stuff to A_X and $A_Y(x)$ will not make them not large
- if $A,B\in\mathcal{U}\otimes\mathcal{V}$, then $A\cap B\in\mathcal{U}\otimes\mathcal{V}$ since intersections of large sets are large

👁 If $\mathcal{U},\mathcal{V}$ are non-principal, then $\mathcal{U}\otimes\mathcal{V}$ is also non-principal

- if $A\in\mathcal{U}\otimes\mathcal{V}$, then A cannot be finite because finite sets are not large

Def: For an ultrafilter $\mathcal{U}$ on X we define $\mathcal{U}^1 := \mathcal{U}$ and $\mathcal{U}^{k+1} := \mathcal{U}\otimes\mathcal{U}^k$.

Proposition: For $\forall$ non-principal ultrafilter $\mathcal{U}$ on ω and $\forall k\in\omega$:

$$\Delta_k^+ := \{(n_1,\dots,n_k) \mid n_1<\dots<n_k\}\in\mathcal{U}^k.$$

Proof: By induction over k we show that $\forall l\in\omega$ we have

$$(\Delta_k^+)_l := \{(n_1,\dots,n_k)\in\Delta_k^+ \mid l<n_1\}\in\mathcal{U}^k$$

Then clearly the superset Δ_k^+ is also $\mathcal{U}^k$-large.

- base case: for $k=1$ we have $(\Delta_1^+)_l = \{n_1\in\omega \mid l<n_1\}$, which is co-finite (its complement is finite), so $(\Delta_1^+)_l\in\mathcal{U}^1=\mathcal{U}$
  - ↪ complement $\notin\mathcal{U} \Rightarrow$ original $\in\mathcal{U}$

- $k \to k+1$: by definition $\Delta^+_{k+1} \in \mathcal{U}^{k+1} = \mathcal{U} \otimes \mathcal{U}^k$ <=> the set $\Gamma := \{n_0 \in \omega \mid (\Delta^+_k)_{n_0} \in \mathcal{U}^k\} \in \mathcal{U}$, and this is true because $\Gamma = \omega$. Indeed, for $\forall n_0 \in \omega$ by the induction hypothesis for k we have

$$(\Delta^+_k)_{n_0} = \{(n_1,\dots,n_k) \in \Delta^+_k \mid n_0 < n_1\} \in \mathcal{U}^k$$ ■

Proposition: Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. Then for $\forall A \in \mathcal{U}^k$ there exists an infinite set H s.t. $[H]^+_k := \{(h_1,\dots,h_k) \in H^k \mid h_1 < \dots < h_k\} \subseteq A$.

Corollary: IRT: $(\forall k)(\forall r): \omega \to (\omega)^k_r$ ... different ultrafilter proof

Proof: Notice that there is a bijection between k-sets $\{n_1,\dots,n_k\} \in [\omega]^k$ and component-wise ordered k-tuples $(n_1,\dots,n_k) \in \Delta^+_k$.

→ hence we can identify $[\omega]^k \simeq \Delta^+_k$ and $[H]^k \simeq [H]^+_k$

⇒ a partition of $[\omega]^k$ into r colors, also partitions $\Delta^+_k = C_1 \dot\cup \dots \dot\cup C_r$

→ pick a nonprincipal ultrafilter $\mathcal{U}$ on ω.

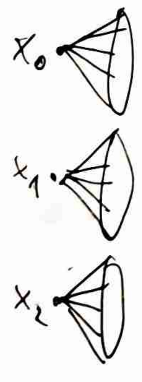
→ we know that $\Delta^+_k \in \mathcal{U}^k$, so there is a unique j s.t. $C_j \in \mathcal{U}^k$

→ use the proposition on $A = C_j$ to get an infinite H s.t. $[H]^k$ has color j ■

Proof of claim: We directly construct the set for $\forall k \geq 1$. For $k=1$:

$\forall A \in \mathcal{U}$ $\exists$ infinite H s.t. $[H]^+_1 = H \subseteq A$ ... true since A is infinite

- $k=2$: $A \in \mathcal{U}^2 \iff \{x \in A_1 \mid A_2(x) \in \mathcal{U}\} \in \mathcal{U}$ ... $A_1 = \{x \mid (x,y) \in A\}$
$A_2(x) = \{y \mid (x,y) \in A\}$



Thus there are $\mathcal{U}$-many $x_i \in A_1$ for which $A_2(x)$ is $\mathcal{U}$-large
⇒ denote the set of all of these x by X

- set $h_0 := x_1$, then $A_2(h_0) \cap X$ is $\mathcal{U}$-large, namely infinite, so $\exists h_1 \in A_2(h_0) \cap X$ s.t. $h_1 > h_0$ ... $(h_0, h_2) \in A$
- to get h_n, let $\Gamma_n = X \cap \left(\bigcap_{i<n} A_2(h_i)\right) \rightsquigarrow \mathcal{U}$-large ⇒ infinite, so $\exists h_n \in \Gamma_n$ s.t. $h_n > h_{n-1} > \dots > h_1 > h_0$

→ since $\forall i < n$: $h_n \in A(h_i)$ we have $(h_i, h_n) \in A$

☼ $H = \{h_i \mid i < \omega\}$ has the property $\forall i < j$: $h_i < h_j$ and $(h_i, h_j) \in A$ ⇒ $[H]^+_2 \subseteq A$

- $k \geq 3$: It starts getting messy for bigger k. For $k=3$ we will need each h_j (for $j \geq 2$) to be contained in all 3 levels at the same time - so that it can appear in each coordinate ■

RAMSEY THEORY AND EQUATIONS

→ statements of the form: given any finite coloring of ω, there exist $x_1, \dots, x_m \in \omega$ with the same color, which satisfy some prescribed condition

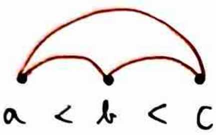
Theorem (Schur, 1916): If ω is finitely colored, there exist $x, y, z \in \omega \setminus \{0\}$ having the same color such that $x + y = z$

↳ not necessarily distinct

Proof: Assume that r colors are used and $\chi : \omega \to r$.

→ define a coloring χ' of $[\omega]^2$ as $\chi'(\{a,b\}) := \chi(|a-b|)$

Ramsey: $\exists \{a,b,c\}$ homogeneous for χ', WLOG $a < b < c$



$$\chi'(a,b) = \chi'(b,c) = \chi'(a,c)$$

$$\chi(\underbrace{b-a}_{x}) = \chi(\underbrace{c-b}_{y}) = \chi(\underbrace{c-a}_{x+y}) \qquad \blacksquare$$

Remark: Schur actually proved this to show the following corollary:

Corollary (Schur): $\forall m$, $\forall$ sufficiently large prime p, $\exists x, y, z \in \mathbb{Z}_p$ s.t.

$x^m + y^m = z^m$ in $\mathbb{Z}_p$ ... Fermats last theorem modulo p

Theorem (Schur, finite version): $(\forall r)(\exists N)$ s.t. if N is r-colored, there $\exists$ nonzero $x, y, z < N$ having the same color such that $x + y = z$

Proof: By compactness, we use the hypergraph

$V = \omega$ ... what we are coloring

$E = \{\{x,y,z\} \in [\omega]^3 \mid x + y = z\}$ ... what we want to find $\blacksquare$

Def: Given $n > 0$, the Schur number $S(n)$ is the smallest integer N s.t. if $\{1, 2, \dots, N\}$ is n-colored, there $\exists$ monochromatic $x, y, x+y \leq N$

Example: We know that

$S(1) = 2$: $\underset{1}{\bullet}\ \underset{2}{\bullet}$ $1 + 1 = 2$

$S(2) = 5$, $S(3) = 14$, $S(4) = 45$, $S(5) = 162$

→ required 2 petabytes of RAM to calculate this number

Theorem (Rado, 1933): Let $S(x_1, \dots, x_m)$ be a linear homogenous constraint:

$$c_1 x_1 + c_2 x_2 + \dots + c_m x_m = 0 , \qquad c_i \in \mathbb{Z}.$$

Then Schur's theorem genearalizes to S, that is: $\forall$ finite coloring of ω there $\exists$ nonzero $x_1, \dots, x_m \in \omega$ with the same color s.t. $S(x_1, \dots, x_m)$,

$\iff$ some nonempty subset of the c_i sums to zero

Schur's theorem corresponds to $S(x_1, x_2, x_3)$: $x_1 + x_2 - x_3 = 0$

Example: Schur's theorem

- genearalizes to $2x + 3y - 5z + w = 0$ as $2+3-5=0$
- does not generalize to $2x + 3y - 6z = 0$

Remark: Rado also proved a similar iff statement for systems of linear homogeneous constrains

HINDMAN'S THEOREM

Schur: $\exists$ nonzero x, y s.t. $x, y, x+y$ are monochr.

? $\exists$ nonzero x, y, z s.t. $x, y, z, x+y, x+z, y+z, x+y+z$ are monochr?

Def: For $X \subseteq \omega$ define the (finite) sum-set of X as $FS(X) \subseteq \omega$:

$$FS(X) := \{ y \in \omega \mid y = \sum_{x \in A} x \text{ for some finite nonempty } A \subseteq X \}$$

Theorem (Folkman, 1970): If ω is finitely colored, there exist arbitrarily large finite sets $X \subseteq \omega$ s.t. $FS(X)$ is monochromatic.

⇑

Theorem (Hindman, 1974): If ω is finitely colored, there exists an infinite set $X \subseteq \omega$ s.t. $FS(X)$ is monochromatic.

Def: $Fin(\omega) := \{ A \subseteq \omega \mid A \neq \emptyset \ \& \ A \text{ is finite} \} = [\omega]^{<\omega} \setminus \{\emptyset\}$

Def: For $D \subseteq Fin(\omega)$ define the finite union-set of D as $FU(D) \subseteq Fin(\omega)$:

$$FU(D) := \{ Y \in Fin(\omega) \mid Y = \bigcup A \text{ for some finite nonempty } A \subseteq D \}$$

Def: $D \subseteq Fin(\omega)$ is a disjoint collection $\equiv \forall X, Y \in D: X \cap Y = \emptyset$

Theorem (FS version of H): If $Fin(\omega)$ is finitely colored, there exists an infinite disjoint collection $D \subseteq Fin(\omega)$ s.t. $FU(D)$ is monochromatic.

Observation: FU version of H $\Rightarrow$ FS version of H

Proof: Given $\chi : \omega \to r$, we want infinite $X \subseteq \omega$ s.t. $FS(X)$ is monochr

→ define an injection $a : Fin(\omega) \to \omega$ as $a(X) := \sum_{i \in X} 2^i$

👁 X, Y disjoint $\Rightarrow a(X) + a(Y) = a(X \dot\cup Y)$

$$\begin{array}{rr} X: & 1001 \\ Y: & 10100 \\ \hline X \dot\cup Y: & 11101 \end{array}$$

$\Rightarrow$ define a r-coloring of $Fin(\omega)$ as $\chi'(X) := \chi(a(X))$

→ FU version of H gives us an infinite disjoint collection $B \subseteq Fin(\omega)$ s.t. $FU(B)$ is monochromatic w.r.t. χ'

→ suppose $B = \{X_1, X_2, X_3, \dots\}$. If $A = X_1 \dot\cup X_3 \dot\cup X_7$, then

$$\left.\begin{array}{l} \chi'(X_1) = \chi'(X_3) = \chi'(X_7) = \chi'(A) \\ \chi(a(X_1)) = \chi(a(X_3)) = \chi(a(X_7)) = \chi(a(X_1) + a(X_3) + a(X_7)) \end{array}\right\} FU \approx FS$$

$\Rightarrow$ take $X := \{a(Y) \mid Y \in B\}$ ∎

PROOF OF FINITE UNIONS VERSION OF HINDMAN

Note: D, D' etc. always denote an infinite disjoint collection

↳ always $D \subseteq Fin(\omega)$

👁 $D \subseteq Fin(\omega)$ & $D' \subseteq FU(D) \Rightarrow FU(D') \subseteq FU(D)$

Def: We define an order on infinite disjoint collections as

$$D' \sqsubseteq D \equiv D' \subseteq FU(D)$$

Def: We say that $S \subseteq Fin(\omega)$ is D-large $\equiv \forall D' \sqsubseteq D : S \cap FU(D') \neq \emptyset$

↳ S is "unavoidable", we cannot miss it for any smaller $D' \sqsubseteq D$

Observations:

- $D' \sqsubseteq D \Rightarrow FU(D') \subseteq FU(D)$
- $D'' \sqsubseteq D' \sqsubseteq D \Rightarrow D'' \sqsubseteq D$
- $FU(D)$ is D-large ... in particular $Fin(\omega)$ is $[\omega]^1$-large ↑ all singletons $\{0\}, \{1\}, \{2\}, \dots$
- S is D-large & $D' \sqsubseteq D \Rightarrow S$ is D'-large
- S is D-large $\Longleftrightarrow$ $S \cap FU(D)$ is D-large
- S is D-large & $S \subseteq S' \Rightarrow S'$ is D-large

Example: $\{A \in Fin(\omega) \mid |A| \text{ even}\}$ is $[\omega]^1$-large

$\{A \in Fin(\omega) \mid |A| \text{ odd}\}$ is not $[\omega]^1$-large

Lemma $\boxed{1}$: If S is D-large and $S = S_1 \cup S_2 \cup \dots \cup S_k$, then $\exists i \in \{1,2,\dots,k\}$, $\exists D' \sqsubseteq D$ s.t. S_i is D'-large.

Intuition: Large sets almost survive partitioning. We prove for $S = S_1 \cup S_2$

Proof: Suppose not. Since S_1 is not D-large, there $\exists D' \sqsubseteq D$ s.t. $S_1 \cap FU(D') = \emptyset$

Since S_2 is not D'-large, there $\exists D'' \sqsubseteq D'$ s.t. $S_2 \cap FU(D'') = \emptyset$

→ so $S_1 \cap FU(D'') = \emptyset$ & $S_2 \cap FU(D') = \emptyset$ $\Rightarrow$ $S \cap FU(D'') = \emptyset$ ↯

- The general case follows by induction ▨

→ our goal will be to prove the following:

Theorem: If S is D-large, then $\exists D^* \sqsubseteq D$ s.t. $FU(D^*) \subseteq S$.

Corollary: This implies FU version of Hindman. Let $Fin(\omega)$ be r-colored, that is partitioned as $Fin(\omega) = S_1 \cup S_2 \cup \dots \cup S_r$.

→ since $Fin(\omega)$ is $[\omega]^1$-large, Lemma $\boxed{1}$ implies that $\exists i\ \exists D \sqsubseteq [\omega]^1$ such that S_i is D-large

→ by the theorem $\exists D^* \sqsubseteq D$ s.t. $FU(D^*) \subseteq S_i$

→ that is $FU(D^*)$ is monochromatic (color i). Hindman is looking for D^* ■

nechápu to ☹

WELL-QUASI ORDERINGS

Def: A relation $\preccurlyeq$ on a set A is a

- preorder (or quasi-order) ≡ it is reflexive and transitive
- partial order ≡ it is an antisymmetric preorder
- well-quasi-order ≡ ∀ infinite sequence $x_1, x_2, \dots, x_n, \dots \subset A$ is good ⊛,
  preorder and ↗ that is: $\exists i<j$ s.t. $x_i \preceq x_j$

Lemma: WQO ⇔ no ∞ decreasing sequences & no ∞ antichains

↳ strictly decreasing: $x_i < x_j \equiv x_i \le x_j \,\&\, x_j \not\le x_i$

Proof: ⇒: obviously

⇐: suppose there $\exists x_1, x_2, \dots x_n, \dots$ s.t. $\forall i<j: x_i \not\le x_j$

👁 all x_i are distinct, otherwise $x_i = x_j$ for some $i<j$

We say that x_i is terminal ≡ $\nexists x_j,\ j>i$ s.t. $x_i > x_j$

👁 there are only finitely many terminal elements, otherwise they form an ∞ antichain: $x_i \not< x_j$ & $x_i \not> x_j$

⇒ WLOG the sequence $(x_i)_i$ contains no terminal elements
↳ otherwise we can just remove them

→ no ∞ antichains: $\exists i_1 < i_2$ s.t. $x_{i_1} > x_{i_2}$

→ x_{i_2} is not terminal ⇒ $\exists i_3 > i_2$ s.t. $x_{i_2} > x_{i_3}$

→ x_{i_3} is not terminal ... we construct an ∞ decreasing sequence ↯ ■

Lemma: WQO ⇒ ∀ infinite sequence $x_1, x_2, \dots, x_n, \dots \subset A$
$\exists i_1 < i_2 < \dots < i_n < \dots$ s.t. $x_{i_1} \leqslant x_{i_2} \leqslant \dots \leqslant x_{i_n} \leqslant \dots$

Intuition: There is an infinite nondecreasing subsequence

Proof: We say that x_i is terminal ≡ $\nexists x_j,\ j>i$ s.t. $x_i \leqslant x_j$

👁 there are only finitely many terminal elements, otherwise the infinite subsequence of them is a bad ⊛ sequence

→ remove all terminal elements from the sequence and take some $i_1 < i_2$ s.t. $x_{i_1} \le x_{i_2}$. Now x_{i_2} is not terminal, so $\exists i_3 > i_2$ s.t. $x_{i_2} \le x_{i_3}$ ... ■

Lemma: $(A, \preccurlyeq)$ WQO ⇒ $\forall B \subseteq A$ has at least one, but only finitely many min. elements.

Proof: If ∞ many then they form an ∞ antichain ↯

- if it had no min. el., then we could construct (assuming AC_ω) an infinite decreasing sequence ↯ ■

Def: Given two preorders $(A, \leq_A)$, $(B, \leq_B)$ define a preorder $\leq$ on $A \times B$ as

$$(a, b) \leq (a', b') \equiv a \leq_A a' \ \& \ b \leq_B b'$$

Def: Given a preorder $(A, \leq)$ we can define a preorder $\leq_n$ on A^n as

$$(a_1, \dots, a_n) \leq_n (a_1', \dots, a_n') \equiv \forall i : a_i \leq a_i'$$

Lemma: If $(A, \leq_A)$ and $(B, \leq_B)$ are WQO, then $(A \times B, \leq)$ is also a WQO.

Proof: Any infinite sequence $(s_i)_i \subset A \times B$ consists of pairs $s_i = (a_i, b_i)$ and defines an ∞ seq. $(a_i)_i \subset A$ and $(b_i)_i \subset B$

- → since $\leq_A$ is a WQO, the ∞ sequence $(a_i)_i$ has an ∞ nondecreasing subsequence $(a_{i_k})_k$, that is, $\exists i_1 < i_2 < \cdots$ s.t. $a_{i_1} \leq_A a_{i_2} \leq_A \cdots$
- → restrict your attention only to the pairs (a_i, b_i) containing these elements – so they are nondecreasing in the first coordinate
- → since the corresponding $(b_{i_k})_k$ is an ∞ seq. and $\leq_B$ is WQO, there $\exists l < k$ s.t. $b_{i_l} \leq b_{i_k}$, hence $(a_{i_l}, b_{i_l}) \leq (a_{i_k}, b_{i_k})$ ■

The lemma extends to any finite number of preorders $(A_i, \leq_i)$ by induction

Corollary (Dickson, 1913): $\forall n : (\omega^n, \leq_n)$ is a WQO. Here ω^n denotes the set of all n-tuples of natural numbers and $\leq$ is the standard order on ω.

HIGMAN'S THEOREM

Def: Let $(A, \leq)$ be a preorder. Denote by $A^* := A^{<\omega}$ the set of all finite strings (words) over the alphabet A, and define a preorder $\sqsubseteq$ on A^* as

$$v \sqsubseteq w \equiv v = v_1 v_2 \dots v_k, \quad w = w_1 w_2 \dots w_m, \quad k \leq m$$
$$\exists i_1 < i_2 < \cdots < i_k \text{ s.t. } v_1 \leq w_{i_1}, \dots, v_k \leq w_{i_k}$$

If $\leq$ is equality $=$, then $\sqsubseteq$ is called string embeddings

Theorem (Higman, 1952): $(A, \leq)$ WQO $\Rightarrow$ $(A^*, \sqsubseteq)$ WQO

Corollary: If A is finite, then string embeddings are a WQO on A^*

↳ equality on any finite set is a WQO from the pigeonhole principle

Note: $\forall w \in A^* : \epsilon \sqsubseteq w$, where ϵ denotes the empty string.

Proof: For contradiction suppose that $\exists$ bad sequence $w_1, w_2, \dots, w_n, \dots \subset A^*$

→ we will construct a minimal bad sequence as follows: ! we need AC_ω

- let w_1^0 be a string of minimal length starting a bad sequence
- look at all bad sequences starting with w_1^0 and from these pick w_2^0, a string w_2 of minimal length

$w_1^0, w_2, w_3, w_4, \dots$

$w_1^0, w_2^0, w_3, w_4, \dots$ ⇒ w_{n+1}^0 is a string of min. length s.t. there

$w_1^0, w_2^0, w_3^0, w_4, \dots$ is a bad sequence whose first n elements are $w_1^0, \dots w_n^0$

→ for $\forall i$ write w_i^0 as $w_i^0 = a_i w_i^1$, where a_i is the first letter of w_i^0, and w_i^1 is the rest of the string

Note: $w_i^0 \neq e$, since otherwise the sequence is not bad ($e \sqsubseteq w$ for $\forall w \in A^*$), but w_i^1 might be empty

→ since $(A, \leq)$ is a WQO and a_i define an ∞ sequence, there $\exists i_1 < i_2 < \cdots$ s.t. $a_{i_1} \leq a_{i_2} \leq \cdots$ is an ∞ nondecreasing subsequence

claim: The infinite subsequence $w_{i_1}^1, w_{i_2}^1, \dots$ is good: $\exists l < k : w_{i_l}^1 \sqsubseteq w_{i_k}^1$

corollary: we have $\left.\begin{array}{l} a_{i_l} \leq a_{i_k} \\ w_{i_l}^1 \sqsubseteq w_{i_k}^1 \end{array}\right\} w_{i_l}^0 \sqsubseteq w_{i_k}^0$ ↯ that $(w_i^0)_i$ is bad and we are done

proof of claim: Suppose $(w_{i_j}^1)_j$ is bad (so all $w_{i_j}^1$ are nonempty)

case 1: $i_1 = 1$, so $w_{i_1}^1 = w_1^1$. Then $w_1^1, w_{i_2}^1, \dots$ is a bad sequence with $|w_1^1| < w_1^0$, contradicting the minimality of w_1^0 ↯

case 2: $i_i > 1$. Then the sequence

$w_1^0, w_2^0, \dots, w_{i_1 - 1}^0, w_{i_1}^1, w_{i_2}^1, \dots$ is also a bad sequence

Indeed, if $w_k^0 \sqsubseteq w_{i_j}^1$, then $w_k^0 \sqsubseteq w_{i_j}^0$ ⇒ $(w_i^0)_i$ is good ↯;

→ but notice that $|w_{i_1}^1| < |w_{i_1}^0|$, contradicting the minimality of $w_{i_1}^0$ ↯

■

Remark: This minimal bad sequence technique is very useful, and we will employ it to prove other WQO statements

Remark: So if A is a finite alphabet, then $(A^*, \sqsubseteq)$ has no ∞ antichain, but it has an antichain of any finite length n

$$\left.\begin{array}{c} 000\dots01 \\ 000\dots11 \\ \vdots \\ \underbrace{111\dots11}_{n} \end{array}\right\} n \longleftarrow$$

($\sqsubseteq$ = string embeddings)

? Higman orders A^* = finite ordered tuples, does it also work for $[A]^{<\omega}$ = finite unordered tuples ?

Def: Given a preorder $(A, \leq)$ define a preorder $\ll$ on $[A]^{<\omega}$ as:

$$X \ll Y \equiv \exists \text{ injection } \varphi : X \hookrightarrow Y \text{ s.t. } \forall x \in X : x \leq \varphi(x)$$

Namely $\emptyset \ll Y$ for $\forall Y \in [A]^{<\omega}$.

Theorem (Nash-Williams): $(A, \leq)$ WQO $\Rightarrow ([A]^{<\omega}, \ll)$ WQO

Proof: For contradiction let $X^0_1, X^0_2, X^0_3, \dots$ be a minimal bad seq. in $[A]^{<\omega}$, constructed the same way as in the proof of Higman's theorem

→ note that all X^0_i have to be nonempty, and let $X^0_i = X^1_i \cup \{a_i\}$

→ since $(A, \leq)$ is a WQO, $\exists i_1 < i_2 < \cdots$ s.t. $a_{i_1} \leq a_{i_2} \leq \cdots$

claim: $X^1_{i_1}, X^1_{i_2}, \dots$ is a good sequence … $\exists l < k$ s.t. $X^1_{i_l} \ll X^1_{i_k}$

corollary: $X^1_{i_l} \ll X^1_{i_k}$ & $a_{i_l} \leq a_{i_k} \Rightarrow X^0_{i_l} \ll X^0_{i_k}$ ↯ and we are done
↪ so $(X^0_i)_i$ is not bad

proof of claim: suppose $(X^1_{i_j})_j$ is bad,

then the sequence $X^0_1, X^0_2, \dots, X^0_{i_1 - 1}, X^1_{i_1}, X^1_{i_2}, \dots$ is also bad

→ indeed, if $X^0_k \ll X^1_{i_j}$, then $X^0_k \ll X^0_{i_j}$ so $(X^0_i)_i$ is not bad ↯

→ but $|X^1_{i_1}| < |X^0_{i_1}|$, contradicting the minimality of $X^0_{i_1}$ ↯

■

Note: we didnt explicitly state the case when $i_1 = 1$ here

Remark: This theorem is in fact a direct consequence of Higman's theorem

→ given a sequence $X_1, X_2, X_3, \dots$ we can order each set X_i arbitrarily to form a tuple W_i, and apply Higman: $W_i \sqsubseteq W_j \Rightarrow X_i \ll X_j$

What about infinite sequences?

Example (Rado): $(Q, \leq)$ WQO $\not\Rightarrow (Q^{\leq\omega}, \sqsubseteq)$ WQO ↱ lexicographical

$Q = \{(i,j) \mid i < j < \omega\}$, $(i,j) \leq (k,l) \equiv (i = k \;\&\; j \leq l) \vee (i < k)$

Def: $Q^{On} := \bigcup_{\alpha \in On} Q^{\alpha}$ … all set-long sequences

Theorem (Nash-Williams, 1968): Q finite $\Rightarrow Q^{On}$ WQO. Q ordered using $=$

↳ in fact he proved something much stronger about so-called better-quasi-orderings

WQOs ON GRAPHS

Def: Undirected graphs G, H

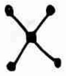
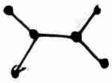
- subgraph $G \subseteq H \equiv V_G \subseteq V_H$ & $E_G \subseteq E_H \cap \binom{V_G}{2}$
- induced subgraph $G \leq H \equiv V_G \subseteq V_H$ & $E_G = E_H \cap \binom{V_G}{2}$
- minor $G \leq_m H \equiv G$ can be obtained from H by { deleting vertices, deleting edges, contracting edges
- topological minor $G \leq_\epsilon H \equiv$ (a subdivision of G) $\subseteq H$

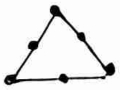
👁 $\subseteq, \leq, \leq_m, \leq_\epsilon$ are all partial orders on the class of all (finite) graphs

Note: We take isomorphic graphs to be identical

👁 $\underline{G \leq H} \Rightarrow \underline{G \subseteq H} \Rightarrow \underline{G \leq_\epsilon H} \Rightarrow \underline{G \leq_m H}$

👁 $G \leq H \overset{1}{\not\Leftarrow} G \subseteq H \overset{2}{\not\Leftarrow} G \leq_\epsilon H \overset{3}{\not\Leftarrow} G \leq_m H$

Proof: ①  $\subseteq$  but $\nleq$ ③  $\leq_m$  but $\nleq_\epsilon$

②  $\leq_\epsilon$  but $\nsubseteq$

↳ max deg $= 4 \Rightarrow$ any subdivision also 4

↳ but  has max deg $= 3$

Def: Given a preorder $(A, \leq)$, we say that a property $\varphi(x)$ is $\leq$-monotone $\equiv$

$$\forall x, y \in A: \varphi(x) \ \& \ y \leq x \Rightarrow \varphi(y)$$

Principle: $(A, \leq)$ WQO $\Rightarrow$ every $\leq$-monotone property $\varphi(x)$ is finitely testable

Explanation: We want to test if a given $x \in A$ has property $\varphi(x)$

→ look at $\bar{A} := \{x \in A \mid \neg\varphi(x)\}$ ... the elements not having $\varphi(x)$ F=FORBIDEN

→ since $\leq$ is WQO, $\bar{A}$ has finitely many minimal elements $F_1, \dots, F_m$

👁 $\varphi(x) \iff \nexists j: F_j \leq x$

$\Leftarrow$: if $\neg\varphi(x)$ then $x \in \bar{A}$ so $\exists j: F_j \leq x$, since there $\exists$ a minimal element bellow x ($\leq$ is WQO) and F_i are all of the min. elements of $\bar{A}$.

$\Rightarrow$: if $F_j \leq x$, then $\varphi(F_j)$ since φ is $\leq$-monotone ∎

Example: G is planar $\iff K_5 \nleq_m G$ & $K_{3,3} \nleq_m G$

Corollary: WQOs on classes of graphs allow us to make finite forbidden graph characterizations of any monotone property

Theorem (Robertson–Seymour): $\leq_m$ is WQO on finite graphs.

Remark: This is considered one of the deepest results of the 20th century, and was proven in a series of articles in years 1983–2004

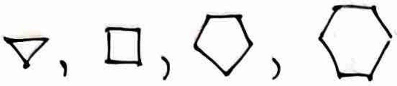
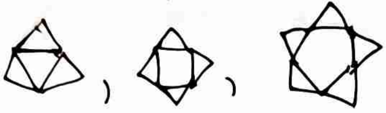
Theorem (Thomas, 1989): $\leq_m$ is not WQO on all (including infinite) graphs.

counterexample used graphs of size $2^{\omega} = \mathfrak{c}$

Conjecture (Thomas, 1989): $\leq_m$ is WQO on countable graphs.

Observation: None of $\subseteq, \leq, \leq_t$ are WQO on all finite graphs

Proof: We find ∞ antichains

- $\subseteq$ →
- $\leq$ ↗ , ...
- $\leq_t$: spiky cycles: , ...

  long lines: , ... ∎

👁 If $\leq_m$ is WQO on all (finite) graphs, then it is also WQO on any smaller class of graphs

KRUSKAL'S TREE THEOREM

Theorem (Kruskal, 1960): $\leq_t$ is WQO on finite trees.

👁 neither $\leq$ nor $\leq$ are WQO on trees

↳ , ... are ∞ antichain

→ we will in fact prove something much more general

Def: Given a set of labels Σ, a Σ-labeled rooted tree is a rooted tree T together with a function $f: T \to \Sigma$. For $x \in T$ we denote $T(x) := f(x)$.

Def: T_Σ := the set of all Σ-labeled (finite) rooted trees.

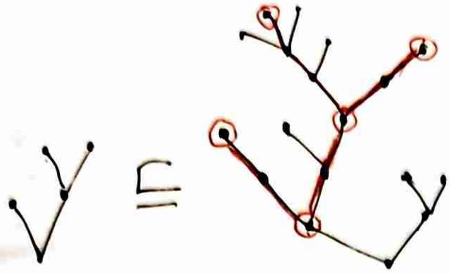
Def: Given a preorder $(\Sigma, \preceq)$ we define a preorder $\sqsubseteq$ on T_Σ as

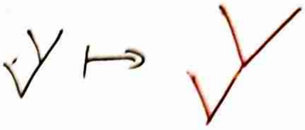
$$T_1 \sqsubseteq T_2 \equiv \exists \text{ injection } \varphi: T_1 \hookrightarrow T_2 \text{ s.t. } \forall x \in T_1: T_1(x) \preceq T_2(\varphi(x))$$
$$\text{and } \forall x, y \in T_1: \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$$

where $x \wedge y$ is the meet of x and y = nearest common ancestor.

Theorem: $(\Sigma, \preceq)$ WQO $\Rightarrow$ $(T_\Sigma, \sqsubseteq)$ WQO.

if $\Sigma = \{\bullet\}$, then $\sqsubseteq$ is the top-minor relation for rooted trees:



the injection  is called an embedding

this implies that $\leq_t$ is WQO on trees

↳ given a sequence $T_1, T_2, \dots$ root each tree in any way, and find an embedding between two of the rooted trees

Kruskal's theorem $\Rightarrow$ Higman's theorem

↳ each $w \in A^*$ can be treated as a tree $T_w \in T_A$

$w = ahoj \quad \Rightarrow \quad T_w =$ 

$v \sqsubseteq w \iff T_v \sqsubseteq T_w$

Proof of Kruskal's thm:

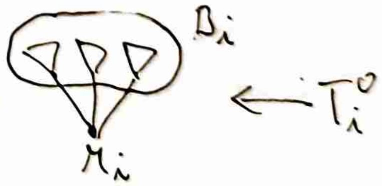
→ for contradiction suppose that $\exists$ a bad sequence in T_Σ and construct a minimal bad sequence $T_1^\circ, T_2^\circ, \dots T_n^\circ, \dots$, where T_{n+1}° is a tree with minimal size $|T_{n+1}|$ s.t. there $\exists$ a bad sequence whose first n elements are $T_1^\circ, \dots, T_n^\circ$ and $(n+1)$-th element is T_{n+1}

$|T_i^\circ| \geq 2$ for all but finitely many i

↳ otherwise there is an ∞ subsequence of one-node trees, and since $(\Sigma, \leq)$ is a WQO, there are $i<j$ s.t. $T_i^\circ \sqsubseteq T_j^\circ$ (T_i, T_j one-node) ↯ $(T_i^\circ)_i$ bad

$\Rightarrow$ WLOG $|T_i^\circ| \geq 2$ for $\forall i$ ... otherwise we just forget them

→ for each T_i° let $r_i \in \Sigma$ be the root of T_i° and $B_i \subseteq T_\Sigma$ be the set of "branches" of T_i°



claim: $\sqsubseteq$ is WQO on $B := \bigcup_{i=1}^{\infty} B_i$

proof: for contradiction let $R_1, R_2, \dots$ be a bad sequence in B

→ $\forall i$ let $f(i)$ be the smallest integer s.t. R_i is a branch of $T_{f(i)}^\circ$

→ let k be such that $f(k)$ is minimal

case 1: if $f(k) = 1$, then $R_1, R_2, \dots$ is a bad sequence in T_Σ with $|R_1| < |T_1^\circ|$, contradicting the minimality of T_1° ↯

__case 2__: $f(k) > 1$: Consider the sequence $T_1^\circ, T_2^\circ, \dots, T^\circ_{f(k)-1}, R_k, R_{k+1}, \dots$

👁 it is bad, contradicting the minimality of $T_{f(k)}$ since $|R_k| < |T^\circ_{f(k)}|$ ↯

↳ indeed, if $i < f(k)$ and $T_i^\circ \sqsubseteq R_j$ for $j \geq k$,

then $T_i^\circ \sqsubseteq T^\circ_{f(j)}$ and $f(j) > i$ since $f(j) \geq f(k) > i$

(✱✱)

__corollary of claim__: Since $(B, \sqsubseteq)$ is WQO, by the Nash-Williams theorem the finite subsets $[B]^{<\omega}$ are WQO by the order induced by $\sqsubseteq$ as:

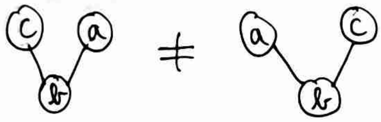
⊛ $\{R_1, \dots, R_n\} \ll \{L_1, \dots L_m\} \iff \exists$ injection $\varphi: R_i \mapsto L_j$ s.t.
$\forall R_i : R_i \sqsubseteq \varphi(R_i)$.

- since $(\Sigma, \leq)$ is WQO there $\exists\, i_1 < i_2 < \cdots$ s.t. the roots $r_{i_1} \leq r_{i_2} \leq \cdots$

$\Rightarrow$ we now use ⊛ on the sequence $B_{i_1}, B_{i_2}, \dots$ to find $l < k$ s.t. $B_{i_l} \ll B_{i_k}$

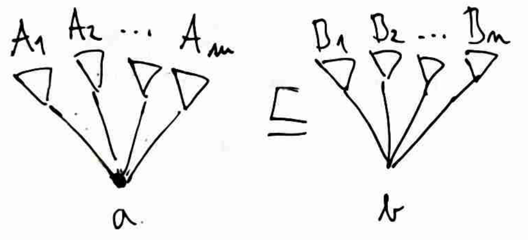
👁 $r_{i_l} \leq r_{i_k}$ & $B_{i_l} \ll B_{i_k} \Rightarrow T^\circ_{i_l} \sqsubseteq T^\circ_{i_k}$ $\Rightarrow (T_i^\circ)_i$ is good ↯ ■

__Remark__: The theorem holds even for __ordered__ labeled rooted trees, that is trees where the order of immediate successors matters:



We need to adjust the definition of the preorder $(T_\Sigma, \sqsubseteq)$ to reflect this,

so we will now also want the following:



$\Longrightarrow$ $\exists\, i_1 < i_2 < \dots < i_m$ s.t.
$A_{i_1} \sqsubseteq B_{i_1}, \dots, A_{i_m} \sqsubseteq B_{i_m}$

👁 This generalized theorem can be proved in the same way, the only difference is that we use Higman's theorem at (✱✱) instead of Nash-Williams.

__What about infinite trees/graphs?__ ...

__Theorem (Nash-Williams, 1965)__: $\leq_t$ is WQO on the class of all (possibly ∞) trees.

__Remark__: These are graph trees, so connected acyclic graphs. They correspond to model-theory trees of height $\leq \omega$. Galvin showed that $\leq_t$ is not WQO for height $> \omega$.

__Theorem (Thomas, 1989)__: $\leq_m$ is WQO on any class of graphs (possibly ∞) with a forbidden finite planar minor. Hence if we chose the minor to be a $n \times m$ grid , we get that $\leq_m$ is WQO on any class of graphs with bounded tree-width.

Kruskals theorem and Γ_0

$\rightarrow (\Sigma, =)$ is WQO

Theorem $\boxed{1}$ **(Kruskal)**: Σ finite $\Rightarrow (T_\Sigma, \preceq)$ is WQO

Theorem $\boxed{2}$ **(Friedman)**: Σ finite $\Rightarrow$ for $\forall k \geq 2$ exists some large $N \geq 2$ s.t. for any finite sequence $t_1, t_2, \dots t_N$ of trees in T_Σ with $\forall m \in [N]: |t_m| \leq m$, there $\exists i_1 < i_2 < \dots < i_k$ s.t. $t_{i_1} \preceq t_{i_2} \preceq \dots \preceq t_{i_k}$.

Idea: Kruskal: $\forall$ infinite sequence in T_Σ contains infinite monotone subsequence (from WQO)

Friedman: if we only want finite monotone subsequences, we do not need to look at infinite sequences - some N is enough

Theorem: Neither $\boxed{1}$ nor $\boxed{2}$ for $\Sigma = \{\bullet\}$ can be proved in PA or even in ATR_0.

Proof: Denote by T the set T_Σ, that is all finite rooted trees.

Fact: $\exists$ surjective mapping $h: T \rightarrow \Gamma_0$ s.t. $\forall s, t \in T$: $\overbrace{s \preceq t \Rightarrow h(s) \leq h(t)}^{\circledast}$.

(rooted) topo-minor

We now show that both $\boxed{1}$ and $\boxed{2}$ imply that Γ_0 is well-founded. Hence no proof of $\boxed{1}$ or $\boxed{2}$ can be formalized in ATR_0 as $ATR_0 \nvdash WF(\Gamma_0)$.

$\rightarrow$ assume for contradiction that $\exists\, \infty$-sequence $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$ of ordinals in Γ_0. Using the fact above and AC_ω there $\exists\, \infty$ sequence $t_1, t_2, t_3, \dots$ of trees in T such that $\forall i: h(t_i) = \alpha_i$. Now:

$\boxed{1}$ $(T, \preceq)$ is WQO $\Rightarrow \exists i < j$ s.t. $t_i \preceq t_j \overset{\circledast}{\Rightarrow} \alpha_i = h(t_i) \leq h(t_j) = \alpha_j$

$\rightarrow$ contradiction since $i < j \Rightarrow \alpha_i > \alpha_j$

$\boxed{2}$ We can only look at the first N trees in the sequence (for $k = 2$)

$\rightarrow$ we again get $i < j$ s.t. $t_i \leq t_j \Rightarrow \alpha_i \leq \alpha_j$ ↯

■

Thus Γ_0 is well founded.

Theorem: The function $F: k \mapsto N$ from theorem $\boxed{2}$ cannot be proved total recursive by PA.

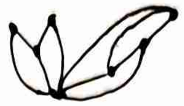
Proof: Recall that PA can prove that F is total recursive $\Leftrightarrow F$ is dominated by some f_α for $\alpha < \varepsilon_0$

Fact: F dominates f_{ε_0}, in fact $F \sim f_\alpha$ for some $\alpha > \Gamma_0$ ■

WQO USING INDUCED SUBGRAPHS

Def: The *tree depth* of a graph G is

$$td(G) := \text{min height of a rooted forrest } F \text{ s.t. } G \overset{\text{subgraph}}{\subseteq} \underset{\text{ancestral relation}}{\text{Closure}(F)}$$

Closure () = 

Closure () =  $= K_4$

tree width ~ similarity to a tree
tree depth ~ similarity to a star

👁 $\forall G$: $td(G) \leq |V_G|$ ... since $\text{Closure}(P_n) = K_n$

👁 $H \subseteq G \Rightarrow td(H) \leq td(G)$

- empty graph E_n: $td(\overbrace{\bullet\bullet\bullet\bullet\bullet}^{n}) = 1$ with forrest $F = \overbrace{\bullet\bullet\bullet\bullet\bullet}^{n}$
- complete graph K_n: $td(K_n) = n$

👁 $\forall G\ \exists$ forrest F with $|V_F| = |V_G|$ of height $td(G)$ s.t. $G \subseteq \text{Closure}(F)$
↳ take any min height forrest – if we did not use some vertex, then we can remove it, and we will still have some forrest, height cannot increase

👁 if G has components $G_1, \dots, G_k$, then $td(G) = \max_i td(G_i)$

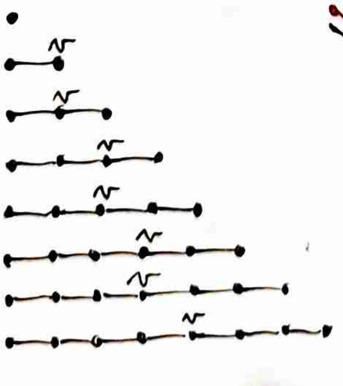
Equivalently: We can define tree depth recursively as

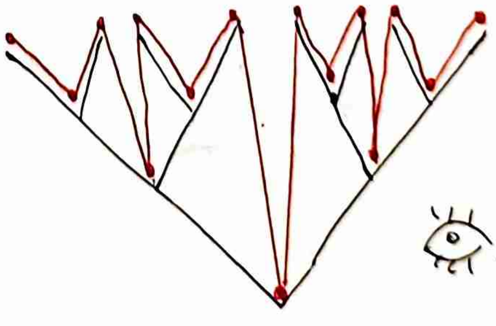
$$td(G) := \begin{cases} 1, & \text{if } |G| = 1 \\ \max\limits_i td(G_i), & \text{if } G \text{ has components } G_1, \dots, G_k \\ 1 + \min\limits_{v \in G} td(G - v), & \text{if } G \text{ is connected} \end{cases}$$

Intuition: We place the vertex v into the root of the tree and place each component G_i into a subtree of height $td(G) - 1$

Example: Paths on n vertices

n	td
1	1
2	1
3	2
4	2
5	2
6	3
7	3
8	3
9	3





👁 $td(P_n) \approx \log_2(n)$

More precisely: $td(P_{2^m - 1}) = m$
$td(P_{2^m}) = m + 1$

Theorem (Ding, 1992): Induced subgraphs $\le$ are WQO on

① every class of graphs with bounded tree depth

② every class of graphs not containing P_n as a subgraph for some n

👁 $\subseteq$ and $\le_t$ are also WQO on these classes as $\le$ is stronger

Observation: Statements ① and ② are in fact equivalent

TD_n := class of graphs G with $td(G) \le n$

NP_n := class of graphs G s.t. $P_n \nsubseteq G$

① $\iff \forall n: (TD_n, \le)$ is WQO ... hence also any subclass

② $\iff \forall n: (NP_n, \le)$ is WQO ... hence also any subclass

- ① $\Rightarrow$ ② since $NP_n \subseteq TD_n$
  ↳ if $P_n \nsubseteq G$, then $\forall$ component G_i of G: $G_i \subseteq K_{n-1} \Rightarrow td(G_i) \le n-1$
- ② $\Rightarrow$ ① since $TD_n \subseteq NP_{2^n}$
  ↳ if $td(G) \le n$, then $P_{2^n} \nsubseteq G$ since $td(P_{2^n}) > n$ ▨

→ we will in fact prove something more general

Def: Given a set of labels Q, a Q-labeled graph is a graph G together with a function $f: V_G \to Q$. We denote for $v \in V_G$: $G(v) := f(v)$.

Def: If $\mathcal{C}$ is a class of graphs, then $\mathcal{C}(Q)$ is the class of all Q-labeled graphs (G, f) s.t. $G \in \mathcal{C}$.

Def: Given a preorder $(Q, \preceq)$ we define a preorder $\ll$ on $\mathcal{C}(Q)$ as

$$G \ll H \quad \equiv \quad G \le H \ \ \& \ \ \forall v \in V_G: G(v) \le H(\sigma(v)), \text{ where}$$

(**) σ is an isomorphism from G to some $H' \le H$ s.t. $G \cong H'$

👁 If $Q = \{\bullet\}$ then $\ll$ is just the induced subgraphs relation.

Theorem: $(\forall n)$: $(Q, \preceq)$ WQO $\Rightarrow$ $(TD_n(Q), \ll)$ WQO.

Corollary: by letting $Q = \{\bullet\}$ we get the result for standard graphs

Proof: Notice that it is enough to prove the theorem for classes of connected graphs

👁 once we have that, Higman's / Nash-Williams will imply it for general graphs

↳ we will know (Connected graphs $\in TD_n(Q)$, $\ll$) is WQO

→ disconnected graph ~ finite set of connected graphs, and

$$G_1 \ll H_1, \dots, G_k \ll H_k \Rightarrow G_1 \dot\cup \cdots \dot\cup G_k \ll H_1 \dot\cup \cdots \dot\cup H_k \checkmark$$

Hence consider only connected graphs. We proceed by induction on n

- $\underline{n=1}$: the only connected graph with $td = 1$ is K_1, so $(TD_1(Q), \ll)$ is nothing but $(Q, \leq)$, which is WQO
- $\underline{n>1}$: let $G_1, G_2, \dots G_m, \dots$ be an infinite sequence of connected $G_i \in TD_n(Q)$

↳ since $td(G_i) \leq n$ there $\exists v_i \in G_i$ s.t. $\forall$ component of $G_i - v_i$ has $td \leq n-1$

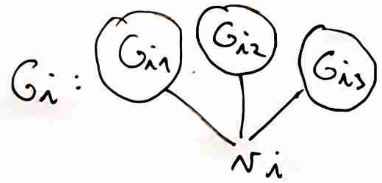
👁 WLOG: $G_1(v_1) \leq G_2(v_2) \leq G_3(v_3) \leq \cdots$

↳ since $(Q, \leq)$ is WQO, $(G_i(v_i))_i$ has an ∞ nondecreasing subsequence

→ for each G_i let $G_{i1}, G_{i2}, \dots, G_{ik_i}$ be the components of G_i

→ let $Q' := Q \times \{0,1\}$ and define a Q'-labeling f_{ij} on G_{ij} as

$$f_{ij}(v) = (G_i(v), e_i(v)), \text{ where } e_i(v) = \begin{cases} 1 & \text{if } vv_i \in E(G_i) \\ 0 & \text{otherwise} \end{cases}$$



and $f_{ij}(v)$ remembers the label of v, but also encodes whether its connected to v_i

→ define a preorder $\leq'$ on Q' as $(q, e) \leq' (q', e') \equiv q \leq q' \;\&\; e = e'$

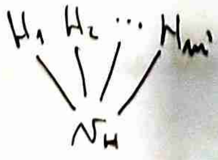
👁 $(Q', \leq')$ is WQO ... since $(Q, \leq)$ and $(\{0,1\}, =)$ are WQO

⇒ induction hypothesis for $n-1$: $TD_{n-1}(Q')$ is WQO by $\ll'$

→ we now use Higman's theorem on the sequence of finite tuples of components $(G_{i1}, \dots, G_{ik_i})$ labeled using Q' by f_{ij}. We obtain $i<j$ s.t.

$$(G_{i1}, \dots, G_{ik_i}) \sqsubseteq (G_{j1}, \dots, G_{jk_j})$$

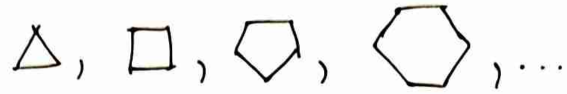
→ to make notation easier let $G_i = G$, $H_i = H$, $v_i = v_G$, $v_j = v_H$, $k_i = m$, and $k_j = m'$. So $\exists i_1 < \cdots < i_m$ s.t. $G_1 \ll' H_{i_1}, \dots, G_m \ll' H_{i_m}$



👁 $G \ll H$ ... $G_i \ll' H_{ij}$ means that $G_i \ll H_{ij}$ and $\forall v \in G_i$: $vv_G \in E(G) \Leftrightarrow \overset{(**)}{\sigma_i(v)} v_H \in E(H)$ ■

→ we have already observed that the graphs ...

C_n:



F_n:

are antichains for both $\subseteq$ and $\leq$

→ we have also seen that the classes of graphs NP_n and TD_n are WQO for both $\subseteq$ and $\leq$

↱ $G \in \mathcal{C}$ & $H \subseteq G \Rightarrow H \in \mathcal{C}$

Theorem (Ding, 1992): Let $\mathcal{C}$ be a subgraph closed class of graphs. Then the following are equivalent:

① $(\mathcal{C}, \subseteq)$ is WQO

② $(\mathcal{C}, \leq)$ is WQO

③ $\mathcal{C}$ contains only finitely many graphs C_n and F_n

Intuition: ② ⇒ ① and ① ⇒ ③ are clear. ③ ⇒ ② is the hard part

👁 TD_n and NP_n are subgraph closed

WQO USING SUBGRAPHS

? are there any interesting classes of graphs which are WQO by $\subseteq$ but not by $\leq$? Clearly such a class can't be $\subseteq$-closed

Recall: $\alpha(G)$ = size of max ind. set
$\omega(G)$ = size of max clique

↱ bounded by some $n \geq 3$

Theorem: $\subseteq$ is WQO on any class of graphs with bounded $\alpha(G)$

Proof: We are given an ∞ sequence $G_1, G_2, \dots, G_n, \dots$

👁 sizes (# vertices) of these graphs are unbounded, otherwise there can't be ∞ many

$\Rightarrow \forall n \; \exists i$ s.t. $K_n \subseteq G_i$ ... from finite Ramsey theorem, we have arbitrarily large graphs with small α ⇒ must be big ω

⇒ hence $G_1 \subseteq G_i$ when we take $n = |V(G_1)|$ ∎

👁 these classes are not $\subseteq$-closed ... deleting all edges boosts α

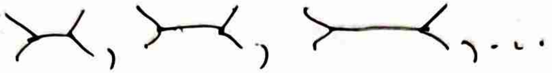
Observation: Classes with bounded α are not WQO by $\leq$.

Proof: $\mathcal{C} = \{G \mid \alpha(G) < n\} = \{G \mid \exists H: K_n \not\leq H \ \& \ G = \overline{H}\}$ $\alpha \geq n$ $\Uparrow$

$= \text{complements of } K_n\text{-free graphs}$

⊛  $G \leq H \iff \overline{G} \leq \overline{H}$ ↳ after taking the complement K_n becomes E_n

↳ ind. subgraphs are injections $G \to H$, where edges go to edges and non-edges go to non-edges

Take complements of  ← K_n-free $\forall n \geq 3$

↳ since  are antichain for $\leq$, their complements are as well, thanks to ⊛ ⇒ we have ∞ antichain in $(\mathcal{C}, \leq)$ ▨

GRAPH WQO SUMMARY

- $\leq_m$: all graphs ... $\leq_\epsilon$ does not work ⇒ neither $\subseteq, \leq$

$\Uparrow$

- $\leq_\epsilon$: all trees ... $\subseteq$ does not work ⇒ neither $\leq$

$\Uparrow$

- $\subseteq$: graphs with bounded $\alpha(G)$ ... $\leq$ does not work

$\Uparrow$

- $\leq$: graphs with bounded $td(G)$

HOMOMORPHISMS

↱ directed / undirected, can have loops

Def: $G=(V_G, E_G)$, $H=(V_H, E_H)$, $f: G \to H$ is a homomorphism $\equiv$

$f: V_G \to V_H$ s.t. $xy \in E_G \Rightarrow f(x)f(y) \in E_H$; equiv: $f \restriction N_G(x) : N_G(x) \to N_H(f(x))$

Def: Isomorphism: $f: V_G \rightarrowtail\!\!\!\!\twoheadrightarrow V_H$ s.t. $xy \in E_G \Leftrightarrow f(x)f(y) \in E_H$;

Lemma: $\exists$ injective homomorphism $G \to H$ $\Leftrightarrow$
$\exists$ isomorphism $G \to H'$, $H' \leq H$

(only take the stuff from H we need)

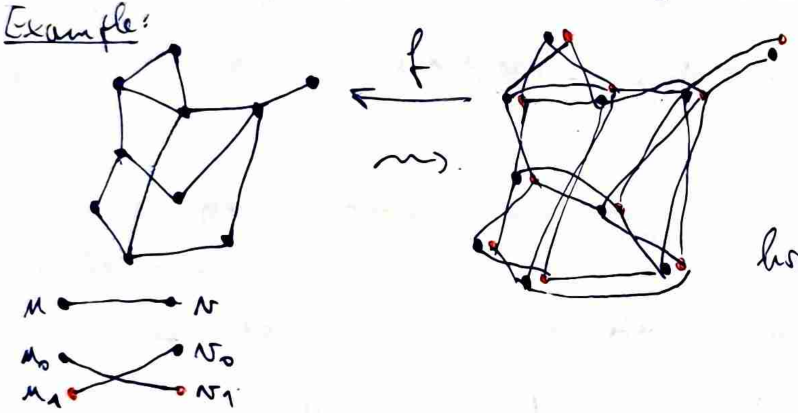
👁 $f: G \to H$ is an isomorphism $\Leftrightarrow$ f is bijective & f, f^{-1} are homomorphisms.

Def: Covering map: $f: V_G \twoheadrightarrow V_H$ s.t. $\forall x \in V_G$: $f \restriction N_G(x) : N_G(x) \rightarrowtail\!\!\!\!\twoheadrightarrow N_H(f(x))$

👁 Covering maps are homomorphisms which are locally isomorphic

👁 Isomorphisms are injective covering maps

Example:



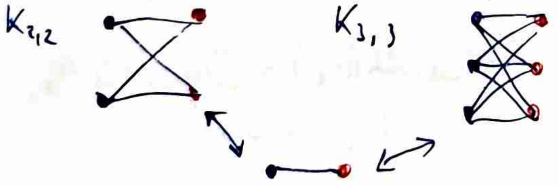
$f: v_0 \mapsto v$, $v_1 \mapsto v$

↳ covering map

homomorphism { surjective ✓
$u_i v_j \in E \Rightarrow uv \in E$ ✓

👁 local isomorphism 

Def: G and H are homomorphically equivalent $\equiv$ $G \to H$ and $H \to G$



... all $K_{n,m}$ are homo-equivalent

Def: $G' \leq G$ is a retract of G $\equiv$ $\exists$ hom. $G \to G'$ s.t. $f \restriction G' = Id$ .. $f(v) = v$ (G')

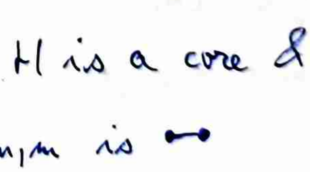
Def: G is a core $\equiv$ G is the only retract of G

$H \to G$ ∵ $H \leq G$

👁 •—• is a core which is a retract of any $K_{n,m}$

Theorem: $\forall$ finite G $\exists$ unique (up to isomorphism) $H \leq G$ s.t. H is a core & $G \to H$

👁 All K_n and C_{2n+1} are cores, the core of any $K_{n,m}$ is •—•

! However, if $\ell < k$, then $C_{2k+1} \to C_{2\ell+1}$ 

Connection to vertex colorings:

$\nearrow V(K_k) = [k]$.

Thm: $f: G \to [k)$ is a k-coloring of G $\iff$ $f: G \to K_k$ is a homomorphism

$\Rightarrow$: u — v $\Rightarrow$ $\overset{u}{\bullet}$ v have different colors c_u, c_v, $f(u) = c_u \in K_k$, $f(v) = c_v \in K_k$ and since $c_u \neq c_v$ $\Downarrow$ $c_u c_v \in E(K_k)$ ∎

$\Leftarrow$ no two adjacent vertices can be mapped on the same color as then $uv \in E \not\Rightarrow c_u c_v \in E$ ↯

Corollary: If $\chi(G) = 3$ & $K_3 \leq G$, then $Core(G) = K_3$
If $\chi(G) = k$ & $K_k \leq G$, then $Core(G) = K_k$

$\boxed{\chi(G) \leq k \iff G \to K_k}$

👁 $\chi(G) \leq k$ & $H \to G$ $\Rightarrow$ $\chi(H) \leq k$ … $H \to G \to K_k \Rightarrow H \to K_k$

👁 $G \to H$ $\Rightarrow$ $\chi(G) \leq \chi(H)$ | $\chi(H) < \chi(G) \Rightarrow G \not\to H$

More general colorings: V_H = available colors, $c_1 c_2 \in E_H$ ≡ colors c_1, c_2 are compatible

$G \to H$ … assignment of colors to V_G s.t. $\forall uv \in E_G$ get compatible colors

→ standard vertex colorings: any two different colors are compatible → $H = K_k$

Def: $\text{OddGirth}(G) := \min\{2k+1 \mid C_{2k+1} \leq G\}$

👁 $G \to H$ $\Rightarrow$ $\text{OddGirth}(H) \leq \text{OddGirth}(G)$

$C_7 \not\to C_9$
(není v C_9 uzavřený walk délky 7)

↳ liché cykly G se zobrazují na kratší (nebo rovné) uzavřené walks in H

Theorem (Erdős): $(\forall k, l < \omega)\ \exists G : \chi(G) = k,\ \text{OddGirth}(G) \geq l$

Corollary: ∃ countably many graphs $(G_n)_{n<\omega}$ s.t. $G_i \not\to G_j$ & $G_j \not\to G_i$ $\forall i \neq j$

↳ we just need $i < j \Rightarrow \chi(G_i) < \chi(G_j)$ & $\text{OddGirth}(G_i) < \text{OddGirth}(G_j)$
$G_j \not\to G_i$ $\qquad$ $G_i \not\to G_j$

Proposition: For digraphs $G \to H$ $\iff$ the vertices of G can be partitioned into sets S_x, $x \in V_H$ s.t. the following holds:

① xx is not a loop in H, then S_x is independent in G

② $xy \notin E(H) \Rightarrow (\forall u \in S_x)(\forall v \in S_y) : uv \notin E_G$

Colorings and digraphs

Def: T_k is the oriented version of K_k … $V = [k]$, $ab \in E \equiv a < b$

↳ the k-coloring characterization for digraphs works with T_k

Lemma: Digraph G satisfies $G \not\to \vec{T}_k \iff \vec{P}_k \to G$.

Theorem: Graph G is k-colorable $\iff$ ∃ acyclic orientation of G not containing $\vec{P}_k$

Def: G is rigid $\equiv$ the only $G \to G$ hom. is identity.

Theorem: There $\exists$ a rigid graph on every vertex set

Theorem: For $\forall$ cardinal κ there is a set $\{G_\alpha \mid \alpha < \kappa\}$ of size κ s.t.:

① all G_α are rigid ② $\forall \alpha \neq \beta$: $G_\alpha \not\to G_\beta$ & $G_\beta \not\to G_\alpha$

Vopěnkas principle: If $\mathcal{C} \subseteq \mathcal{G}$ is a proper class, then $\exists H, G \in \mathcal{C}$: $H \to G \vee G \to H$.

Theorem: Equivalently: no sequence of graphs $\langle G_\alpha \mid \alpha \in On \rangle$ has both of the properties:

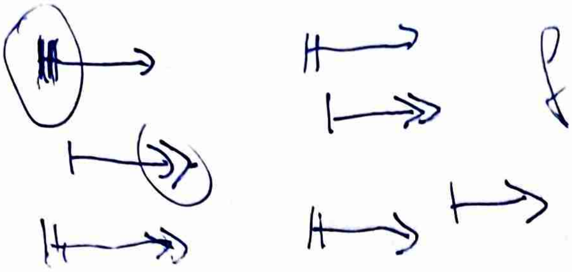
① $\alpha \leq \beta \Rightarrow G_\alpha \to G_\beta$ uniquely ② $\alpha < \beta \Rightarrow G_\beta \not\to G_\alpha$

all G_α are rigid

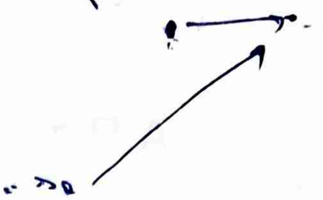
Weak Vopěnkas principle: No sequence of graphs $\langle G_\alpha \mid \alpha \in On \rangle$ has both of the properties

① $\alpha \leq \beta \Rightarrow G_\beta \to G_\alpha$ (uniquely) ② $\alpha < \beta \Rightarrow G_\alpha \not\to G_\beta$

↳ not needed



$f: x \mapsto x^2$ $f: A \hookrightarrow B$



A Rigid Graph for Every Set

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Abstract: A graph G is called *rigid* if the identical mapping $V(G) \to V(G)$ is the only homomorphism $G \to G$. In this note we give a simple construction of a rigid oriented graph on every set. © 2002 John Wiley & Sons, Inc. J Graph Theory 39: 108–110, 2002

Keywords: *rigid graphs; homomorphisms; set-theory*

1. CONSTRUCTION

Let X be an infinite set and assume that X is an ordinal $X = \{\beta; \beta \leq \alpha\}$. Let $X' = \{\beta'; \beta' \leq \alpha'\}$ be a disjoint copy of X. Further let $\{a, b, c, a', b', c'\}$ be six vertices disjoint with X and X'. For every ordinal $\beta \leq \alpha$ with countable cofinality let us choose a sequence $\{\beta_n\}$ such that supp $\beta_n = \beta$. We define the oriented graph (V, E) by the following set of arcs:

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$(0,a)$, (a,b), (b,c), $(c,0)$ $(b,0)$ and $(0',a')$, (a',b'), (b',c'), $(c',0')$, (a',c');
(β,γ) and (β',γ') for all $\beta < \gamma \leq \alpha$;
(β,β') for all $\beta \leq \alpha$;

for every ordinal $\beta \leq \alpha$ with countable cofinality let β be joined with β'_n by an oriented path of length $n+2$. (All these paths are supposed to be vertex disjoint.)

Let V be the set of all vertices thus obtained. Clearly V is a countable union of sets of cardinality $\leq \alpha$ and thus V and X are in 1–1 correspondence.

Theorem 1.1. *The oriented graph $G=(V,E)$ is rigid.*

2. PROOF

Let $f: V \to V$ be a homomorphism. As X and X' are transitive orientations of complete graphs, f restricted to both X and X' is an injection (Fig. 1). The graph G is acyclic with the exception of vertices $\{a,b,c,0\}$ and $\{a',b',c',0'\}$. The graphs induced by these sets are isomorphic, but there is no homomorphism between them which would preserve vertices 0 and $0'$. However, the mapping f restricted to $\{a,b,c,0\}$ satisfies either $f(0)=0$ or $f(0)=0'$ (as vertices $0,0'$ are distinguished by large clique which includes them). This shows that both $f(0')=0$ and $f(0)=0'$ are impossible and thus $f(0)=0, f(0')=0'$. Consequently f restricted to the set $\{a,a',b,b',c,c'\}$ is the identity. It follows that f maps X to X and X' to X'. As pairs (β,β') are the only arrows between X and X' we have $(f(\beta))'=f((\beta)')$ for every $\beta<\alpha$. Let $\beta(0)$ be the smallest β for which $f(\beta)\neq\beta$. Then necessarily $f(\beta(0))>\beta(0)$ (as, if $f(\beta(0))<\beta(0)$ then $f(f(\beta(0)))<f(\beta(0))$ which violates that $\beta(0)$ was minimal). Thus $\beta(0)<f(\beta(0))$. We put $f(\beta(0))=\beta(1)$ and $\beta(n)=f^n(\beta(0))$ (the n-times iterated mapping f). Let $\beta=\operatorname{supp}\beta(n)$. β is also the limit of the sequence $\{\beta_n\}$. However as the sequences $\beta(n)$ and β_n are interlacing and as f maps the set $\{\beta(n)\}$ into a subset we get by monotonicity that that the sets $\{\beta(n)\}$ and

$f(\Delta)=\omega$

$\beta\Omega$

β_0

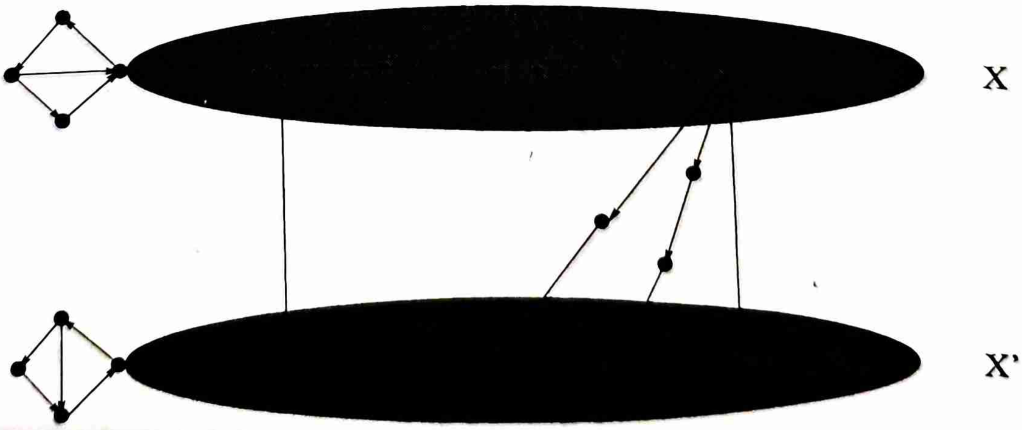

FIGURE 1. The oriented graph $G=(V,E)$ is rigid.

$\{f(\beta_n)\}$ are interlacing again and thus by the definition of the graph G we get $f(\beta) = \beta$ (as β is the only vertex joined to the set $\{\beta'_n\}$ by directed paths). But then $f(\beta_n) = \beta_n$ for every n (as in our situation the mapping f has to preserve the length of paths between β and β'_n). This is a contradiction as if we choose m and n such that $\beta(m) < \beta_n < \beta(m+1)$ then $f(\beta_n) > \beta(m+1)$, a final contradiction.

3. REMARKS

The existence of a rigid graph on every set is an important result which lies in the heart of several combinatorial and non-combinatorial embeddings (see [4]). The problem of existence of rigid relations appeared first in [2] and the existence on every set has been proved by Vopěnka, et al. [5] and reproved in [1] and [4]. All these proofs are modification of the original proof in that they either use embeddings for relational systems with countably many relations (as in [1]) or depend more on the ordinal arithmetic (as in [4], pp. 63–65). This is not necessary as shown by our direct construction.

Perhaps the importance of this result justifies yet another simpler proof. Let us remark that all the proofs (including the present one) are using "fixing" ordinals with countable cofinality (this original idea of [5] is credited to Vopěnka in [4]).

Finally, let us remark that this is an infinite problem as all finite directed paths are rigid. These simplest finite rigid graphs serve as building blocks of our construction. It is important that one can prove the existence of a *rigid graph* on every set in ZFC. This is in a sharp (and surprising) contrast with difficulties when one wants to construct a proper class of *mutually rigid* graphs (see [4] for a disscussion of this).

If we want to construct a rigid undirected graph (or graphs with some special properties) we can replace each edge by one suitable finite undirected rigid graph. As can be expected the same is true if we want to get an acyclic or well founded relation (we use, e.g., rigid balanced orientation of a path, see [3] and [4]).

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A Rigid Graph for Every Set

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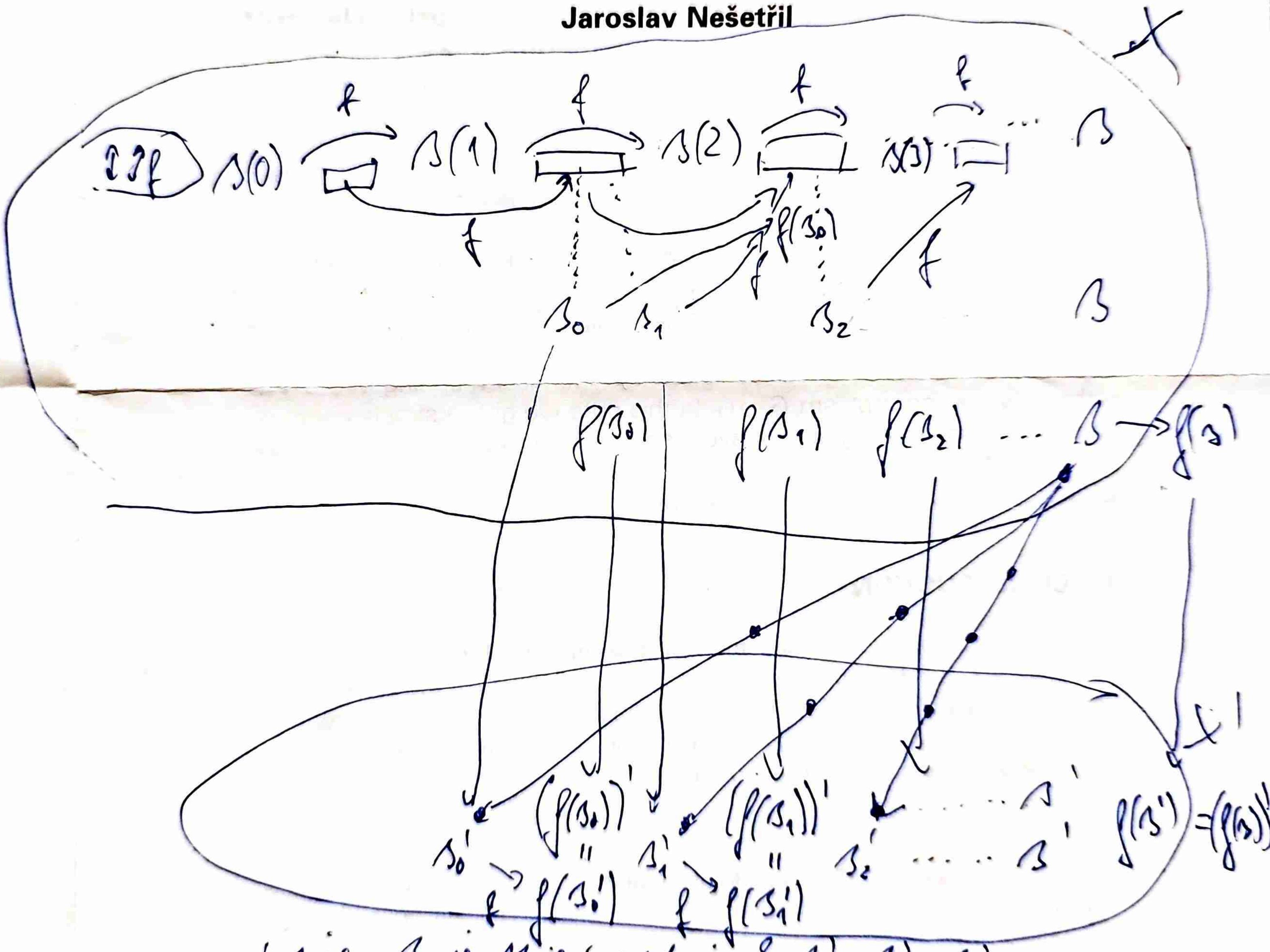

→ protože β je spojený cestou k β_0', β_1', β' ...
$f(\beta)$ musí být k $f(\beta_n') = (f(\beta_n))'$, kofinální posloupnosti $(f(\beta))'$
→ tato posloupnost ale konverguje k β' ∵ $f(\beta_n) \to \beta$, tedy $(f(\beta))' = \beta'$
⇒ $f(\beta) = \beta$

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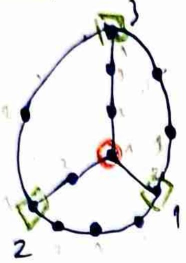
Many pairwise rigid graphs

Def: $\{G_i \mid i \in I\}$ are pairwise rigid $\equiv$ $\forall i$: G_i is rigid & $\forall i \neq j$: $G_i \not\to G_j$

Examples: Finite rigid graphs:



→ there are three C_7 s, each has to map to a C_7

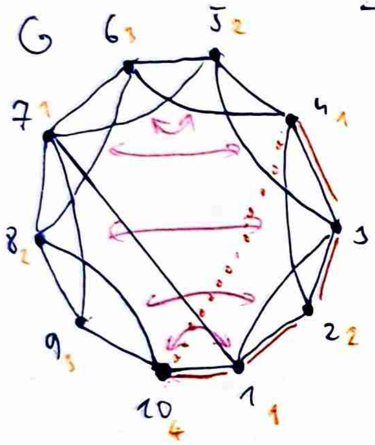
→ C_7 has to map to a closed walk of length 7 ... only C_7 s here, because there is not shorter cycle

- ○ → ○ ∵ it is the only node included in each C_7
- (□, □, □) → (□, □, □) ∵ the only nodes included in two C_7s
- $\square_1 \to \square_1$ ∵ the only □ with $d(○, □) = 1$
  ⇒ from this $\square_2 \to \square_2$ and $\square_3 \to \square_3$

PROBABLY NOT SUFFICIENT ARGUMENTATION

→ from this we do the rest of the nodes

10 vertices



$\chi(G) = 4$, 👁 remove any vertex ⇒ $\chi = 3$

colors

⇒ any $f: G \to G$ is surjective (⇒ bijective)

↳ we have to use all vertices as if $\chi(G') < \chi(G)$ then $G \not\to G'$

also maybe here

👁 $1 \mapsto 1$ or 10, $2 \mapsto 2$ or 9, $3 \mapsto 3$ or 8, ...

↳ triangles have to go to triangles and
1, 10 ∈ only 1 Δ
2, 9 ∈ only 2 Δ
3...8 ∈ 3 Δ

⇒ we want to prevent the homomorphism shown as ↷

⇒ add the edge {1, 7}, everything said before still works ∵ the new cycle is = mean = triangle

→ the pink homomorphism fails because the red ... edge is missing → $f(1) f(7) \notin E$

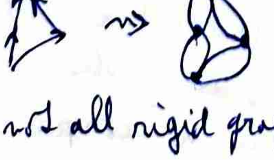
👁 This construction allows us to do arbitrary large rigid graphs, just make the cycle longer ... 7, 10, 13, 16, 19, 22, ...

Replacing edges by indicator graphs

Def: For digraph $\vec{G}$ and (connected) rigid graph I define by $\vec{G} * I$ the graph obtained from $\vec{G}$ by replacing each arc by I

→ indicator

Idea: when making a $\vec{G} \to \vec{G}$-hom, arcs are fundamental units, now I is the fundamental unit and I has to map to another I in the same way



I serves as an oriented edge

→ probably not all rigid graphs work as indicators and how we find them matters

👁 # countable graphs $= 2^\omega$ ... functions $[\omega]^2 \to (0,1)$

Theorem: There is a set of 2^ω countable graphs ↳ 2 element sets, $[\omega]^2 \approx \omega$
s.t. all graphs are pair-wise rigid (and themselves rigid)

Pf: Take any rigid digraph of size ω

$\overrightarrow{H}$

Take $G := \overrightarrow{H} * I$ for any indicator I to get a rigid graph G

Define $\mathcal{G} := \{ \overrightarrow{G}_\sigma \mid \overrightarrow{G}_\sigma$ is an orientation of $G\}$

👁 all $\overrightarrow{G}_\sigma$ are rigid ... otherwise the underlying G would not be

👁 $\overrightarrow{G}_\sigma$ are pairwise-rigid ... if not, after forgetting the orientation we would get a homomorphism $G \to G$

Define ⊛ $\{ \overrightarrow{G}_\sigma * I \mid \overrightarrow{G}_\sigma \in \mathcal{G}\}$ ... still all pairwise rigid
↳ the size of this set $= |\mathcal{G}| = 2^\omega$

▨

👁 If we started with a graph of size $\mathcal{K}$, then we would get that:

Thm: There are $2^\mathcal{K}$ graphs of size $\mathcal{K}$ & there $\exists$ a set of $2^\mathcal{K}$ pairwise-rigid graphs of size $\mathcal{K}$.
↳ just use a starting graph of size $\mathcal{K}$

! Nešetřil uses a clever construction to define a $\overbrace{\text{rigid}}$ graph on $\mathcal{K}$

→ idea of taking the 2^ω many countable graphs from ⊛ and making a disjoint union to get a rigid graph of size 2^ω and iterating this process of the above proof to get large rigid graphs fails

→ suppose we have 2^ω many pairwise rigid digraphs $\mathcal{G}$ as in the proof

⇒ define $\overrightarrow{H}_1$ as the disjoint union of these digraphs ... 👁 $|\overrightarrow{H}_1| = 2^\omega$

$\overrightarrow{H}_1 = 0000$ ... only 4 for simplicity ↪ 👁 $\overrightarrow{H}_1$ is rigid

→ iterating: $G_1 := \overrightarrow{H}_1 * I$ and $\mathcal{G}_1 := \{ \overrightarrow{G}_{1\sigma} \mid \overrightarrow{G}_{1\sigma}$ orientation of $G_1\}$

→ all good so far. However, imagine we want to go to $\overrightarrow{H}_2$

$\overrightarrow{H}_2 = \overbrace{\underset{\sigma_1}{0}000}\ \overbrace{\underset{\sigma_1}{0}000}\ 0000 \ \dots$ The blocks ~ orientations of H_1

$\underset{\sigma_1}{0} \curvearrowright \underset{\sigma_1}{0}$ the total orientation of ⊓ and ⊓ is different,
→ even though the first cell is oriented the same → can swap them
⇒ $\overrightarrow{H}_2$ is no longer rigid

Def: Relační struktura na množině X je $(X \mid \varepsilon_1, \varepsilon_2, \dots \varepsilon_n)$ kde ε_i jsou relace s aritami $ar_1, \dots ar_n$.

Def: Homomorfismus mezi dvěma strukturami X, Y se stejnou signaturou je

$f : X \to Y$ že $\forall (x' \subseteq X) : x' \in \varepsilon_i^X \Rightarrow f[x'] \in \varepsilon_i^Y$

Thm: $\exists$ relační struktura t.ž. jedna z arit je neomezená, ale můžu jich

Pf: udělat vlastní třídu mnoho a všechny to bude navzájem strnulé

$S_\alpha = (\alpha \mid R_\alpha, K_\alpha, \{\alpha\})$ → relace arity α obsahující jediný prvek, a sice $\{\delta \mid \delta < \alpha\}$

kardinál ↳ strnulá na α ↳ úplný graf na α

Suppose $S_\alpha \to S_\beta$! α, β jsou kardinály

a) $\alpha = \beta$, pak je to identita $\because$ R_α je strnulá

b) $\alpha < \beta$, pak bychom chtěli aby $\forall x \in \{\alpha\}$ bylo $f[x] \in \{\beta\}$,

čili $\alpha = \{\delta \mid \delta < \alpha\} \rightsquigarrow \{f(\delta) \mid \delta < \alpha\} \in \{\beta\}$ což nejde $\because$

$|\beta| > |\alpha| \rightsquigarrow$ kardinály

c) $\alpha > \beta$, pak bychom chtěli aby

$K_\alpha \to K_\beta$ ... graf s větším X na graf s menším X,

to ale nejde → v rámci Car ▨

Thm: VP $\Longleftrightarrow$ pro relační struktury s omezenými aritami nejde udělat vlastní třídu vzájemně strnulých struktur

→ relační struktura s omezenými aritami lze embedovat do grafů

→ neomezené arity $\Rightarrow$ lze sestrojit vlastní třídu mnoho (nahoře)

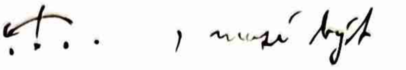
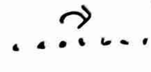
(VP$_1$): $\forall \langle G_\alpha \mid \alpha \in On \rangle$ ~~G_α strnulé~~ $\exists \alpha < \beta$: $G_\alpha \to G_\beta$ } WQO třídně

(VP$_2$): $\forall \langle G_\alpha \mid \alpha \in On \rangle$ ~~G_α strnulé~~ $\exists \alpha < \beta$: $G_\alpha \subseteq G_\beta$ nebo $G_\alpha \leq G_\beta$ (*)

(VP$_3$): $\forall \langle G_\alpha \mid \alpha \in On \rangle$ G_α strnulé $\exists \alpha < \beta$ $G_\alpha \to G_\beta \lor G_\beta \to G_\beta$

(VP$_1$): $\nexists$  , musí být 

↳ vlastní třída vzájemně rigid

(VP$_3$) $\nexists$  , musí být  nebo 

(*) silnější, protože $\leq$ jsou obecnější

verze $\subseteq$ $\Leftarrow$ verze $\leq$

Thm: (VP$_1$) $\Longleftrightarrow$ (VP$_2$) $\Longleftrightarrow$ (VP$_3$)

② $\Rightarrow$ ① , ① $\Rightarrow$ ③ obviously

③ $\Rightarrow$ ② ... obměnou: $\neg$② $\Rightarrow$ $\neg$③

→ předpokládejme že $\exists \langle G_\alpha \mid \alpha \in On \rangle$ t.ž. $\forall \alpha < \beta$: $G_\alpha \nleq G_\beta$ dokazujeme to silnější

→ pak najdeme vlastní třídu vzájemně strnulých grafů

→ $G_\alpha = (V_\alpha, E_\alpha)$ rozšířím o jiné druhy "barvy" hrany

- $E^1_\alpha = \binom{V_\alpha}{2}$ ... úplný graf
- $E^2_\alpha = \binom{V_\alpha}{2} \setminus E_\alpha$ ... doplněk G_α
- E^3_α = strnulík na V_α

} celkem supergraf G^*

↳ různé hrany lze kódovat přes indikátory

- pokud mám $f: G^*_\alpha \to G^*_\alpha$ homomorfismus, pak je to identita díky E^3_α
- pokud mám $f: G^*_\alpha \to G^*_\beta$ homomorfismus: ↳ G^*_α jsou strnulé
  - díky E^1_α to musí být injekce... nemůžu zobrazit dva vrcholy na stejný vrchol, jinak by mi vznikla smyčka, nejde kvůli úplnému grafu
  - homomorfismus zachovává hrany, tedy
    - nalevo hrana ⇒ napravo hrana (**)

  E^2_α vynucuje: nalevo nehrana ⇒ napravo nehrana

! omezím se jenom na G_α kde α je kardinál, počet jich je On mnoho

- pokud $G^*_\alpha \to G^*_\beta$, tak $\beta \geq \alpha$, protože to je injekce a α, β kardinály

→ $\alpha = \beta$ již vyřešené, zajímá nás $\alpha < \beta$

(**) G^*_α je indukovaný podgraf G^*_β podle toho homomorfismu, takže $G^*_\alpha \leq G^*_\beta$ pro $\alpha < \beta$, spor ↯

- kdybych dokazoval místo $\leq$ pro $\subseteq$ tak nepotřebuju E^2_α
- kdyby pro homomorfismy (vlastně ③ → ①) tak taky nepotřebuju E^2_α ▨