ABSOLUTENESS

<u>Goal</u>: Given a model V of ZFC, modify it to get a model of ZFC + φ

$\Rightarrow$ showing that ZFC + φ is consistent relative to ZFC

for example: $\varphi = \neg CH$

<u>How to make a new model</u>

① remove some sets $\rightsquigarrow$ L = <u>constructible universe</u>

② add new sets $\rightsquigarrow$ forcing

<u>Def</u>: Let V be a model of ZFC. We say that $W \subseteq V$ is a <u>transitive class</u> if

$$x \in W \quad \& \quad V \models y \in x \quad \Rightarrow \quad y \in x$$

<u>Def</u>: A formula φ in the language of ZFC is called <u>Δ_0</u> if it is equivalent to a <u>bounded formula</u>, i.e. one in which quantifiers have the form $\forall x \in y$ or $\exists x \in y$.

<u>Example</u>: The following statements are Δ_0:

- <u>x is an ordinal</u> ... transitive set well-ordered by $\in$
- "x is a function", "x is a function and $y = \mathrm{Dom}(x)$", <u>"x is a real number"</u> $\hookrightarrow$ a function $\omega \to 2$
- $x = n$ for some fixed $n \in \omega$
- <u>$x = \omega$</u> ... x is inductive, an ordinal, and every $\alpha \in \omega$ is finite

<u>Example</u>: The following are <u>not</u> Δ_0:

- $x = \omega_1$
- $x = P(y)$

<u>Proposition</u>: Let $\varphi(u_1, \dots, u_n)$ be a Δ_0 formula and $W \subseteq V$ a transitive class.

Then $\forall x_1, \dots, x_n \in W$: $\quad V \models \varphi(x_1, \dots, x_n) \iff W \models \varphi(x_1, \dots, x_n)$

$\rightarrow$ we say that Δ_0 formulas are <u>absolute</u>

<u>Idea</u>: If $\alpha \in V$ is an ordinal and $W \subseteq V$ is a transitive submodel of V containing α, then α is an ordinal in W as well, but it could have gotten bigger or smaller. But since ω and $n \in \omega$ are absolute, their values cannot change.

<u>Proof</u>: By structural induction on φ. For atomic formulas $x \in y$ and $x = y$ this is obviously true and checking connectives $\neg\varphi$, $\varphi \wedge \psi$ is easy (Note that $\varphi \vee \psi \sim \neg(\neg\varphi \wedge \neg\psi)$ and $\varphi \to \psi \sim \neg\varphi \vee \psi$)

If $\varphi \sim (\forall y \in x)\psi$ and $x \in W$, then by transitivity $x \subseteq W$.
Thus $W \models \psi(y) \iff V \models \psi(y)$, so $W \models \varphi(x) \iff V \models \varphi(x)$
Similarly for $\varphi \sim (\exists y \in x)\psi$. ■

Def: A formula φ is called

Lévy hierarchy

1) Σ_0 or Π_0 if it is Δ_0
2) $\underline{\Sigma_{n+1}}$ if there exists a Π_n formula ψ s.t. $\varphi \sim (\exists x)\psi$
3) $\underline{\Pi_{n+1}}$ if there exists a Σ_n formula ψ s.t. $\varphi \sim (\forall x)\psi$

Def: φ is $\underline{\Delta_n}$ if it is equivalent to both a Σ_n and Π_n formula
↳ Δ_1 is both $(\forall x)\psi_1$ and $(\exists x)\psi_2$ where ψ_1, ψ_2 are Δ_0

Examples of Δ_1 formulas:
- R is a well-order on X
- x is a finite set

Proposition: Δ_1 formulas are absolute.

Proof: Let $W \subseteq V$ be transitive. We will show that $\Big\langle$ Σ_1 are upward absolute; Π_1 are downward absolute

- let $(\exists x)\psi$ be Σ_1 with $\psi \in \Delta_0$. Assume that $W \models (\exists x)\psi(x)$.
  Then there is some $a \in W$ s.t. $W \models \psi(a)$. Since ψ is absolute, also $V \models \psi(a)$ and $V \models (\exists x)\psi(a)$
- let $(\forall x)\psi$ be Π_1 with $\psi \in \Delta_0$. Assume that $V \models (\forall x)\psi(x)$
  Then for every $a \in W$ we have $V \models \psi(a)$ and by absoluteness $W \models \psi(a)$ and thus $W \models (\forall x)\psi(x)$ ■

Fact: Δ_2 formulas are no longer guaranteed to be absolute
↳ "$x = \omega_1$" is Δ_2 but not absolute

- x is an ordinal ... Δ_0
- x is uncountable ... Π_1 ... $(\forall f)\neg(f$ is function with $\mathrm{Dom}(f) = \omega \wedge \mathrm{Rng}(f) = x)$
- every $\alpha \in X$ is countable ... Σ_1 ... $(\exists F)$ (F is a function with $\mathrm{Dom}(F) = X$ and $\forall \alpha \in X$ $F(\alpha)$ is a function with $\mathrm{Dom} = \omega \wedge \mathrm{Rng} = \alpha$)

IDEA OF FORCING

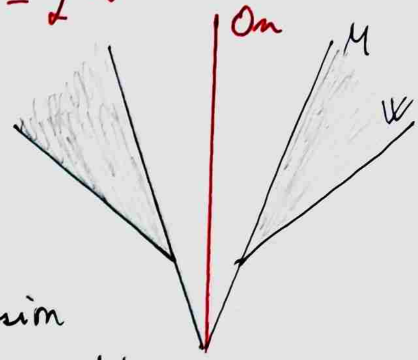
→ given $M \models ZFC$, we will make a transitive extension $W \supseteq M$ s.t.

$W \models ZFC + \varphi$ → φ for example CH or $\neg$CH
↳ $\aleph_1 = 2^{\aleph_0}$

→ but always so that $On^{(M)} = M \cap On^{(W)} = On^{(W)}$

↳ we do not add any new ordinals

⇒ we make the universe "wider", not "taller"



Strategy: Pick a set $P \in M$ and a suitable new object $G \subseteq P$ outside of M. From the we build an extension $M[G] \supseteq M$ consisting of all sets definable from G over M.

? How to choose G so that $M[G] \models ZFC$?

- How do we ensure that $M[G] \models \varphi$?

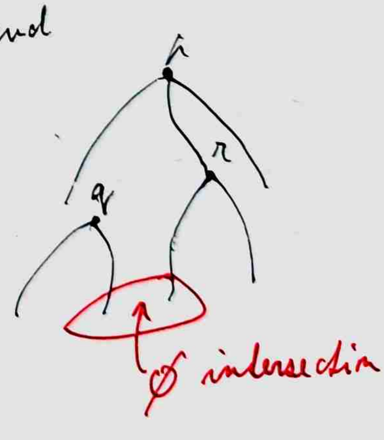
Theorem (Cohen, 1963): If P is a poset and G a V-generic filter on P, then $V[G] \models ZFC$.

Theorem (Balcar–Štěpánek, 1967) If $V \subseteq W$ are transitive models of ZFC with the same ordinals and W was generated from V by adding some G, then (essentially) G is a V-generic filter on some $P \in V$

POSETS

Def: $(\mathbb{P}, \leq)$ is a poset if $\leq$ is reflexive and transitive.

- $p \parallel q$ are compatible if $\exists r \in P$ s.t. $r \leq p$ & $r \leq q$
- $p \perp q$ are orthogonal otherwise ... no common lower bound



Def: $(\mathbb{P}, \leq)$ is a nice poset if it is also

- antisymmetric ... $p \leq q \leq p \Rightarrow p = q$
- separative ... $p \not\leq q \Rightarrow \exists r \leq p$ s.t. $r \perp q$
  ↳ specifically: $q < p \Rightarrow \dots$
- has a maximum denoted $\mathbb{1}_{\mathbb{P}}$
  ↳ $p \leq \mathbb{1}$ for $\forall p$

Def: Let $(\mathbb{P}, \leq)$ be a poset. A subset $F \subseteq P$ is

- upwards closed $\equiv$ $p \in F$ & $p \leq q \Rightarrow q \in F$
- downwards closed $\equiv$ $p \in F$ & $q \leq p \Rightarrow q \in F$ ← also open
- dense $\equiv (\forall p \in \mathbb{P})(\exists d \in F): d \leq p$
- open dense $\equiv$ it is dense & downwards closed

- *predense* $\equiv (\forall p \in \mathbb{P})(\exists d \in F): d \parallel p$ → p, q are compatible in F
- a *filter* $\equiv$ it is upwards closed and $(\forall p, q \in F)(\exists r \in F)$ s.t. $r \le p$ & $r \le q$
- an *antichain* $\equiv \forall p \neq q$ in F are orthogonal in $\mathbb{P}$

Def: $a \in \mathbb{P}$ is an *atom* if it is a minimal element.

Intuition: In forcing, $\mathbb{P}$ represents a set of *conditions* $p \in \mathbb{P}$, and we call $\mathbb{P}$ itself a "forcing notion" using which we "force".

- $p \le q$ ... condition p is *stronger* than condition q
- $p \parallel q$ ... p and q can be extended into a stronger condition carrying information of both p and q
- $p \perp q$ ... p and q contradict each other ... they are *incompatible*
- $\mathbb{1}$ ... *trivial condition* carrying no information
- atom ... a condition so strong that it cannot be extended
- separative poset ... if p is not stronger than q then p can be extended into a condition r that is incompatible with q
- dense set ... containing arbitrarily strong conditions
- predense set ... every condition is compatible with something in the set

Convention: From now on, by "poset" we mean "nice poset" unless stated otherwise.

Def: Let $(\mathbb{P}, \le)$ be a poset. A filter $G \subseteq \mathbb{P}$ is *V-generic* if $G \cap D \neq \emptyset$ for every dense $D \subseteq \mathbb{P}$ with $D \in V$

If $a \in \mathbb{P}$ is an atom, then the principal filter $F_a := \{p \in \mathbb{P}, p \ge a\}$ is V-generic
↳ every dense set contains all atoms

! From now on filters cannot contain any atoms

Proposition: If $\mathbb{P}$ is atomless, then there is no V-generic filter $F \in V$.

Proof: Let $F \in V$ be a filter on $\mathbb{P}$. We claim that $D := \mathbb{P} \setminus F$ is dense so F is not V-generic

→ take any $r \in \mathbb{P}$ ... since $\mathbb{P}$ is atomless there $\exists\, a < r$

→ since $r \not\le a$, by separativity there is $b \le r$ with $b \perp a$ ↗ $a, b \le r$

→ since a and b are incompatible, F cannot contain both so at least one $\in D$ ∎

⇒ therefore if we choose $\mathbb{P}$ atomless and $G \subseteq \mathbb{P}$ a V-generic filter on $\mathbb{P}$, then we can be sure that G is outside the model V !

Fact: We can assume that for $\forall \mathbb{P}$ and $p \in \mathbb{P}$ there $\exists$ V-generic G (not in V) s.t. $p \in G$.

THE FORCING THEOREM

→ this will all be very informal and imprecise

Fact: For every $x \in V[G]$ there is a name $\dot{x} \in V$.

↳ name ~ recipe how to build x from G over V

→ a poset $\mathbb{P}$ defines a class $V^{\mathbb{P}} \subseteq V$ of $\mathbb{P}$-names

→ we also define a check function $\check{}: V \to V^{\mathbb{P}}$, $z \mapsto \check{z}$

→ given a V-generic filter $G \subseteq \mathbb{P}$ define an equivalence $=_G$ on $V^{\mathbb{P}}$

↳ our model is $V[G] \cong (V^{\mathbb{P}}/=_G, \in_G)$ ... elements = equivalence class of $V^{\mathbb{P}}$

↳ $V \subseteq V[G]$ by the check function $z \mapsto$ equivalence class containing $\check{z}$

→ by "evaluating a name $\dot{x}$" we mean taking the equivalence class containing $\dot{x}$

⇒ more formally later, not important right now

Theorem (The Forcing Theorem): Consider a formula $\varphi(u_1, \dots, u_n)$ and a poset $\mathbb{P}$. Then for every fixed choice of $\mathbb{P}$-names $\dot{x}_1, \dots, \dot{x}_n \in V^{\mathbb{P}}$, there exists a set $D \subseteq \mathbb{P}$ with $D \in V$ s.t. for every V-generic filter G on $\mathbb{P}$ we have

$$V[G] \models \varphi(\dot{x}_1/G, \dots, \dot{x}_n/G) \iff G \cap D \neq \emptyset$$

↳ evaluation of names

Idea: There are many choices of G so it seems like the relationship of whether φ holds in $V[G]$ should be very complicated

⇒ but if we fix φ and names in the ground model V, then the relationship is remarcably simple and can be witnessed from V via D

Notation: The set D is the "set of conditions $p \in \mathbb{P}$ that force $\varphi(\dot{x}_1, \dots, \dot{x}_n)$"

↳ we denote it by $D_{\varphi(\dot{x}_1, \dots, \dot{x}_n)}$ or simply D_φ

informally: $D_\varphi = \{p \in \mathbb{P} \mid p \in G \Rightarrow V[G] \models \varphi(\dot{x}_1/G, \dots, \dot{x}_n/G)\} \in V$

If $p \in D_\varphi$, we write $p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$ and read "p forces $\varphi(\dot{x}_1, \dots, \dot{x}_n)$"

Note: Abuse of notation ... if φ has a free variable u, so $\varphi = \varphi(u)$ and we write $p \Vdash \varphi$ then we mean "for every name $\dot{x} \in V^{\mathbb{P}}$: $p \Vdash \varphi(\dot{x})$"

! φ will usually have no free variables so it does not matter

Observation:

① D_ψ is downwards closed ... so stronger conditions also force ψ

② $p \in D_\psi$ & $q \in D_{\neg\psi} \Rightarrow p \perp q$ ... compatible conditions cannot force contradictory statements

③ D_ψ dense $\Rightarrow V[G] \models \psi$ for every choice of G

④ $D_\psi \dot\cup D_{\neg\psi}$ is dense

⑤ if $\varphi \Rightarrow \psi$ then $D_\varphi \subseteq D_\psi$

② more generarily: $p \in D_\psi$, $q \in D_\varphi$ and ψ, φ are mutually exclusive

Proof: ① because filters are upwards closed - if $p \in D_\psi$ and $q \leq p$ then $q \in G \Rightarrow p \in G \Rightarrow V[G] \models \psi$

② if there was some $r \leq p, q$ then $r \in D_\psi \cap D_{\neg\psi}$, so $r \in G \Rightarrow V[G] \models \psi \wedge \neg\psi$, which is impossible

③ because a V-generic filter intersects every dense set

④ fix $p \in \mathbb{P}$ and let G be a V-generic filter containing p ← see Fact two pages back

→ since $V[G]$ is a specific model, either $V[G] \models \psi$ or $V[G] \models \neg\psi$

WLOG $V[G] \models \psi$, so $G \cap D_\psi \neq \emptyset$ ... let $q \in G \cap D_\psi$

→ since $p, q \in G$, there $\exists r \in G$ s.t. $r \leq p, q$. Since $q \in D_\psi$, by ① also $r \in D_\psi$

⑤ If $p \in G \Rightarrow V[G] \models \varphi$ and $\varphi \Rightarrow \psi$ then $p \in G \Rightarrow V[G] \models \psi$ ▨

THE CONTINUUM HYPOTHESIS

We will try to prove the consistency of $ZFC + \varphi$ where φ is

CH : There exists a surjection $f: \omega_1 \to \mathbb{R}$

$\neg$CH : There exists $C \subseteq \mathbb{R} \approx 2^\omega$ such that $|C| = \aleph_2$

exactly what we want

! From the two cases

① $\varphi = (\exists x)\psi$

② $\varphi = (\forall x)\psi$

→ forcing is good at solving case ①

↳ we want G to be an object witnessing the $\exists x$ in φ

↳ choose $\mathbb{P}$ as a set of "approximations" of G

→ but we have no control of what G we get, so we need to choose $\mathbb{P}$ so that every V-generic filter G has the desired propery

→ we than need to check that by introducing this G, we did not also break something else

Consistency of CH

CH: $\exists$ surjection $f : \omega_1 \to \mathbb{R}$

$\nearrow$ poset for collapsing 2^ω to ω_1

$\mathbb{P} := \{ f : A \to \mathbb{R} \mid A \subseteq \omega_1, \ |A| \le \omega \}$ ordered as $f \le g$ iff $g \subseteq f$

$\to$ stronger conditions = functions that know more

Check that $\mathbb{P}$ is nice:

- $\mathbb{1}_{\mathbb{P}}$ = empty mapping
- antisymmetry ✓
- separativity ... if $f \not\le g$ then extend f to h s.t. $\exists x \in \omega_1 : g(x) \neq h(x)$ $\Rightarrow h \perp g$
- atomless ... we can always extend further

If $F \subseteq \mathbb{P}$ is a filter, then $\bigcup F$ is a partial function $\omega_1 \to \mathbb{R}$

! **Def:** A partial function $f ; A \to B$ is a function $f : A' \to B$ with $\mathrm{Dom} f = A' \subseteq A$

↳ we use semicolon notation

$\to$ we need to check that if $f, g \in F$, then f and g agree on $B := \mathrm{Dom} f \cap \mathrm{Dom} g$

↳ F is a filter ... $\exists h \le f, g$ in F ... so $f, g \subseteq h$ ✓

Defining dense sets ... V-generic G will intersect them all

- for $\forall r \in \mathbb{R}$ let $D_r := \{ f \in \mathbb{P} \mid r \in \mathrm{Rng}(f) \}$ ... D_r is dense since we can extend any $g \in \mathbb{P}$ by choosing $\alpha \in \omega_1 \setminus \mathrm{Dom}(g)$ and setting $g(\alpha) := r$
- if $F \cap D_r \neq \emptyset$, then $r \in \mathrm{Rng}(\bigcup F)$

$\Rightarrow$ if G is a V-generic filter then $r \in \mathrm{Rng}(\bigcup G)$ for every $r \in \mathbb{R} \cap V$

↳ ground model real

$\Rightarrow$ the surjection we are after is $f = \bigcup G$

Note: Strictly speaking, f may not be defined on the entire ω_1 ... but it is:

↳ we can fix this by defining $D_\alpha := \{ f \in \mathbb{P} \mid \alpha \in \mathrm{Dom}(f) \}$ for $\alpha \in \omega_1$

$\to D_\alpha$ are dense so G has to intersect them all $\Rightarrow$ now $\mathrm{Dom}(\bigcup G) = \omega_1^{(V)}$

We have shown: $V[G] \models f : \omega_1^{(V)} \to \mathbb{R} \cap V$ is surjective

↳ but ω_1 could have changed and we could have added some new reals

Note: Since V and $V[G]$ have the same ordinals, $\omega_1^{(V)}$ could not have become bigger, but it could have collapsed to a countable ordinal in $V[G]$

↳ maybe we introduced a bijection between ω and $\omega_1^{(V)}$.

Def (**σ-closed**): A poset $\mathbb{P}$ is σ-closed if for $\forall$ sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ with $p_n \in \mathbb{P}$, there $\exists$ universal lower bound $p_\omega \in \mathbb{P}$ s.t. $\forall n: p_\omega \leq p_n$

↳ p_ω is stronger than all p_n

$\mathbb{P}$ above is σ-closed

↳ take a union of all functions in that chain

Theorem: If $\mathbb{P}$ is σ-closed, then $\mathbb{R}^{V[G]} = \mathbb{R}^V$ ... σ-closed does not add new real numbers

Proof: Fix a real $c \in \mathbb{R}^{V[G]}$ and investigate a name $\dot{c} \in V^{\mathbb{P}}$ for c.

goal: show that $c \in V$, so $c \in \mathbb{R}^{(V)}$ since "x is a real" is Δ_0 and absolute

→ we view c as a function $\omega \to \{0,1\}$

⇒ denote by $\dot{c}(n)$ a name for the n-th bit of c

Consider the set

$$D_c := \bigcup_{x \in \mathbb{R}^V} D_{\dot{c}=x} \quad \text{where } D_{\dot{c}=x} = \underbrace{\{ p \in \mathbb{P} \mid p \in G \Rightarrow V[G] \models \dot{c}/_G = \check{x} \}}_{p \Vdash \dot{c} = x \;\to D_\varphi} \in V$$

name $\dot{c}$ will evaluate as x by definition

Note: $D_c \in V$ since $D_{\dot{c}=x} \in V$ and the union is over $\mathbb{R}^V \in V$

claim: $D_c \subseteq \mathbb{P}$ is dense

↳ then, since $D_c \in V$, every V-generic G must intersect D_c, so $\exists p \in G \cap D_c$, hence $p \in D_{\dot{c}=x}$ for some $x \in \mathbb{R}^V$ and thus $V[G] \models \dot{c}/_G = \check{x}$, so $c = x \in \mathbb{R}^V$

~~proof~~: let $p \in \mathbb{P}$ ... want $q \leq p$ s.t. $q \in D_c$

$c: \omega \to 2$ ... ω and 2 are absolute, so c is still a map from old ω to old 2

We construct a sequence $p = p_0 \geq p_1 \geq p_2 \geq \dots$ by recursion inside V together with bits $b_0, b_1, \dots \in \{0,1\}$ s.t. $\forall n \in \omega: p_{n+1} \Vdash \dot{c}(n) = b_n$

→ suppose p_n has already been defined

↳ we know that $D_{\dot{c}(n)=0} \cup D_{\dot{c}(n)=1}$ is dense, so we may choose $p_{n+1} \leq p_n$ from this union, and it determines the bit $0/1$

→ φ and $\neg\varphi$

⇒ by σ-closure $\exists$ universal lower bound $p_\omega \in \mathbb{P}$

→ since the recursion was carried out in V, the set representing the sequence $\langle b_n \mid n \in \omega \rangle \in V$ as well

⇒ define $\bar{x} \in \mathbb{R}^V$ by $\bar{x}(n) := b_n$

$p_\omega \Vdash \dot{c} = \bar{x}$, so $p_\omega \in D_{\dot{c}=\bar{x}} \subseteq D_c$ and $p_\omega \leq p$, so D_c is dense ∎

↳ since D_φ is downwards closed, so $\forall n: p_\omega \Vdash \dot{c}(n) = b_n$ and $\forall n: \bar{x}(n) = b_n$

Theorem: If $\mathbb{P}$ is σ-closed, then $\omega_1^{V[G]} = \omega_1^V$ ... σ-closed does not collapse ω_1

Proof sketch: We show that no surjection $f: \omega \to \omega_1^V$ was added in $V[G]$

→ suppose that $f \in V[G]$ is a function $f: \omega \to \omega_1^V$ with name $\dot{f} \in V^{\mathbb{P}}$

→ similarly to last time, for any given $p \in \mathbb{P}$ we build two sequences: of conditions $p = p_0 \geq p_1 \geq \dots$ and ordinals $\alpha_0, \alpha_1, \dots \in \omega_1^V$ s.t.

$$\forall n \in \omega: p_{n+1} \Vdash \dot{f}(n) = \alpha_n$$

→ by σ-closure $\exists$ lower bound $p_\omega \in \mathbb{P}$ ... $\forall n: p_\omega \Vdash \dot{f}(n) = \alpha_n$

→ by construction $\langle \alpha_n \mid n \in \omega \rangle \in V$, so $p_\omega \Vdash \dot{f} = \langle \alpha_n \mid n \in \omega \rangle$

→ we could again consider $D_{\dot{f}} := \bigcup_{\substack{X: \omega \to \omega_1^V \\ \in V}} D_{\dot{f} = \check{X}}$, and $p_\omega \in D_{\dot{f} = \langle \alpha_n \rangle} \subseteq D_{\dot{f}}$

⇒ so $D_{\dot{f}}$ is dense and every V-generic G intersects it

⇒ every function $\omega \to \omega_1^V$ in $V[G]$ was already in V

Remark: The argument actually proves that $\forall X \in V$:

σ-closed forcings do not add any new functions $f: \omega \to X$

→ no new reals $\sim X = 2$, no collapsing $\omega_1 \sim X = \omega_1^V$

Corollary: $Con(ZFC) \Rightarrow Con(ZFC + CH)$.

Consistency of $\neg CH$

$\neg CH: \exists C \subseteq \mathbb{R}, |C| = \aleph_2$

finite

$$\mathbb{P} := \{ f: A \to 2 \mid A \subseteq \omega_2 \times \omega, |A| < \omega \} \text{ ordered as } f \leq g \text{ iff } g \subseteq f$$

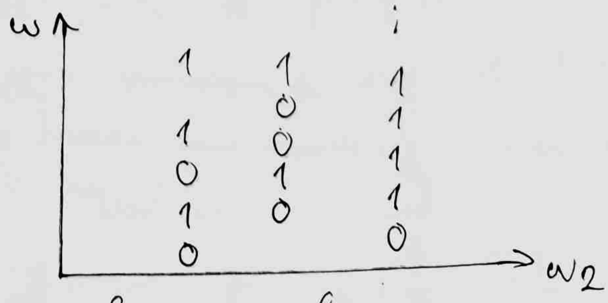
Exercise: $\mathbb{P}$ is nice & atomless

- atomless ✓
- antisymmetric ✓
- $\mathbb{1} \in \mathbb{P}$ ... $f = \emptyset$
- separable $f \nleq g$ ... extend $h \perp g$

If $F \subseteq \mathbb{P}$ is a filter, then $\bigcup F$ is a partial function $\omega_2 \times \omega \to 2$

↳ fills in a part of the grid

Intuition:



columns = reals

→ entire grid $\sim C \subseteq \mathbb{R}$ of size $\aleph_2$

↳ if all columns distinct !

Defining dense sets: for $\alpha \in \omega_2$, $n \in \omega$, put

$D_{\alpha,n} := \{ f \in \mathbb{P} \mid (\alpha,n) \in \mathrm{Dom}(f) \}$ ... dense in $\mathbb{P}$

if F is a filter and $F \cap D_{\alpha,n} \neq \emptyset$, then $(\alpha,n) \in \mathrm{Dom}(\bigcup F)$

$\Rightarrow G$ V-generic filter $\Rightarrow \bigcup G : \omega_2 \times \omega \to 2$ is a total function $\Rightarrow$ the grid is full

. For $\alpha \in \omega_2$ define $C_\alpha : \omega \to 2$ by $C_\alpha(n) := (\bigcup G)(\alpha, n)$ ... C_α = column at index $\alpha \in \omega_2$

! we need to argue that in each column we have a different real

• For $\alpha \neq \beta$ in ω_2 let ↗ approximations of the grid that disagree on columns α and β

$E_{\alpha,\beta} := \{ f \in \mathbb{P} \mid \exists n \in \omega : f(\alpha,n) \neq f(\beta,n) \}$ ... dense since $\mathrm{Dom}(f)$ is finite

$\Rightarrow G$ intersects each $E_{\alpha,\beta}$, so $\forall \alpha \neq \beta : V[G] \models C_\alpha \neq C_\beta$

If we set $C := \{ C_\alpha \mid \alpha \in \omega_2^V \}$, we have $V[G] \models |C| = |\omega_2^V|$

! we need to argue that ω_2 did not collapse to something smaller

Def (C.C.C.): A poset $\mathbb{P}$ is ccc if it contains no uncountable antichains.

! antichain $\neq$ incomparable elements, but elements without common lower bounds in $\mathbb{P}$

Note: ccc = countable chain condition ... because historically, forcing was done on complete boolean algebras and there: $\exists$ uncountable chain $\Leftrightarrow$ $\exists$ uncountable antichain

Proposition: $\mathbb{P}$ above is ccc.

→ the proof uses the Δ-system lemma:

Lemma (Δ-system lemma): Let $\mathcal{F}$ be a family of finite sets s.t. $|\mathcal{F}|$ is < uncountable regular &.
Then $\exists\, \mathcal{F}' \subseteq \mathcal{F}$ s.t. $|\mathcal{F}'| = |\mathcal{F}|$, and a finite (possibly empty) set R s.t.

$$\forall A \neq B \text{ in } \mathcal{F}' : A \cap B = R \quad \leftarrow R = \text{root of the sunflower}$$

Def: A family $\mathcal{F}'$ with this property is called a Δ-system with root R

Proof of Proposition: Suppose that $\{ f_\alpha \mid \alpha \in \omega_1 \} \subseteq \mathbb{P}$ were an uncountable antichain.

→ each $A_\alpha := \mathrm{Dom}(f_\alpha)$ is finite, by the Δ system lemma $\exists$ uncountable $I \subseteq \omega_1$ and $\exists$ finite $R \subseteq \omega_2 \times \omega$ s.t.
$A_\alpha \cap A_\beta = R$ for $\forall \alpha \neq \beta$ in I

→ there are only finitely many functions $R \to 2$, so by pigeonhole $\exists$ uncountable $J \subseteq I$ s.t. the restrictions $f_\alpha \upharpoonright R$ for $\alpha \in J$ all coincide

$\Rightarrow$ for $\forall \alpha, \beta \in R$ is the set $f_\alpha \cup f_\beta$ again a partial function $\omega_2 \times \omega \to 2$, so $\in \mathbb{P}$

↳ but $f_\alpha \cup f_\beta$ is a common lower bound of f_α and f_β. ↯ ... $\{ f_\alpha \mid \alpha \in \omega_1 \}$ not antichain ▨

Theorem: If $\mathbb{P}$ is ccc, then $V \models |\lambda| < |\kappa| \iff V[G] \models |\lambda| < |\kappa|$ ⊛

and $V \models |\lambda| = cf(\kappa) \iff V[G] \models |\lambda| = cf(\kappa)$

$\Rightarrow$ ccc posets do not collapse any cardinals & preserve cofinalities

$\Rightarrow$ V and V[G] have the exact same cardinal structure

Recall: V and V[G] have the exact same ordinals

Corollary: $\omega_2^V = \omega_2^{V[G]}$, so $V[G] \models |C| = \aleph_2$, and we have:

Theorem: $Con(ZFC) \Rightarrow Con(ZFC + \neg CH)$

Proof of Theorem: We will focus on ⊛ ... note that "$\Leftarrow$" is trivial because it says:

$V \models |\kappa| \leq |\lambda| \Rightarrow V[G] \models |\kappa| \leq |\lambda|$ ✓

For "$\Rightarrow$" we will treat the special case

$\lambda = \omega$ and $\kappa = \omega_1^V$, as the general case is similar

Claim: V[G] contains no surjection $b : \omega \to \omega_1^V$ → want: $Rng(b)$ countable

↳ let $b : \omega \to \omega_1^V$ in V[G], and let $\dot{b} \in V^{\mathbb{P}}$ be a name for it

- for $\forall n \in \omega$ let $\dot{b}(n)$ be a name for $b(n) \in \omega_1^V$
- for $\alpha \in \omega_1^V$ let $D_{\dot{b}(n)=\alpha} := \{p \in \mathbb{P} \mid p \Vdash \dot{b}(n) = \alpha\}$ — $\dot{b}(n)$ will evaluate as α

→ since for $\alpha \neq \beta$ are the conditions $\dot{b}(n) = \alpha$ and $\dot{b}(n) = \beta$ mutually exclusive, we have

$\boxtimes$ $p \in D_{\dot{b}(n)=\alpha}$ & $q \in D_{\dot{b}(n)=\beta} \Rightarrow p \perp q$

- for $n \in \omega$ define $R_n := \{\alpha \in \omega_1^V \mid D_{\dot{b}(n)=\alpha} \neq \emptyset\}$ ... set of all $\alpha \in \omega_1^V$ that have a chance to be $b(n)$

R_n is countable

↳ otherwise for $\forall \alpha \in R_n$ pick $p_\alpha \in D_{\dot{b}(n)=\alpha}$ ... by $\boxtimes$ $\{p_\alpha \mid \alpha \in R_n\}$ is an antichain ↯ ccc

→ $\forall n : b(n) \in R_n$, so $Rng(b) \subseteq \bigcup_{n \in \omega} R_n =: R$

→ since each $R_n \in V$, also $R \in V$ and R is countable (in V), so b is not surjective

General case: $b : \lambda \to \kappa$, $Rng(b) \subseteq \bigcup_{\delta < \lambda} R_\delta$ has size $\leq \lambda \cdot \omega = \lambda \Rightarrow b$ not surjective ▨

Proof of cofinality preservation: Suppose that $V \models cf(\kappa) = \lambda$ but $V[G] \models cf(\kappa) = \mu < \lambda$

→ clearly $cf(\kappa) > \omega$. Let $f : \mu \to \kappa$ be a function cofinal in κ in V[G]

→ let $\dot{f}$ be a name for f and $\dot{f}(\alpha)$ a name for $f(\alpha)$

- for $\forall \beta \in \kappa$ let $D_{\dot{f}(\alpha)=\beta} := \{p \in \mathbb{P} \mid p \Vdash \dot{f}(\alpha) = \beta\}$
- for $\forall \alpha \in \mu$ let $R_\alpha := \{\beta \in \kappa \mid D_{\dot{f}(\alpha)=\beta} \neq \emptyset\}$ ... possible values of $f(\alpha) < \kappa$

→ as before each R_α is countable because we can define an antichain of the same size

- for $\alpha \in \mu$ let $\gamma_\alpha := \sup R_\alpha$ ... $\gamma_\alpha < \kappa$ as $cf(\kappa) > \omega$
- now: $\sup(Rng\, f) \leq \sup_{\alpha \in \mu} \gamma_\alpha < \kappa$ because the sequence $\langle \gamma_\alpha \mid \alpha \in \mu \rangle$ exists in V

by our construction, and $\mu < cf(\kappa)$. Hence f is no cofinal in κ ↯ ▨

POSET PROPERTIES — already know σ-closed & ccc

- we introduce two local notions: linked and centered
  and global notions derived from them

Def: A subset $A \subseteq \mathbb{P}$ is

- linked if $(\forall p, q \in A)(\exists r \in \mathbb{P}): r \leq p, q$ ... every 2 elements are compatible in $\mathbb{P}$
- centered if $(\forall \text{ finite } B \subseteq A)(\exists r \in \mathbb{P}): (\forall p \in B): r \leq p$ ... every finite set of conditions has a common lower bound

centered $\Rightarrow$ linked ! "$\Leftarrow$" does not hold in general

Def: A poset $\mathbb{P}$ is

- σ-linked if $\mathbb{P} = \bigcup_{n<\omega} L_n$, where each L_n is linked in $\mathbb{P}$
- σ-centered if $\mathbb{P} = \bigcup_{n<\omega} C_n$, where each C_n is centered in $\mathbb{P}$

} any countable poset is σ-centered since it is a count. union of singletons

σ-centered $\Rightarrow$ σ-linked

Def: A poset $\mathbb{P}$ is

- Knaster if $\forall$ uncountable $A \subseteq \mathbb{P}$ contains an uncountable linked subset
- has precaliber ω_1 if $\forall$ uncountable $A \subseteq \mathbb{P}$ contains an uncountable centered subset

precaliber ω_1 $\Rightarrow$ Knaster

Proposition:
① σ-linked $\Rightarrow$ Knaster ② σ-centered $\Rightarrow$ precaliber ω_1 ③ Knaster $\Rightarrow$ c.c.c.

Proof: ① and ② hold from the pigeonhole principle, ③ is obvious

Hence:

$$\begin{array}{ccccc} \sigma\text{-centered} & \Rightarrow & \text{precaliber } \omega_1 & \searrow & \\ \Downarrow & & \Downarrow & & \text{c.c.c.} \\ \sigma\text{-linked} & \Rightarrow & \text{Knaster} & \nearrow & \end{array}$$

Note. The Δ-system lemma proof that the poset for adding reals is ccc actually shows that it has precaliber ω_1

FORCING AXIOMS

→ to avoid the entire construction of finding a V-generic $G \subseteq \mathbb{P}$ to get $V[G]$, we consider forcing axioms, which assert that for a prescribed class of posets, a sufficiently generic filter already exists in V

- κ = how many dens sets we intersect
- $\mathcal{P}$ = the class of posets we talk about

$\boxed{FA_\kappa(\mathcal{P})}$: For $\forall \mathbb{P} \in \mathcal{P}$ and $\forall$ family $\mathcal{D}$ of dense subsets with $|\mathcal{D}| \leq \kappa$, there $\underbrace{\exists \text{ filter } G \subseteq \mathbb{P} \text{ intersecting every } D \in \mathcal{D}}_{G \text{ is a } \mathcal{D}\text{-generic filter}}$

Theorem (Rasiowa–Sikorski): $ZFC \vdash FA_\omega(\text{all posets})$

Proof: Let $\mathcal{D} = \langle D_n \mid n \in \omega \rangle$ and using density choose $p_0 \geq p_1 \geq \cdots$ $p_n \in D_n$
$G :=$ upward closure of $\{p_i \mid i \in \omega\}$ ... G is a $\mathcal{D}$-generic filter ▨

Proposition: $ZFC \vdash \forall \kappa : FA_\kappa(\text{posets with atoms})$

Proof: If $\mathbb{P}$ has an atom $a \in \mathbb{P}$, then the principal filter $F_a := \{p \in \mathbb{P} \mid p \geq a\}$ intersects every dense subset D since $a \in D$ ▨

Proposition: $ZFC \vdash FA_{\omega_1}(\sigma\text{-closed posets})$

Proof: As in Rasiowa–Sikorski; given $\langle D_\alpha \mid \alpha \in \omega_1 \rangle$ we build a sequence $\langle p_\alpha \mid \alpha \in \omega_1 \rangle$ s.t. $p_\alpha \in D_\alpha$ and $\alpha < \beta \Rightarrow p_\alpha \geq p_\beta$

- $p_\alpha \rightsquigarrow p_{\alpha+1}$ by using the density of $D_{\alpha+1}$
- at limit steps $\delta \in \omega_1$, we first get a cofinal subsequence of $\langle p_\alpha \mid \alpha < \delta \rangle$ of length ω and apply σ-closure to get a universal lower bound p'_δ

→ we then get $p_\delta \leq p'_\delta$ s.t. $p_\delta \in D_\delta$ using density of D_δ ▨

Proposition: $ZFC \vdash \forall$ countable atomless $\mathbb{P} : \neg FA_{2^\omega}(\{\mathbb{P}\})$

Proof: If $\mathbb{P}$ is (nice) atomless and $F \subseteq \mathbb{P}$ a filter, then $\mathbb{P} \setminus F$ is dense
↪ we already proved this earlier, given $p \in \mathbb{P}$ take $q < p$ and by separativity $\exists r \leq p$ s.t. $r \perp q$. F cannot contain both r and q, so one $\in \mathbb{P} \setminus F$

$\Rightarrow \mathcal{D} := \{\mathbb{P} \setminus F \mid F \text{ is a filter on } \mathbb{P}\} \subseteq \mathcal{P}(\mathbb{P}) \xRightarrow{|\mathbb{P}| = \omega} |\mathcal{D}| \leq 2^\omega$

→ but no filter F can intersect every $D \in \mathcal{D}$ ... $F \cap (\mathbb{P} \setminus F) = \emptyset$ ▨

? Is there some $\mathbb{P}$ s.t. $ZFC \vdash \neg FA_{\omega_1}(\{\mathbb{P}\})$? $\quad$ $\ulcorner f \leq g$ iff $g \subseteq f$

Def: The collapse poset is $Coll(\omega,\omega_1) := (\{f; \omega \to \omega_1 \mid |f| < \omega\}, \supseteq)$

↳ forcing with it would collapse ω_1 to ω

↳ partial functions

Exercise: This is a nice atomless poset

Proposition: $ZFC \vdash \neg FA_{\omega_1}(\{Coll(\omega,\omega_1)\})$

Remark: We say that no suitable filter exists in V, but if we were to force with this poset, we would get one outside of V ... V-generic G

↳ then $\bigcup G : \omega \to \omega_1^V$ is surjective & collapses ω_1

Remark: $Coll(\omega,\omega_1)$ is neither ccc nor σ-closed

Proof: Put $\mathbb{P} := Coll(\omega,\omega_1)$. As before, if $F \subseteq \mathbb{P}$ is a filter then $\bigcup F ; \omega \to \omega_1$.

- for $n \in \omega$ define $D_n := \{f \in \mathbb{P} \mid n \in Dom f\}$
- for $\alpha \in \omega_1$ define $R_\alpha := \{f \in \mathbb{P} \mid \alpha \in Rng f\}$

→ all D_n, R_α are dense and there are only $\omega + \omega_1 = \omega_1$ many of them

⇒ suppose that $\exists F \subseteq \mathbb{P}$ intersected them all ... $D_n \to \bigcup F$ total, $R_\alpha \to \bigcup F$ surjective $\Big\}$ $\omega \approx \omega_1$ ↯ ∎

MARTIN'S AXIOM

Let $\varkappa < 2^\omega$ be a cardinal and $\mathcal{P}$ a class of ccc posets. Define statements:

$$MA_\varkappa \equiv FA_\varkappa(\text{all ccc posets})$$

$$MA_\varkappa(\mathcal{P}) \equiv FA_\varkappa(\mathcal{P})$$

weaker version

Note: For $\varkappa \geq 2^\omega$ this would fail because any countable atomless poset is ccc.

$CH \Rightarrow \forall \varkappa < 2^\omega : MA_\varkappa$ ... due to Rasiowa-Sikorski

Axiom (Martin's axiom): $\boxed{MA} \equiv \neg CH \wedge (\forall \varkappa < 2^\omega) MA_\varkappa$.

Theorem (Solovay-Tennenbaum): $Con(ZFC) \Rightarrow Con(ZFC + MA)$

↳ this is proved using iterated forcing on ccc posets

→ but once we have this, we can often avoid forcing and work with MA insted

Using MA to prove consistency results: Given $\varphi = (\exists x)\psi$, we design a ccc poset $\mathbb{P}$ of approximations for the witness x, and show that a sufficiently generic filter G would yield this witness. Then we apply MA to get G

PROPER FORCING AXIOM

Idea: When proving the independence of CH, we saw that both σ-closed and ccc forcing notions preserve $\aleph_1$ ... $\omega_1^V = \omega_1^{V[G]}$

→ Shelah, Baumgartner and Todorcevic isolated in the 1980s a more general property preserving $\aleph_1$

Def (informal): $\mathbb{P}$ is proper if it preserves $\aleph_1$ → ccc and σ-closed are proper

Axiom: $\boxed{\text{PFA}} \equiv FA_{\omega_1}$(proper posets)

Fact: If $\mathbb{P}$ is a countable support iteration of proper forcings, then it is proper

Consequences of PFA:

- $2^{\aleph_0} = \aleph_2$ ... so PFA $\Rightarrow \neg$CH
- there are no Suslin trees
- SCH holds

PFA $\Rightarrow$ MA ... if $2^\omega = \aleph_2$, then MA only talks about $\kappa = \omega$ and $\kappa = \omega_1$

- $\kappa = \omega$ holds from Rasiowa-Sikorski Theorem
- $\kappa = \omega_1$ holds due the PFA since ccc posets are proper

Consistency strength of PFA: Not fully known – bounds:

Supercompact cardinal $\Rightarrow$ PFA

PFA $\Rightarrow$ Woodin cardinal that is a limit of Woodin cardinals

CARDINAL CHARACTERISTICS OF THE CONTINUUM

$\mathfrak{c} := |\mathbb{R}| = 2^\omega$

→ since $\mathbb{R} \approx 2^\omega \approx \omega^\omega$, we can work in the Baire space ω^ω of functions $\omega \to \omega$

Notation: $\alpha < \omega^\omega$ denotes an ordinal, $f \in \omega^\omega$ denotes a function $\omega \to \omega$

→ a cardinal characteristic of the continuum is a cardinal κ defined using $\mathbb{R}$ that is known to be between ω_1 and 2^ω

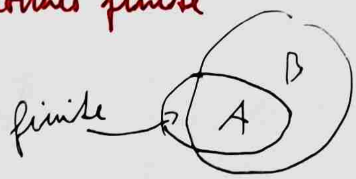
→ it is the the first cardinal for which some property of ω^ω fails

Def: We define two orders on ω^ω:

- domination $f \leq g$ iff $\forall n \in \omega: f(n) \leq g(n)$ → g dominates f
- eventual domination $f \leq^* g$ iff $\exists N$ s.t. $\forall n > N: f(n) \leq g(n)$
- strict versions: $f < g$ and $f <^* g$ ... just replace $\leq$ in the definition by $<$

Notation: $A \subseteq^* B$ means that $A \setminus B$ is finite ... subset modulo finite

↳ in general, * means "up to finitely many mistakes"



Def: A family of functions $\mathcal{B} \subseteq \omega^\omega$ is unbounded if there is no $g \in \omega^\omega$ that would eventually dominate every $f \in \mathcal{B}$

Def: A family $D \subseteq \omega^\omega$ is dominating if $\forall f \in \omega^\omega \; \exists g \in D$ s.t. $f \leq^* g$

↳ every f is eventually dominated by something from D

Def: The bounding number $\mathfrak{b}$ and dominating number $\mathfrak{d}$ are:

$$\mathfrak{b} := \min\{|B| \mid B \subseteq \omega^\omega \text{ is unbounded}\}$$

$$\mathfrak{d} := \min\{|D| \mid |D| \subseteq \omega^\omega \text{ is dominating}\}$$

Observation: $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^\omega$

Proof: $\omega_1 \leq \mathfrak{b}$ by a simple diagonalization argument

↳ given $\langle f_n \mid n \in \omega \rangle$ take $g : n \mapsto \max\limits_{i \leq n} f_i(n)$ ... eventually dominates every f_n

- $\mathfrak{b} \leq \mathfrak{d}$ because every dominating familly is trivially unbounded

↳ suppose D was bounded by some g and define $g^+ : n \mapsto g(n)+1$

→ then $\exists f \in D$ s.t. $g^+ \leq^* f$ and g does not eventually dominate $f \in D$ ↯

We also study $[\omega]^\omega \approx 2^\omega$ the set of infinite subsets of ω

Def: $P \in [\omega]^\omega$ is a pseudointersection of a familly $\mathcal{C} \subseteq [\omega]^\omega$ if $\forall C \in \mathcal{C} : P \subseteq^* C$.

Def: The pseudointersection number $\mathfrak{p}$ is

$$\mathfrak{p} := \min\left\{|\mathcal{C}| \;\middle|\; \mathcal{C} \subseteq [\omega]^\omega,\ \begin{array}{l}\mathcal{C} \text{ is centered in the poset } ([\omega]^\omega, \subseteq) \text{ and} \\ \mathcal{C} \text{ has no pseudointersection}\end{array}\right\}$$

Note: $\mathcal{C}$ centered $\iff$ intersecting finitely many elements of $\mathcal{C}$ is always infinite

$\omega_1 \leq \mathfrak{p}$

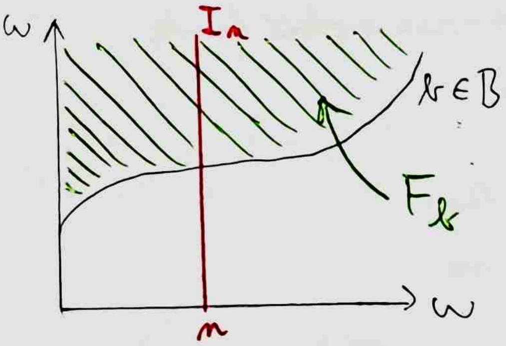
↳ again we diagonalize ... given centered $\mathcal{C} = \langle C_n \mid n \in \omega \rangle$, pick $p_0 \in C_0$ and for $n > 0$ pick $p_n \in (C_0 \cap \dots \cap C_n) \setminus \{p_0, \dots, p_{n-1}\}$ ... $P := \{p_n \mid n \in \omega\}$ is pseudoint.

Proposition: $\mathfrak{p} \leq \mathfrak{b}$

Proof: We must show that if $B \subseteq \omega^\omega$ has size $|B| < \mathfrak{p}$, then it is bounded

WLOG: each $b \in B$ is non-decreasing ... we can replace b by $n \mapsto \max\limits_{i \leq n} b(i)$

↳ continues on next page



- For $b \in B$, put $F_b := \{(n,m) \mid m > b(n)\} \subseteq \omega \times \omega$
- For $n \in \omega$, put $I_n := \{(n,m) \mid m \in \omega\}$ ... n^{th} column

→ consider $\mathcal{C} := \{F_b \mid b \in B\} \cup \{\omega\times\omega \setminus I_n \mid n \in \omega\}$

↳ this is a system of subsets of $\omega \times \omega$!

→ the injection $\varphi : (n,m) \mapsto 2^n \cdot 3^m$ has the property that

$$\forall A, B \subseteq \omega\times\omega : \varphi[A \cap B] = \varphi[A] \cap \varphi[B],$$

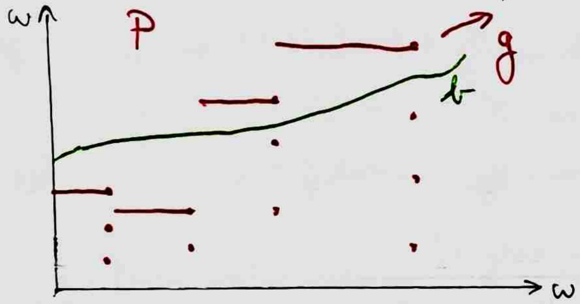
so if $\mathcal{C}$ is centered in $([\omega\times\omega]^\omega, \subseteq)$, then $\{\varphi[C] \mid C \in \mathcal{C}\}$ is centered in $([\omega]^\omega, \subseteq)$

$\mathcal{C}$ is centered ... intersecting the graph above finitely many functions is infinite, and then we only delete finitely many columns

$\Rightarrow \mathcal{C}$ has a pseudointersection $P \subseteq \omega\times\omega$ ... since $|\mathcal{C}| = |B| + \omega < \mathfrak{p}$

- P is infinite, but $\forall n$: $P \cap I_n$ is finite $\because P \subseteq^* \omega\times\omega \setminus I_n$

$\Rightarrow \Pi := \{n \mid P \cap I_n \neq \emptyset\}$ is infinite ↳ only finitely many points in each column



- for $n \in \omega$ let n' be the least $n' \in \Pi$ s.t. $n' \geq n$
- define $g : \omega \to \omega$ by

$$g(n) := \max\{m \mid (n', m) \in P\}$$

Note: well-defined $\because 0 < |P \cap I_n| < \omega$

$\forall b \in B : b \leq^* g$ ... $P \subseteq^* F_b$, so up to finitely many mistakes we have

$$b(n) \underset{\text{monotonicity}}{\leq} b(n') \overset{*}{<} g(n)$$ ∎

Corollary: $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^\omega$

→ what we will prove:

Theorem: $MA \Rightarrow \omega_1 < \mathfrak{p} = \mathfrak{b} = \mathfrak{d} = 2^\omega$.

→ the main theorem we will prove is Bell's theorem:

Theorem (Bell's theorem) $\forall \kappa$: $MA_\kappa(\sigma\text{-centered}) \iff \mathfrak{p} > \kappa$.

Notation: For functions $f, g \in \omega^{<\omega}$, we write

$$f \sqsubseteq g \quad \text{if } f \text{ is an initial segment of } g$$

→ if $f : n \to \omega$, $g : m \to \omega$, we mean by $f \sqsubseteq g$ that $n \leq m$ and $g \restriction n = f$

$f \sqsubseteq g$ is actually the same as $f \subseteq g$

HECHLER FORCING – for adding a dominating function

$\mathbb{H} := \{(p,f) \mid p \in \omega^{<\omega},\ f \in \omega^\omega\}$

Note: there are many "equivalent" definitions (with the same forcing consequences)

ordered by: $(p,f) \leq (p',f')$ iff

i) $p' \sqsubseteq p$

ii) $f(n) \geq f'(n)$ for $\forall n \in \omega$

iii) $p(k) \geq f'(k)$ for $\forall k \in \mathrm{Dom}(p) \setminus \mathrm{Dom}(p')$

Intuition:

- p = stem, f = side function
- f constrains how the stem can be extended

Proposition: $\mathbb{H}$ is σ-centered (and thus also ccc).

Proof: $|\omega^{<\omega}| = \omega$, so if we for $\forall p \in \omega^{<\omega}$ define $C_p := \{(p,f) \mid f \in \omega^\omega\} \subseteq \mathbb{H}$, we have countably many sets that make up $\mathbb{P}$

each C_p is centered ... given $(p,f_0),\dots,(p,f_m)$, we find a universal lower bound $(p,\bar{f})$ where $\bar{f}(n) := \max_{i \leq m} f_i(n)$ ∎

Theorem: $MA_\kappa(\sigma\text{-centered}) \Rightarrow \mathfrak{b} > \kappa$

Proof: Let $B \subseteq \omega^\omega$ s.t. $|B| \leq \kappa$. To show that it is bounded, we need to find a dominating function g. Define dense sets:

- for $\forall n \in \omega$: $D_n := \{(p,f) \in \mathbb{H} \mid n \in \mathrm{Dom}\, p\}$
- for $\forall b \in B$: $R_b := \{(p,f) \in \mathbb{H} \mid b < f\}$

↳ f strictly dominates b

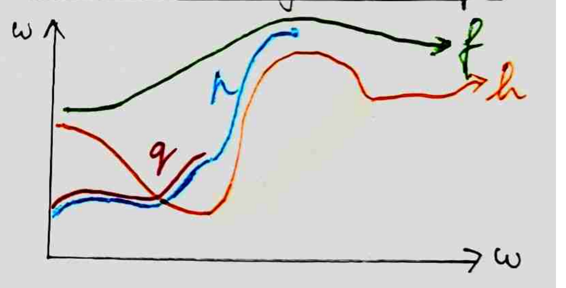
→ in total $\leq \kappa$ many dense sets

$\Rightarrow$ by MA_κ, there is a filter G intersecting each

→ define $g: \omega \to \omega$ by $g := \bigcup \{p \mid (p,f) \in G\}$

- g is a function from i) because G is a filter
- g is total $\because$ G intersects $\forall D_n$

Visualization of $(p,f) \leq (q,h)$:



want: $\forall b \in B$: $b \leq^* g$ since G intersects R_b

↳ fix b and let $(p,f) \in G \cap R_b$. Note that f dominates b ... definition of R_b

claim: $b(n) < g(n)$ for every $n \in \omega \setminus \mathrm{Dom}(p)$

- fix $n \in \omega \setminus \mathrm{Dom}(p)$ and consider $g(n)$ ... $g(n) = q(n)$ for some $(q,h) \in G$

→ since G is a filter, there $\exists (p',f') \in G$ extending both (p,f) and (q,h)

→ because it extends (q,h), we know that $p'(n) = q(n) = g(n)$

→ because it extends (p,f), we know that $p'(n) \geq f(n) > b(n)$

$\Big\}$ $g(n) > b(n)$

definition of R_b ↑

∎

MATHIAS-PŘÍKRÝ FORCING – for adding a pseudointersection

Def: Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter on ω extending the Fréchet filter

↳ $\mathcal{F}_{co} = \{A \subseteq \omega \mid \omega \setminus A \text{ finite}\}$
↳ cofinite sets

$$\mathbb{M}(\mathcal{F}) := \{(a,u) \mid a \in [\omega]^{<\omega},\ u \in \mathcal{F},\ a < u\}$$

where $a < u$ means $\max(a) < \min(u)$, $\max(\emptyset) = -1$, ordered by

$$(a,u) \le (a',u') \text{ iff } a' \sqsubseteq a,\quad u \subseteq u',\quad a \setminus a' \subseteq u'$$

Note: $a' \sqsubseteq a$ ... we interpret $a \in [\omega]^{<\omega}$ as an increasing function $|a| \to \omega$

↳ equiv: $a' \subseteq a$ & $\max(a') < \min(a \setminus a')$

Intuition: (a,u) ... a is a finite stem and u is an $\mathcal{F}$-large set lying entirely above it, and serving as a reservoir for extensions

If $\mathcal{F}$ extends the Fréchet filter, then it contains no finite sets

↳ suppose $u \in \mathcal{F}$ is finite ... then $\omega \setminus u \in \mathcal{F}_{co} \subseteq \mathcal{F}$, so $u, \bar{u} \in \mathcal{F} \implies u \cap \bar{u} = \emptyset \in \mathcal{F}$ ↯

Proposition: $\mathbb{M}(\mathcal{F})$ is σ-centered → for every $\mathcal{F}$ extending $\mathcal{F}_{co}$

Proof: Since $|[\omega]^{<\omega}| = \omega$, we can define for each $a \in [\omega]^{<\omega}$ a set $C_a := \{(a,u) \mid u \in \mathcal{F}\}$

C_a is centered ... given $(a,u_0), \dots, (a,u_m)$, take $(a, \underbrace{u_0 \cap \dots \cap u_m}_{\text{remains in } \mathcal{F}}) \in \mathbb{M}(\mathcal{F})$ $\blacksquare$

Theorem: $MA_\kappa(\sigma\text{-centered}) \implies \mathfrak{p} > \kappa$

Corollary: $MA \implies \mathfrak{p} = 2^\omega$ ... since $\forall \kappa < 2^\omega$: $\mathfrak{p} > \kappa$, but $\mathfrak{p} \le 2^\omega$

Proof: Let $\mathcal{C} \subseteq [\omega]^\omega$ be centered and have size $|\mathcal{C}| \le \kappa$.

→ we want to find a pseudointersection P

Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be the filter on ω generated by $\mathcal{C}$ together with all cofinite sets

→ consider $\mathbb{M}(\mathcal{F})$ ... σ-centered

↳ take upwards closure
↳ this produces a filter ∵ $\mathcal{C}$ is centered

Define dense sets:

- for $m \in \omega$: $D_m := \{(a,u) \in \mathbb{M}(\mathcal{F}) \mid \max(a) > m\}$
- for $C \in \mathcal{C}$: $R_C := \{(a,u) \in \mathbb{M}(\mathcal{F}) \mid u \subseteq C\}$

R_C is dense ... given (a,u), take $(a, u \cap C) \in R_C$ ... $u \cap C \in \mathcal{F}$ ∵ $\mathcal{F}$ is a filter

D_m is dense ... given (a,u), pick $k \in u$ s.t. $k > m$ ← u is infinite

define $\underbrace{(a',u')}_{\in D_m} \le (a,u)$ by $a' := a \cup \{k\}$, $u' := u \setminus \{0,1,\dots,k\}$

→ in total $\le \kappa$-many dense sets

$\implies MA_\kappa$: $\exists$ filter $G \subseteq \mathbb{M}(\mathcal{F})$ intersecting them all

We claim that $P := \bigcup\{a \mid (a,u) \in G\}$ is a pseudointersection of $\mathcal{C}$

- $|P| = \omega$ ... since G intersects every D_n
- $\forall C \in \mathcal{C}: P \subseteq^* C$ ... since G intersects R_C

→ fix C and pick $(a,u) \in R_C \cap G$ ... $u \subseteq C$ ⊛

<u>claim:</u> $P \setminus a \subseteq C$ ... since a can be extended only from u

- fix $k \in P \setminus a$ ⇝ by definition of P there $\exists (b,v) \in G$ s.t. $k \in b$
- because G is a filter, there is $(a',u') \in G$ extending both (a,u) and (b,v)

→ $k \in a'$ since (a',u') extends (b,v)

→ $k \in u \subseteq C$ since (a',u') extends (a,u) and $k > \max(a)$ ∎

<u>Note:</u> Since $\mathbb{M}(F)$ is σ-centered, it is ccc and preserves all cardinals

MATHIAS FORCING

$\mathbb{M} := \{(a,u) \mid a \in [\omega]^{<\omega},\ u \in [\omega]^{\omega},\ a < u\}$ ordered the same as $\mathbb{M}(\mathcal{F})$

$\mathbb{M}$ is not σ-closed ... $\forall n$ take (a_n, u_n) where $a_n = n$, $u_n = \omega \setminus n$

$\mathbb{M}$ is not ccc ... there is an antichain of size 2^ω ... ! antichain = no lower bounds

↳ take $(\emptyset, B_\alpha)$ where B_α are the branches of the cantor tree $(2^{<\omega}, \subseteq)$

↳ $\forall \alpha \neq \beta$: $B_\alpha \cap B_\beta$ is finite, so there can be no common extension

<u>Fact:</u> $\mathbb{M}$ does <u>not collapse ω_1</u>, but it does <u>collapse 2^ω to ω_1</u>

<u>Theorem:</u> Forcing with $\mathbb{M}$ adds a new subset $X \in [\omega]^\omega$ with the <u>reaping property</u>:

for $\forall A \in V \cap [\omega]^\omega$: $X \subseteq^* A$ or $X \subseteq^* (\omega \setminus A)$

→ X cannot be split into 2 infinite parts by any ground model subset $A \subseteq \omega$

<u>Proof:</u> We define dense sets D_n as before and any V-generic filter $G \subseteq \mathbb{M}$ thus produces an infinite set $X := \bigcup\{a \mid (a,u) \in G\}$

→ fix $A \in V \cap [\omega]^\omega$, we will show that the set of conditions that force that $X \subseteq^* A$ or $X \subseteq^* (\omega \setminus A)$ is dense, so every G forces this behaviour

→ let $q = (a,u) \in \mathbb{M}$ be a condition ... we want a stronger p forcing this

<u>case 1,</u> <u>$u \cap A$ is infinite</u>: Consider the condition $p := (a, u \cap A) \leq q$

<u>claim:</u> $p \in G \Rightarrow X \subseteq^* A$

<u>proof:</u> the same as ⊛ with p playing the role of $(a,u) \in R_C \cap G$

<u>case 2,</u> <u>$u \setminus A$ is infinite</u>: Consider $p := (a, u \setminus A) \leq q$

<u>claim:</u> $p \in G \Rightarrow X \subseteq^* (\omega \setminus A)$

<u>proof:</u> again the same as ⊛ ∎

COHEN FORCING

$Fn(I,2) := \{p ; I \to 2 \mid |p| \text{ is finite}\}$ ordered by $p \le q$ iff $p \supseteq q$

↳ finite partial functions from I to 2

$\mathbb{C} := Fn(\omega, 2)$ ... Cohen forcing → adds a new Cohen real

→ when proving the consistency of $\neg CH$, we forced with $Fn(\omega_2 \times \omega, 2)$

If I is infinite, then $I \approx I \times \omega$, so $Fn(I,2) \cong Fn(I \times \omega, 2)$

Fact: Every countable atomless poset has the same forcing consequences as $\mathbb{C}$, so one may also write e.g. $\mathbb{C} = (2^{<\omega}, \supseteq)$ ... nodes of the Cantor tree

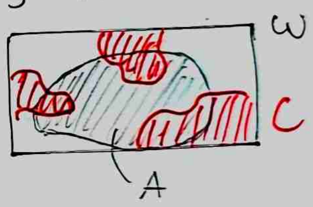
Dense sets:

- for $n \in \omega$: $D_n := \{p \in \mathbb{C} \mid n \in \mathrm{Dom}\, p\}$ ... V-generic $G \subseteq \mathbb{C}$ defines a total function (real number) $c = \bigcup G : \omega \to 2$
- for $r \in V \cap {}^{\omega}2$: $R_r := \{p \in \mathbb{C} \mid \exists k \in \mathrm{Dom}\, p \;\; p(k) \neq r(k)\}$ ... Cohen real c differs from $\forall$ old real

  ↳ for every old real number

Proposition: The Cohen real c defines a set $C := \{n \in \omega \mid c(n) = 1\} \in [\omega]^{\omega}$ that splits every ground model $A \in V \cap [\omega]^{\omega}$

↳ both $A \cap C$ and $A \setminus C$ are infinite



Proof: Fix A and for every $n \in \omega$ define

$E^A_n := \{p \in \mathbb{C} \mid \exists k \geq n, k \in A \cap \mathrm{Dom}\, p, p(k) = 1\}$ ... dense ∵ A is infinite ⇒ unbounded

⇒ G intersects every E^A_n, so C agrees with A arbitrarily high ⇒ $A \cap C$ is infinite ∎

→ we can do a similar argument for the zeroes $p(k) = 0$

Adding κ many Cohen reals

→ forcing with $Fn(\kappa \times \omega, 2)$ adds κ many distinct Cohen reals

↳ just reuse the exact same arguments we used when working with $Fn(\omega_1 \times \omega, 2)$ to prove the consistency of $\neg CH$

→ $Fn(I,2)$ is always ccc ... by the Δ-system lemma

⇒ it preserves the cardinal structure of the universe

Corollary: For $\forall \alpha$ it is consistent that $2^{\omega} \geq \aleph_{\alpha}$.

ALMOST DISJOINT SYSTEMS & UNIFORMIZATION

Def: For functions g, h with common domain D we write

$$g =^* h \quad \text{iff} \quad \{x \in D \mid g(x) \neq h(x)\} \text{ is finite}$$

→ they agree, modulo finite exceptions

Def: For a function f and a constant k we write

$$f \equiv^* k \quad \text{iff} \quad \{x \in \operatorname{Dom} f \mid f(x) \neq k\} \text{ is finite}$$

→ f is the constant function k, modulo finitely many exceptions

Def: A family $\mathcal{A} \subseteq [\omega]^\omega$ is

- an AD system $\equiv$ for all $A \neq B$ in $\mathcal{A}$: $|A \cap B| < \omega$
- uniformizable $\equiv$ for $\forall f: \mathcal{A} \to 2$ perscription of values 0 and 1 to $A \in \mathcal{A}$, there $\exists u: \omega \to 2$ s.t. $\forall A \in \mathcal{A}$: $u \restriction A \equiv^* f(A)$
- strongly uniformizable $\equiv$ for every family $\{f_A : A \to 2 \mid A \in \mathcal{A}\}$ there $\exists u: \omega \to 2$ s.t. $\forall A \in \mathcal{A}$: $u \restriction A =^* f_A$

strongly unif. $\Rightarrow$ unif. $\Rightarrow$ AD system

- unif ~ strongly unif where we only allow f_A to be constant functions 0/1
- if there were $A \neq B$ in $\mathcal{A}$ with $|A \cap B| = \omega$, take $f: \mathcal{A} \to 2$, $\begin{array}{l} A \mapsto 0 \\ B \mapsto 1 \end{array}$

Exercise: If $|\mathcal{A}| \leq \omega$, then the reverse implications also hold.

There exists an AD system of size $2^\omega = |[\omega]^\omega|$

↳ take the branches of the Cantor tree ... $A \in [\omega]^\omega$ ~ its char. function $\omega \to 2$

Observation: If $\mathcal{A}$ is uniformizable, then $2^{|\mathcal{A}|} \leq 2^\omega$

Note: By Easton's theorem, it is consistent that for example $2^\omega = 2^{\aleph_1} = 2^{\aleph_2} = 2^{\aleph_3} = \aleph_4$

Proof: There are $2^{|\mathcal{A}|}$ functions $f: \mathcal{A} \to 2$ and we assume that each has $u_f\ \omega \to 2$ as above

claim: distinct $f, f': \mathcal{A} \to 2$ cannot share the same uniformization u.

$\Rightarrow$ hence $f \mapsto u_f$ is an injection from $2^{|\mathcal{A}|}$ to 2^ω ... the end

proof: Suppose u uniformizes both f and f' and pick $A \in \mathcal{A}$ s.t. $f(A) \neq f'(A)$

↳ then $u \restriction A \equiv^* f(A)$ but also $u \restriction A \equiv^* f'(A)$ $\lightning$ ■

Fact: ZFC cannot prove the other direction.

↳ proof idea on the next page

Def: Given an AD system $\mathcal{A} \subseteq [\omega]^\omega$, we say that a set $X \in [\omega]^\omega$ separates two subsystems $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ if $(\forall A \in \mathcal{B})\, A \subseteq^* X$ & $(\forall A \in \mathcal{C})\, A \cap X =^* \emptyset$.

Def: An AD system is inseparable if no two uncountable subsystem can be separated.

inseparable $\Rightarrow$ not uniformizable ... take any $f: \mathcal{A} \to 2$ that assigns both 1 and 0 to uncountably many $A \in \mathcal{A}$ ... defines two uncountable subsystems ... a uniformization would separate them

Def: An AD system $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ is a Luzin gap if

(*) $(\forall \alpha < \omega_1)(\forall n \in \omega)$: $\{\beta < \alpha \mid A_\alpha \cap A_\beta \subseteq n\}$ is finite ... (*) is $\Delta_0 \Rightarrow$ absolute

Theorem: $\boxed{1}$ Every Luzin gap is inseparable $\boxed{2}$ ZFC $\vdash$ there $\exists$ Luzin gap

Corollary: ZFC $\vdash$ there $\exists$ AD system $\mathcal{L}$ of size ω_1 that is not uniformizable

$\hookrightarrow$ This gives the fact from the end of the previous page since by Easton's theorem there is a model where $2^{\omega_1} = 2^\omega$ but $|\mathcal{L}| = \omega_1$ is not unif.

Proof of $\boxed{1}$: Suppose that there are two uncountable subsystems $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ and a set $X \in [\omega]^\omega$ separating them ... in other words:

- $\forall B \in \mathcal{B}$: $B \setminus X$ is finite
- $\forall C \in \mathcal{C}$: $C \cap X$ is finite

$\Rightarrow$ by pigeonhole there $\exists$ uncountable $\mathcal{B}' \subseteq \mathcal{B}$, $\mathcal{C}' \subseteq \mathcal{C}$ and numbers $m, n \in \omega$ s.t.

- $\forall B \in \mathcal{B}'$: $B \setminus X \subseteq m$
- $\forall C \in \mathcal{C}'$: $C \cap X \subseteq k$

$\to$ let $n := \max(m, k)$ and take any $A_\alpha \in \mathcal{B}'$ and $A_\beta \in \mathcal{C}'$ ... then $A_\alpha \cap A_\beta \subseteq n$

$\hookrightarrow A_\alpha \cap A_\beta = (A_\alpha \cap A_\beta \cap X) \cup (A_\alpha \cap A_\beta \setminus X) \subseteq n$

$\Rightarrow$ since $|\mathcal{B}'| = \omega_1$, its set of indices is cofinal and we can pick $A_\alpha \in \mathcal{B}'$ with infinitely many $\beta < \alpha$ s.t. $A_\beta \in \mathcal{C}'$, violating $\{\beta < \alpha \mid A_\alpha \cap A_\beta \subseteq n\}$ being finite ∎

Proof of $\boxed{2}$: We will construct a Luzin gap $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ by transfinite recursion

$\to$ first choose a family $\{A_n \mid n < \omega\}$ of pairwise disjoint infinite subsets of ω

$\hookrightarrow$ this satisfies (*) for any $\alpha < \omega$

$\to$ suppose we already constructed an AD system $\{A_\beta \mid \beta < \alpha\}$ for some $\omega \leq \alpha < \omega_1$ and want to add a new set A_α

$\to$ because $|\alpha| = \omega$, we can enumerate $\{A_\beta \mid \beta < \alpha\}$ as $\{B_n \mid n < \omega\}$

- for $k \in \omega$ consider $R_k := B_k \setminus \bigcup_{i<k} B_i$ ... since B_k is infinite & the sets are almost disjoint, the set R_k is infinite $\Rightarrow \exists x_k \in R_k$ s.t. $x_k > k$. Let $A_\alpha := \{x_k \mid k \in \omega\}$

- almost disjoint: A_α is almost disjoint from every A_β, $\beta < \alpha$

  $\hookrightarrow A_\beta = B_i$ for some i and $A_\alpha \cap B_i \subseteq \{x_0, \dots, x_i\}$ since x_k for $k > i$ avoid B_i

- Luzin property: $\forall n \in \omega$: $\{\beta < \alpha \mid A_\alpha \cap A_\beta \subseteq n\}$ is finite

  $\hookrightarrow$ suppose $A_\alpha \cap B_k \subseteq n$ ... since $x_k \in B_k$, then $k < x_k < n$

  $\Rightarrow$ this can happen only when $k < n$ ... finitely many times ∎

UNIFORMIZING THE CANTOR TREE

→ in the definition of an AD system / unif. system we had $A \subseteq [\omega]^\omega$

↳ but we could equivalently have $A \subseteq [Z]^\omega$ for any countable Z

→ consider $Z = 2^{<\omega}$ ... the Cantor tree

<u>Notation</u>: $[2^{<\omega}]$ is the set of <u>branches</u> of Z

↳ we can view $B \in [2^{<\omega}]$ as $B \in [2^{<\omega}]^\omega$ ... the set of finite initial segments of B

! In this section: $Z = 2^{<\omega}$, $A \subseteq [2^{<\omega}]$

<u>Theorem</u>: Let $A \subseteq [2^{<\omega}]$ have size $|A| =: \kappa < 2^\omega$

$MA_\kappa(\sigma\text{-centered}) \Rightarrow A$ is uniformizable.

<u>Corollary</u>: $\kappa < \mathfrak{p} \Rightarrow 2^\kappa = 2^\omega$ → from monotonicity $\omega \leq \kappa \Rightarrow 2^\kappa \geq 2^\omega$

<u>Proof</u>: By Bell's theorem, $\mathfrak{p} > \kappa \Rightarrow MA_\kappa(\sigma\text{-centered})$

→ we can take any $A \subseteq [2^{<\omega}]$ of size $|A| = \kappa$ and by MA it is uniformizable

→ but by an earlier observation any uniformizable family A satisfies $2^{|A|} \leq 2^\omega$ ■

<u>Corollary</u>: $MA \Rightarrow \forall \kappa < 2^\omega : 2^\kappa = 2^\omega$

↳ since $MA \Rightarrow \mathfrak{p} = 2^\omega$

<u>Proof of Theorem</u>: Fix A and a function $f : A \to 2$ we will want to uniformize.

→ we define the <u>uniformization forcing</u> for f as follows: coherence condition

$$\mathbb{P}_f := \{ p = (m_p, v_p, B_p) \mid m_p \in \omega,\ v_p : 2^{\leq m_p} \to 2,\ B_p \in [A]^{<\omega},\ \circledast \}$$

↳ binary tree of height $m_p + 1$, labeled by 0s and 1s ↳ finite set of branches

→ ordered by $p \leq q$ iff

- $m_p \geq m_q$, $v_p \supseteq v_q$, $B_p \supseteq B_q$
- $v_p(s) = f(A)$ for all branches $A \in B_q$ and their initial segments $s \sqsubseteq A$ s.t. $s \in 2^k$ where $k \in (m_q, m_p]$

↳ formally $s \in A \in [2^{<\omega}]^\omega$

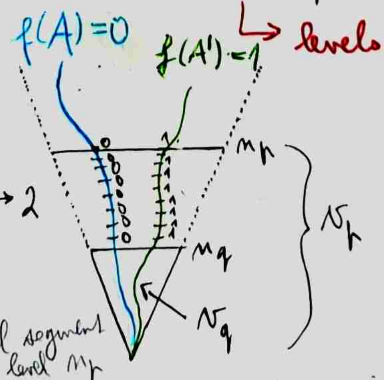
ends at level k ↳ levels of v_p above v_q

<u>Intuition</u>:

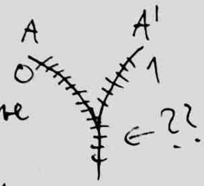
m_p ... marker for how far v_p goes

v_p ... approx. of the desired $u : Z \to 2$

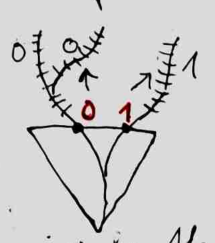
B_p ... list of <u>promises</u>



! problem: what if we have



Solution:



$\circledast$ $(\forall A \in B_p) : v_p(A \upharpoonright 2^{m_p}) = f(A)$ ... so the extension color is determined by the color of the longest initial segment of A in v_p ⇒ compatibility ✓

$A \upharpoonright 2^{m_p}$: $\in 2^{m_p}$, initial segment up to level m_p

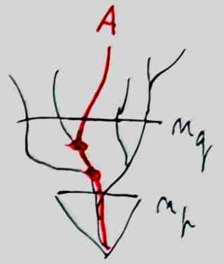
$\mathbb{P}_f$ is σ-centered ... for $\forall n\in\omega$ and $v: 2^{\le n}\to 2$, the set $C_v := \{p\in\mathbb{P}_f \mid v_p = v\}$ is centered, and there are countably many such v

↳ just merge all promises, keeping the tree $= v$

Defining dense sets:

- for $n\in\omega$ let $D_n := \{p\in\mathbb{P}_f \mid n_p \ge n\}$ ... obviously dense
- for $A\in\mathcal{A}$ let $R_A := \{p\in\mathbb{P}_f \mid A\in B_p\}$ ... can we always add A as a promise?

→ to see that R_A is dense, given $p\in\mathbb{P}_f$ s.t. $A\notin B_p$, we choose n_q high enough so that A separates from every $A'\in B_p$ ... now the coherence condition ⊛ can be satisfied



We invoke MA_κ ... $\omega + |\mathcal{A}| = \omega+\kappa = \kappa$ dense sets

→ let $G\subseteq \mathbb{P}_f$ be a filter intersecting all these dense sets

⇒ define $u: 2^{<\omega}\to 2$ by $u := \bigcup\{v_p \mid p\in G\}$

- u is a legal function ∵ G is a filter
- u is total ∵ G intersects every D_n
- $\forall A\in\mathcal{A}: u\restriction A =^* f(A)$ ... u is the desired uniformization

↳ because G intersects R_A ... let $p\in G\cap R_A$ ... $A\in B_p$

claim: $\forall k > n_p: u(A\restriction k) = f(A)$ ... after n_p, u adheres to the promise A

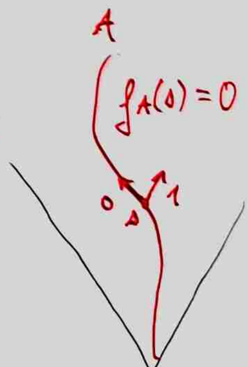
↳ take $q\in G$ s.t. $n_q \ge k$ ... since G is a filter, there $\exists p'\in G, p'\le p,q$

→ $v_{p'}(A\restriction k) = u(A\restriction k)$ and p' extends p so it adheres to A ⇒ $v_{p'}(A\restriction k)=f(A)$ ▨

STRONGLY UNIFORMIZING THE CANTOR TREE

Recall: Every countable AD system is strongly uniformizable

Theorem: No $\mathcal{A}\subseteq[2^{<\omega}]$ of size $|\mathcal{A}|=\omega_1$ is strongly uniformizable

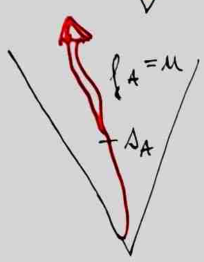


Proof: For each branch $A\in\mathcal{A}$ define $f_A: A\to 2$ by $f_A(s)=i$, where i = unique bit s.t. $s^\frown i \in A$

claim: $\mathcal{F} := \{f_A \mid A\in\mathcal{A}\}$ has no uniformization

→ suppose that $u: 2^{<\omega}\to 2$ uniformizes $\mathcal{F}$

⇒ so $(\forall A\in\mathcal{A})(\exists s_A\in A)$ s.t. $\forall t\in A, t\supseteq s_A: u(t)=f_A(t)$



→ since $|\mathcal{A}|=\omega_1$ but $|2^{<\omega}|=\omega$, by pigeonhole there are $A\ne B$ in $\mathcal{A}$ s.t. $s_A=s_B$ ... so these branches both contain s_A and u correctly predicts f_A and f_B from this point on

→ but $A\ne B$, so the labeling will diverge once A and B split ↯ ▨

? maybe if we tried something more messy than branches, we could construct an uncountable, strongly uniformizable system? Actually, it is independent:

Fact: MA $\Rightarrow$ every strongly uniformizable system $\mathcal{A} \subseteq [\omega]^{\omega}$ is countable.

Fact: GCH $\Rightarrow \exists$ ccc $\mathbb{P}$ s.t. forcing with $\mathbb{P}$ produces an uncountable, strongly unif system

$\Rightarrow$ MA is not a full substitute for ccc forcing as assuming MA might cause some poset whose ccc is independent to not be ccc

LEBESGUE MEASURE

$\rightarrow$ we consider the Lebesgue measure λ on $[0,1]$

$\mathcal{I} := \{A \subseteq [0,1] \mid A$ is a finite union of intervals with rational endpoints$\}$

Note: $\mathcal{I}$ is countable.

Def: The Lebesgue null ideal is $\mathcal{N}_{\lambda} := \{A \subseteq [0,1] \mid A$ is measurable & $\lambda(A) = 0\}$

this is a σ-ideal ... closed under countable unions from the σ-additivity of λ

Def: The additivity of $\mathcal{N}_{\lambda}$ is $\quad \rightarrow$ add$(\mathcal{N}_{\lambda})$ = largest κ s.t. $\mathcal{N}_{\lambda}$ is κ-complete ideal

$$\text{add}(\mathcal{N}_{\lambda}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{N}_{\lambda} \ \& \ \bigcup \mathcal{A} \notin \mathcal{N}_{\lambda} \text{ ... it is not null}\}$$

$\hookrightarrow$ unmeasurable or has positive measure

$\omega < \text{add}(\mathcal{N}_{\lambda}) \leq 2^{\omega}$

$\hookrightarrow \omega < \mathcal{N}_{\lambda}$ from additivity of λ, and $\leq 2^{\omega}$ since $[0,1] = \bigcup_{r \in [0,1]} \{r\}$

Fact: ZFC $\vdash$ add$(\mathcal{N}_{\lambda}) \leq \mathfrak{b}$

Theorem: $MA_{\kappa}(\sigma\text{-linked}) \Rightarrow \text{add}(\mathcal{N}_{\lambda}) > \kappa$

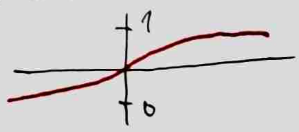
Corollary: ZFC cannot prove that $\exists \mathcal{A} \subseteq \mathcal{N}_{\lambda}$ of size $|\mathcal{A}| < 2^{\omega}$ s.t. $\bigcup \mathcal{A}$ is not null

$\hookrightarrow$ in other words, it is consistent that the ideal $\mathcal{N}_{\lambda}$ is 2^{ω}-complete

closed under unions of size $< 2^{\omega}$

Remark: This also follows from the consistency of CH ... since then $2^{\omega} = \omega_1$ ... but now we know it would be true even if CH were not consistent

? Aren't we making things easier by restricting our attention to $[0,1]$?

$\hookrightarrow$ NO! ∴ $(0,1)$ and $\mathbb{R}$ are isomorphic through a map like arctan 

$\rightarrow$ we prove the theorem on the next page

Proof of the Theorem ... want: $\mathrm{add}(\mathcal{N}_\lambda) > \kappa$

finite unions of intervals with rational endpoints

$\to$ we will need two facts about λ

Fact $\boxed{1}$: If $X \subseteq [0,1]$ is measurable, then for $\forall \varepsilon > 0$ there $\exists$ compact $I \in \mathcal{I}$ and open $\sigma \subseteq (0,1)$ s.t.

$$I \subseteq X \subseteq \sigma \quad \text{and} \quad \lambda(\sigma \setminus I) < \varepsilon$$

Fact $\boxed{2}$: For any sequence $\langle X_i \mid i \in \omega \rangle$ of measurable sets of $[0,1]$, we have

$$\lambda\Big(\bigcup_{i<\omega} X_i\Big) = \lim_{n\to\infty} \lambda\Big(\bigcup_{i<n} X_i\Big)$$

$\to$ Let $\mathcal{A} \subseteq \mathcal{N}_\lambda$ be a family of null sets s.t. $|\mathcal{A}| \leq \kappa$

Plan: show that $\forall \varepsilon > 0$: $\lambda^*(\bigcup \mathcal{A}) \leq \varepsilon$, so $\lambda^*(\bigcup\mathcal{A}) = 0$, so $\bigcup \mathcal{A}$ is measurable and $\lambda(\bigcup\mathcal{A}) = 0$

$\hookrightarrow$ outer measure

$\Rightarrow$ Fix $\varepsilon > 0$ and consider the poset

$$\mathbb{P}_\varepsilon := \{ u \subseteq [0,1] \mid u \text{ open } \& \ \lambda(u) < \varepsilon \} \text{ ordered by } u \leq v \text{ iff } v \subseteq u$$

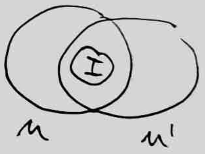
Claim: $\mathbb{P}_\varepsilon$ is σ-linked.

this is possible from Fact $\boxed{1}$

proof: For each $u \in \mathbb{P}_\varepsilon$ choose $I_u \in \mathcal{I}$ s.t. $I_u \subseteq u$ and $\lambda(u \setminus I_u) < \dfrac{\varepsilon - \lambda(u)}{2}$

- for $I \in \mathcal{I}$ define $A_I := \{ u \in \mathbb{P}_\varepsilon \mid I_u = I \}$ ... countable partition of $\mathbb{P}_\varepsilon$ since $|\mathcal{I}| = \omega$
- each A_I is linked.

$\hookrightarrow$ let $u, u' \in A_I$ ... since $I \subseteq u \cap u'$, we have



$$u \cup u' = (u \setminus I) \cup (u' \setminus I) \cup I$$

$\Rightarrow$ by subadditivity:

$$\lambda(u \cup u') \leq \lambda(u \setminus I) + \lambda(u' \setminus I) + \lambda(I)$$
$$< \frac{\varepsilon - \lambda(u)}{2} + \frac{\varepsilon - \lambda(u')}{2} + \lambda(I) = \varepsilon + \overbrace{\frac{\lambda(I) - \lambda(u)}{2}}^{\leq 0} + \overbrace{\frac{\lambda(I) - \lambda(u')}{2}}^{\leq 0} \leq \varepsilon$$

$\Rightarrow$ $u \cup u' \in \mathbb{P}_\varepsilon$, and it is a common lower bound of u and u'

Remark: $\mathbb{P}_\varepsilon$ is in fact σ-n-linked for every $n \in \omega$.

$\hookrightarrow$ $A \subseteq \mathbb{P}$ is n-linked if $\mathbb{P}$ contains common lower bounds for all n-tuples from A

$\hookrightarrow$ $\mathbb{P}$ is σ-n-linked if $\mathbb{P} = \bigcup_{n<\omega} A_n$ where each A_n is n-linked in $\mathbb{P}$

To see this, we simply estimate $\lambda(u \setminus I_u) < \dfrac{\varepsilon - \lambda(u)}{n}$

! σ-n-linked for $\forall n$ $\not\Rightarrow$ σ-centered

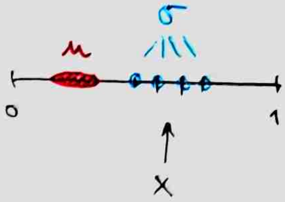
$\hookrightarrow$ different n may require different decompositions of $\mathbb{P}$

Defining dense sets: ↗ all ε-small open sets containing the null-set X

- for $\forall X \in \mathcal{A}$ let $D_X := \{ u \in \mathbb{P}_\varepsilon \mid X \subseteq u \}$

→ D_X is dense in $\mathbb{P}_\varepsilon$ ... given open u with $\lambda(u) < \varepsilon$, use Fact $\boxed{1}$ and let $\sigma \supseteq X$ be open with $\lambda(\sigma) < \varepsilon - \lambda(u)$

⇒ then $u \cup \sigma \in D_X$ and $u \cup \sigma \leq u$



$MA_{\aleph_1}$ (σ-linked):

→ let $G \subseteq \mathbb{P}_\varepsilon$ be a filter intersecting each D_X

👁: $\bigcup \mathcal{A} \subseteq \bigcup G$ ... since $G \cap D_X \neq \emptyset$

claim: $\lambda^*(\bigcup G) \leq \varepsilon$ ... so $\lambda^*(\bigcup \mathcal{A}) \leq \varepsilon$ and we are done

proof: Note that every open set is a (countable) union of intervals with rational endpoints

⇒ $u \in G$ is a union of (a,b):

↳ $(a,b) \in \mathbb{P}_\varepsilon$ and $u \leq (a,b)$, so $(a,b) \in G$ ← upward closure of G

⇒ $\bigcup G = \bigcup \{ u \in G \mid u \in \mathcal{I} \}$

Since $\mathcal{I}$ is countable, we can enumerate $G \cap \mathcal{I}$ as $\langle u_i \mid i \in \omega \rangle$

- $\forall n$: $u_0, \dots, u_{n-1} \in G$ have a common lower bound $w \in G$ ∵ G is a filter

↳ $w \supseteq u_0 \cup \dots \cup u_{n-1}$ and $\lambda(w) < \varepsilon$ since $w \in \mathbb{P}_\varepsilon$

By Fact $\boxed{2}$: $\lambda^*(\bigcup G) = \lambda\left(\bigcup_{i<\omega} u_i\right) = \lim_{n\to\infty} \lambda\left(\bigcup_{i<n} u_i\right) \leq \lim_{n\to\infty} \lambda(w) < \varepsilon$ ■

↳ countable union of measurable sets is measurable

THE DIAMOND PRINCIPLE ◇

→ we first review some combinatorics of ω_1

Def: Let $C, S \subseteq \omega_1$. ↗ if we omit "limit" here, nothing changes as $\sup(\alpha+1) = \alpha$

- C is closed if $\forall$ limit $\alpha < \omega_1$ with $\sup(C \cap \alpha) = \alpha$, we have $\alpha \in C$
- C is unbounded if $\forall \beta < \omega_1$ $\exists \alpha \in C$ s.t. $\alpha > \beta$
- C is a club if it is both closed & unbounded
- S is stationary if $S \cap C \neq \emptyset$ for every club $C \subseteq \omega_1$

Fact:

- every stationary set is unbounded ... since $\omega_1 \setminus \alpha$ is a club for $\forall \alpha < \omega_1$
- intersection of countably many clubs is a club ⇒ every club is stationary
- if $S \subseteq \omega_1$ is stationary and $S = \bigcup_{n<\omega} S_n$, then some S_n is stationary
- if $A \subseteq \omega_1$ is unbounded, then $acc(A) := \{ \alpha < \omega_1 \mid \sup(A \cap \alpha) = \alpha \}$ is a club

↳ accumulation points

Def: The diamond principle $\diamondsuit$ is the statement that there exists a $\diamondsuit$-sequence $\langle A_\alpha \subseteq \alpha \mid \alpha < \omega_1 \rangle$ s.t.

$$\forall A \subseteq \omega_1 : \{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\} \text{ is stationary in } \omega_1$$

$\rightarrow$ so we can "predict" every $A \subseteq \omega_1$ with stationary precision

Example: We can note for instance

- $A = \emptyset$ ... stationary-many A_α are empty
- $A = \omega$ ... if $\alpha \geq \omega$ and $A \cap \alpha = A_\alpha$, then $A_\alpha = \omega$
  $\Rightarrow$ stationary-many A_α are $= \omega$
- in general if $A \subseteq \omega_1$ is bounded, so $\sup(A) < \omega$, then there are stationary-many A_α equal to A

Observation: $\diamondsuit \Rightarrow CH$

Proof: Every $A \subseteq \omega$ is bounded in ω_1, so in particular $\exists \alpha < \omega_1$ s.t. $A_\alpha = A$
$\Rightarrow$ hence $P(\omega) \subseteq \{A_\alpha \mid \alpha < \omega_1\}$ ∎

HAJNAL-MÁTÉ GRAPHS

Def: A family $\{C_\alpha \subseteq \alpha \mid \text{limit } \alpha < \omega_1\}$ is a ladder system on ω_1 if

$\forall$ limit $\alpha < \omega_1$: $\sup C_\alpha = \alpha$ & C_α has order-type ω

$\rightarrow$ essentially, we pick a fundamental sequence for $\forall$ limit $\alpha < \omega_1$

Def: Fix a ladder system $\{C_\alpha\}$. The associated Hajnal-Máté graph G has vertices $= \omega_1$, edges: for every limit α, draw edges from α to all $\beta \in C_\alpha$

Theorem: $\diamondsuit \Rightarrow \exists$ ladder system $\{C_\alpha\}$ whose HM graph G has $\chi(G) = \omega_1$

Theorem: $MA_{\omega_1} \Rightarrow \chi(G) \leq \omega$ for every HM graph G.

$\rightarrow$ to prove the $\diamondsuit$ theorem, we will need an equivalent reformulation of $\diamondsuit$

Fact: $\diamondsuit \iff \exists$ sequence $\langle D_\alpha : \alpha \to \omega \mid \alpha < \omega_1 \rangle$ s.t.

$$\forall F : \omega_1 \to \omega : \{\alpha < \omega_1 \mid f \restriction \alpha = D_\alpha\} \text{ is stationary}$$

Note: Clearly "$\Leftarrow$" since a subset $A_\alpha \subseteq \alpha$ is just a function $\alpha \to 2$
$A \subseteq \omega_1$ $\qquad$ $\omega_1 \to 2$

<u>Proof of the $\diamondsuit$ Theorem:</u> Fix a sequence $\langle D_\alpha : \alpha \to \omega \mid \alpha < \omega_1 \rangle$

→ we can construct a ladder system $\{C_\alpha\}$ s.t. $\forall$ limit $\alpha < \omega_1$ and
$\forall n \in N_\alpha := \{n \in \omega \mid D_\alpha^{-1}(n)$ is cofinal in $\alpha\}$ we have $C_\alpha \cap D_\alpha^{-1}(n) \neq \emptyset$

<u>Intuition:</u> for each limit $\alpha < \omega_1$ we consider its labeling by $n \in \omega$
→ if a label $n \in \omega$ reaches all the way to α, then we want our fundamental sequence $C_\alpha \subseteq \alpha$ to use this label at least once

this is easy to arrange... we want a cofinal ω-sequence $C_\alpha = \{\gamma_\xi \mid \xi < \omega\} \subseteq \alpha$
↳ enumerate $N_\alpha = \{n_\xi \mid \xi < |N_\alpha|\}$ and for $\xi < |N_\alpha|$ choose $\gamma_\xi \in D_\alpha^{-1}(n_\xi)$
→ if N_α is finite, chose the other γ_ξ arbitrarily
→ we can make C_α cofinal since all $D_\alpha^{-1}(n)$ for $n \in N_\alpha$ are cofinal

<u>claim:</u> The ladder system described above yields a HM graph G with $\chi(G) = \omega_1$

<u>Proof:</u> Suppose for contradiction that $F : \omega_1 \to \omega$ is a proper coloring of G

- by $\diamondsuit$, the set $S = \{\alpha < \omega_1 \mid F \restriction \alpha = D_\alpha\}$ is stationary
- since F decomposes ω_1 into countably many color classes $F^{-1}(n)$, there $\exists n$ s.t.
  $S_n = S \cap F^{-1}(n) = \{\alpha \in S \mid F(\alpha) = n\}$ is stationary (by an earlier fact)
- in particular, the color class $F^{-1}(n)$ is unbounded in ω_1

⇒ by an earlier fact, the set $C = \{\alpha < \omega_1 \mid F^{-1}(n) \cap \alpha$ is cofinal in $\alpha\}$ is a club

- since S_n is stationary and C a club, there $\exists \alpha \in S_n \cap C$
  - $\alpha \in S_n$ ... we know: $F \restriction \alpha = D_\alpha$ & the color $F(\alpha) = n$
  - $\alpha \in C$ ... $F^{-1}(n) \cap \alpha = D_\alpha^{-1}(n)$ is cofinal in α ... therefore $n \in N_\alpha$

⇒ by our construction there exists $\beta \in C_\alpha \cap D_\alpha^{-1}(n)$
- since $F \restriction \alpha = D_\alpha$, the vertex β has color $F(\beta) = D_\alpha(\beta) = n$
- since $\beta \in C_\alpha$, it is connected to vertex α, which also has color $F(\alpha) = n$ ■

⇒ so F is not a proper coloring ↯

Proof of the $MA\omega_1$ Theorem

Fix a ladder system $\{C_\alpha\}$ and let G be its δM-graph

→ we want a proper coloring $f : \omega_1 \to \omega$ ... consider proper colorings of finite $H \subseteq G$:

$$\mathbb{P} := \{ \underset{\text{partial function}}{p ; \omega_1 \to \omega} \mid |\mathrm{Dom}\, p| < \omega ,\ \forall \alpha, \beta \in \mathrm{Dom}\, p : \beta \in C_\alpha \Rightarrow p(\alpha) \neq p(\beta) \}$$

→ ordered by $p \le q$ iff $p \supseteq q$ ... p knows more

if $G \subseteq \mathbb{P}$ is a filter, then $\bigcup G ; \omega_1 \to \omega$, and if G intersects each

$D_\alpha := \{ p \in \mathbb{P} \mid \alpha \in \mathrm{Dom}\, p \}$ ... dense ... then $\bigcup G$ is total

⇒ if we show that $\mathbb{P}$ is ccc, then we can use $MA\omega_1$ and we are done

Let $A \subseteq \mathbb{P}$ have size $|A| = \omega_1$ ... we will find compatible $p, q \in A$

→ we apply the Δ-system lemma to the family $\{ \mathrm{Dom}\, p \mid p \in A \}$ and get

$A_0 \subseteq A$ of size $|A_0| = \omega_1$ and a finite root $\Delta \subseteq \omega_1$ s.t.

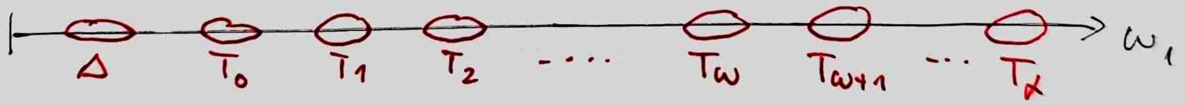
$$\mathrm{Dom}\, p \cap \mathrm{Dom}\, q = \Delta \quad \text{for all } p \neq q \text{ in } A_0$$

__Lemma:__ There is $A_1 \subseteq A_0$ of size $|A_1| = \omega_1$ that is a __block scheme__:

If we enumerate A_1 as $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ and define tails $T_\alpha := \mathrm{Dom}\, p_\alpha \setminus \Delta$, then

$$\forall \alpha : \Delta < T_\alpha \quad \& \quad \forall \alpha < \beta : T_\alpha < T_\beta$$

finite



__Proof:__ First enumerate A_0 as $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ and define tails as above

→ since Δ is finite, $\gamma := \max \Delta$ is countable. Also, the tails are pairwise disjoint

⇒ there are only countably many α with $\min T_\alpha < \gamma$, so we can drop them

→ now all T_α are above Δ, but they might interlace ... we want a block scheme

⇒ we construct a subsequence $\langle T_{\alpha_\delta} \mid \delta < \omega_1 \rangle$ by transfinite recursion:

Having already chosen $\langle T_{\alpha_\delta} \mid \delta < \beta \rangle$ for some $\beta < \omega_1$, consider how far this goes:

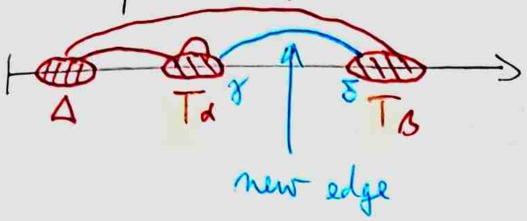
$\Gamma := \sup \{ \max T_{\alpha_\delta} \mid \delta < \beta \} < \omega_1$ ... countable

→ as with γ above, since the tails are disjoint, only countably many can intersect

Γ, so when we drop them, we still have an uncountable reservoir ahead ∎

There are only countably many functions $\Delta \to \omega$, so by pigeonhole there is $A_1' \subseteq A_1$ of size $|A_1'| = \omega_1$ s.t. $\forall p,q \in A_1' : p \upharpoonright \Delta = q \upharpoonright \Delta$. WLOG $A_1' = A_1$

→ so all $p \in A_1$ agree on $\Delta \subseteq \mathrm{Dom}\, p$, and we want compatible $p_\alpha, p_\beta \in A_1$



But there could be a new edge introduced when we consider $p_\alpha \cup p_\beta \le p_\alpha, p_\beta$

⇒ <u>want</u>: $p_\alpha, p_\beta \in A_1$ s.t. there is no $T_\alpha - T_\beta$ edge

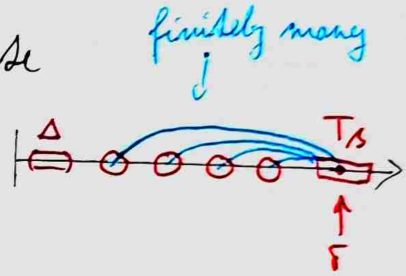
↳ there can be an edge only if $\gamma \in C_\delta$ and $\gamma \in T_\alpha$, $\delta \in T_\beta$

Put $A_2 := \{ p_\alpha \in A_1 \mid \alpha \text{ is an isolated ordinal} \}$ ... $|A_2| = \omega_1$

<u>Claim</u>: For $\forall p_\beta \in A_2$ and $\forall$ limit $\delta \in \overset{\text{finite}}{T_\beta}$, the set of bad edges $\gamma \in C_\delta$

$$E_\delta = \{ \gamma \in C_\delta \mid \exists \alpha < \beta : \gamma \in T_\alpha \} \quad \text{is finite}$$

→ once we have this, take any $p_\beta \in A_2$ with $\beta \geq \omega$



$$E := \bigcup_{\text{limit } \delta \in T_\beta} E_\delta \quad \text{is finite} \quad \dots \text{ since } T_\beta \text{ is finite}$$

$E =$ all edges into T_β from the left

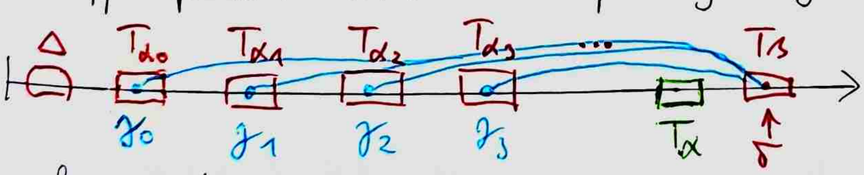
⇒ there are only finitely many bad edges from the left into T_β, but there are infinitely many blocks T_α, $\alpha < \beta$ to the left of T_β, so $\exists \alpha < \beta$ with no $T_\alpha - T_\beta$ edge ⇒ p_α, p_β are compatible

so A is not an antichain ↯

<u>Proof of claim:</u>

• If only finitely many α contribute to E_δ, we are good since the tails T_α are finite

→ suppose for contradiction that infinitely many $\alpha < \beta$ contribute



↳ α_1

→ from each T_{α_n}, choose some $\gamma_n \in C_\delta$

→ since C_δ has order-type ω, we have $\sup \gamma_n = \sup C_\delta = \delta$

→ let <u>$\alpha := \sup \alpha_n$</u> and notice that $\alpha < \beta$ since β is isolated and $\forall n : \alpha_n < \beta$

⇒ $\forall n : T_{\alpha_n} < T_\alpha < T_\beta$, so

• $\forall n : \gamma_n < \min T_\alpha \implies \sup \gamma_n = \delta \leq \min T_\alpha$

→ since $T_\alpha < T_\beta$, this places $\delta \in T_\beta$ strictly before T_β ↯

∎

EQUIVALENT FORMULATIONS OF MARTIN'S AXIOM

Def: A subset $D \subseteq \mathbb{P}$ is

OPEN = DOWNWARDS CLOSED

- open dense $\equiv$ it is dense & downwards closed
- pre-dense $\equiv (\forall p \in \mathbb{P})(\exists t \in D): p \parallel t$

Def: Let $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$. A filter $G \subseteq \mathbb{P}$ is $\mathcal{D}$-generic if $\forall D \in \mathcal{D}: G \cap D \neq \emptyset$

Lemma: In the statement of MA_κ,

$(\forall \text{ccc poset } \mathbb{P})(\forall \mathcal{D} \subseteq \mathcal{P}(\mathbb{P}), |\mathcal{D}| \leq \kappa)(\forall D \in \mathcal{D} \text{ is dense}): \exists\ \mathcal{D}\text{-generic filter } G,$

we can replace "dense" with any of the following:

(1) open dense (2) pre-dense (3) $\subseteq$-maximal antichain (in $\mathbb{P}$)

Every maximal antichain is pre-dense $\rightarrow$ otherwise we could extend it

Proof: Obviously $MA \Rightarrow$ (1) and (2) $\Rightarrow$ (3)

(1) $\Rightarrow$ (2): If D is pre-dense, then its downward closure $\downarrow D := \{p \in \mathbb{P} \mid \exists q \in D: p \leq q\}$ is open dense. If G is a filter intersecting $\downarrow D$, then it also intersects D since G is upwards closed

(3) $\Rightarrow MA$: If D is dense, and $A \subseteq D$ is a maximal antichain in D, then A is also a maximal antichain in $\mathbb{P}$

$\hookrightarrow$ if we could add $p \in \mathbb{P}$, there is $q \in D$, $q \leq p$, and we could add q

Corollary: For generocity, we can use: dense sets, open dense, pre-dense, maximal antichains.

PRODUCT ORDERING POSETS

Knaster = every uncountable $A \subseteq \mathbb{Q}$ contains an uncountable linked subset $B \subseteq A$

? $\mathbb{P}, \mathbb{Q}$ ccc $\overset{?}{\Longrightarrow}$ $\mathbb{P} \times \mathbb{Q}$ ccc ?

$\hookrightarrow$ product order $(p,q) \leq (p',q')$ iff $p \leq p'$ & $q \leq q'$

Proposition: $\mathbb{P}$ ccc, $\mathbb{Q}$ Knaster $\rightarrow \mathbb{P} \times \mathbb{Q}$ ccc

Proof: Let $\{(p_\alpha, q_\alpha) \mid \alpha < \omega_1\} \subseteq \mathbb{P} \times \mathbb{Q}$ be an uncountable subset ... want compatible elements

Case A) If $\exists q \in \mathbb{Q}$ s.t. $|\{\alpha \mid q_\alpha = q\}| = \omega_1$, then invoke the ccc of $\mathbb{P}$ and get $\alpha_1 \neq \alpha_2$ s.t. $p_{\alpha_1} \parallel p_{\alpha_2}$ ... then $(p_{\alpha_1}, q) \parallel (p_{\alpha_2}, q)$

Case B) Otherwise each $q \in \mathbb{Q}$ repeats only countably many times, so # distinct $q_\alpha = \omega_1$

$\Rightarrow$ use Knaster of $\mathbb{Q}$ to get $A \in [\omega_1]^{\omega_1}$ s.t. $\forall \alpha, \beta \in A: q_\alpha \parallel q_\beta$

$\rightarrow$ since $|\{p_\alpha \mid \alpha \in A\}| = \omega_1$, by ccc of $\mathbb{P}$ there $\exists \alpha \neq \beta$ s.t. $p_\alpha \parallel p_\beta$

$\Big\}$ happy

Theorem: Under MA_{ω_1}, the following holds:

① $\mathbb{P}$ ccc $\Rightarrow$ $\mathbb{P}$ Knaster

② $\mathbb{P}$ ccc $\Rightarrow$ $\mathbb{P}$ has precaliber ω_1

Recall that:

Knaster $\Rightarrow$ ccc

$\Uparrow$

precaliber ω_1 $\Rightarrow$ ccc

Under MA_{ω_1}: all are $\Leftrightarrow$

Corollary: Under MA_{ω_1}: $\mathbb{P}, \mathbb{Q}$ ccc $\Rightarrow$ $\mathbb{P} \times \mathbb{Q}$ ccc.

② $\Rightarrow$ ① since precaliber ω_1 $\Rightarrow$ Knaster

Remark on consistency strength:

Def: $\mathbb{P}$ is ω_1-n-Knaster if $\forall$ uncountable $A \subseteq \mathbb{P}$ contains an uncountable subset $B \subseteq A$ that is n-linked.

precaliber ω_1 $\Rightarrow$ $\forall n$: ω_1-n-Knaster

Note: Knaster $\equiv$ ω_1-2-Knaster

Fact: ② $\Rightarrow$ MA_{ω_1}

Open Problem: ① $\overset{?}{\Longrightarrow}$ MA_{ω_1}

→ recently it was proved that "$\mathbb{P}$ ccc $\Rightarrow$ $\mathbb{P}$ ω_1-3-Knaster" implies MA_{ω_1}

Proof of ②

- First note that if $G \subseteq \mathbb{P}$ is a filter, then any $A \subseteq G$ is centered in $\mathbb{P}$

→ let $Q = \{q_\alpha \mid \alpha < \omega_1\} \subseteq \mathbb{P}$ ... want uncountable $A \subseteq Q$ centered in $\mathbb{P}$

- for $\forall \beta < \omega_1$ let $D_\beta := \{q_\alpha \mid \alpha > \beta\}$

→ if all D_β were pre-dense in $\mathbb{P}$, we could use MA_{ω_1} to intersect each with a filter $G \subseteq \mathbb{P}$ ... then $G \cap Q$ would be uncountable centered subset of Q

↳ because ω_1 is regular

Problem: D_β might not be pre-dense in $\mathbb{P}$

Lemma: There $\exists p \in \mathbb{P}$ s.t. $\forall r \leq p$ is compatible with uncountably many $q_\alpha \in Q$

→ we then restrict ourselves to $\mathbb{P} \upharpoonright p := \{r \in \mathbb{P} \mid r \leq p\}$

$(\forall r \in \mathbb{P} \upharpoonright p)(\forall \beta)(\exists q \in D_\beta)$: $r \parallel q$ since $\{q_\alpha \mid \alpha \leq \beta\}$ is countable for $\beta < \omega_1$

→ for $\beta < \omega_1$ let $E_\beta = \{r \in \mathbb{P} \upharpoonright p \mid r \leq q_\alpha \text{ for some } \alpha > \beta\}$

↳ downward closure of D_β restricted to $\mathbb{P} \upharpoonright p$

↳ the sets E_β are dense since the lower bound $s \leq r, q$ is in E_β

Since $\mathbb{P} \upharpoonright p$ is also ccc, we can invoke MA_{ω_1} to get a filter $G \subseteq \mathbb{P} \upharpoonright p$ intersecting each E_β. Its upwards closure $\overline{G}$ in $\mathbb{P}$ is also a filter and it intersects each D_β

$\Rightarrow$ hence $\overline{G} \cap Q$ is an uncountable centered subset of Q

Proof of lemma: Assume for contradiction that for $(\forall p \in \mathbb{P})(\exists r \leq p)$ s.t.

$C_r := \{q \in Q \mid r \| q\}$ has size $|C_r| \leq \omega$

$\rightarrow$ in other words: $D := \{r \in \mathbb{P} \mid |C_r| \leq \omega\}$ is dense in $\mathbb{P}$

- let $A \subseteq D$ be a maximal antichain (both in D and $\mathbb{P}$ as D is dense)

$\rightarrow$ since A is maximal, it is predense, and also $|A| \leq \omega$ since $\mathbb{P}$ is ccc

$\Rightarrow \forall q \in Q$ there $\exists a \in A$ s.t. $a \| q$ ... so $q \in C_a$ and $|C_a| \leq \omega$

write $Q \subseteq \bigcup_{a \in A} C_a$ ... countable union of countable sets, so Q is countable, a contradiction ↯ ∎

Normalizing a Suslin tree

① trees with this condition have no short sprouts. Let T be an ω_1-tree and $T' := \{t \in T \mid |[t, \rightarrow)| = \omega_1\}$, where $[t, \rightarrow) := \{s \in T \mid s \geq t\}$

T' is downwards closed & nonempty since some $t \in T_0$ must be in T' from countable levels

- given $t \in S'_\alpha$ and $\beta > \alpha$, the union $\bigcup_{s \in T_\beta} [s, \rightarrow)$ covers the entire $[t, \rightarrow)$ up to countably many exceptions (descendants of t below level β)

$\rightarrow$ since $[t, \rightarrow)$ is uncountable, there $\exists s \in T_\beta$ s.t. $[s, \rightarrow)$ is uncountable, so $s \in T'$

② Let S be a Suslin tree with no short sprouts. We will define a closed unbounded sequence of levels $C = \{\alpha_\delta \mid \delta < \omega_1\} \subseteq \omega_1$ s.t. $\forall t \in S' := \bigcup_{\alpha \in C} S_\alpha$ has ω immediate successors ... since C is unbounded, S' still has height ω_1 and is Suslin

Fact: If a tree of height ω_1 has all levels finite, then it contains a cofinal branch

- for $t \in S$ consider the subtree  $\{s \in S \mid s \| t\}$. Since it has no cofinal branch it must have a level $\alpha(t)$ of size ω. Since S has no short sprouts, all subsequent levels also have size ω

$\Rightarrow (\forall t \in S)(\exists \alpha(t) < \omega_1)(\forall \beta > \alpha(t)) : |S_\beta \cap [t, \rightarrow)| = \omega$

- Set $\alpha_0 := 0$. In limit stages λ we let $\alpha_\lambda := \sup\{\alpha_\delta \mid \delta < \lambda\}$.

In successor stages $\delta \rightarrow \delta + 1$ we let $\alpha_{\delta+1} := \sup\{\alpha(t) \mid t \in S_{\alpha_\delta}\}$

PROPERTIES OF ω_1-TREES

Def: An ω_1-tree is a tree T of height ω_1 where every level T_α is countable

Def: An Aronszajn tree is an ω_1-tree with no cofinal branch (branch of length ω_1).

Fact: ZFC $\vdash$ there $\exists$ Aronszajn tree.

$\Rightarrow$ so ω_1 does not have the tree property ... König's lemma does not work for it

Def: A tree T of height ω_1 with no cofinal branch is special if we can write

$$T = \bigcup_{n<\omega} A_n$$, where each $A_n \subseteq T$ is an *antichain ... normal antichain
↳ incomparable elements

Equiv.: There $\exists f: T \to \omega$ injective on chains ... $x <_T y \Rightarrow f(x) \neq f(y)$

→ if $|T| = \omega_1$, some *antichain A_i is uncountable

Fact: ZFC $\vdash$ there $\exists$ special Aronszajn tree

Def: A Suslin tree is an Aronszajn tree with no uncountable *antichain.

Suslin and Special Aronszajn trees are mutually exclusive

Axiom: Suslin Hypothesis SH: There are no Suslin trees

Fact: $V=L$ as well as $\diamondsuit$ imply that Suslin trees exist

Theorem: MA_{ω_1} implies that all Aronszajn trees are special, so no Suslin trees exist
↳ we will prove this later

$\Rightarrow$ SH is independent of ZFC

↱ recall that it held under MA_{ω_1}

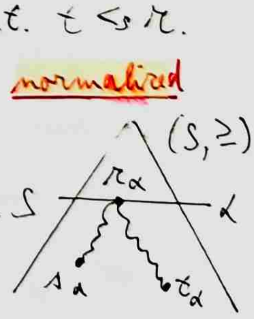
Proposition: If a Suslin tree exists, then "$\mathbb{P}, \mathbb{Q}$ ccc $\Rightarrow \mathbb{P}\times\mathbb{Q}$ ccc" does not hold

Proof: Let S be a suslin tree, WLOG ① $(\forall t \in S)(\forall \alpha > |t|_S)(\exists r \in S_\alpha)$ s.t. $t <_S r$.
② every $t \in S$ is branching ... has ≥ 2 immediate successors } S is normalized

← explanation

↗ $s \| t$ means that $s \leq_S t$ or $t <_S s$

→ let $\mathbb{P} = (S, \geq)$ with the reverse order ... so stronger conditions are higher in S



antichains in $\mathbb{P}$ $\equiv$ *antichains in S , so $\mathbb{P}$ is ccc

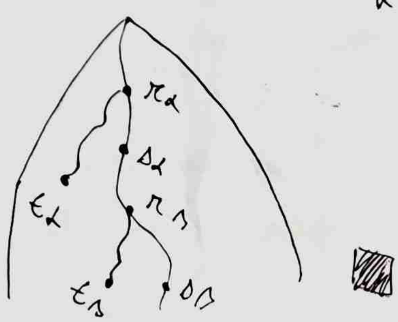
claim: $\mathbb{P}\times\mathbb{P}$ is not ccc ... for $\forall \alpha < \omega_1$ pick① $r_\alpha \in S_\alpha$ and $s_\alpha, t_\alpha >_S r_\alpha$ s.t. $s_\alpha \wedge_S t_\alpha = r_\alpha$

→ $\{(s_\alpha, t_\alpha) \mid \alpha < \omega_1\} \subseteq \mathbb{P}\times\mathbb{P}$ is an uncountable antichain in $\mathbb{P}\times\mathbb{P}$

$(s_\alpha, t_\alpha) \| (s_\beta, t_\beta)$ for $\alpha < \beta$ iff $s_\alpha \| s_\beta$ & $t_\alpha \| t_\beta$

→ $s_\alpha \| s_\beta$ together with $r_\alpha < s_\alpha$ and $r_\beta < s_\beta$ give that $s_\alpha, s_\beta, r_\alpha, r_\beta$ lie on a common branch

→ since t_α and t_β branch off at levels $\alpha \neq \beta$ we have $t_\alpha \not\| t_\beta$ ↯



? What if we force with $\mathbb{P} = (S, \geq)$?

👁 filters = chains ... the only common lower bound of s, t is s or t themselves

→ define dense sets $D_\alpha := \{t \in S \mid |t|_S \geq \alpha\}$ for $\alpha < \omega_1$ ... dense from ②

⇒ a V-generic filter $G \subseteq \mathbb{P}$ is a cofinal branch ∵ it intersects every D_α

Moreover, since $\mathbb{P}$ is ccc, it preserves all cardinals, so ω_1 stays ω_1

⇒ forcing with $(S, \geq)$ adds a cofinal branch to S, destroying it

PRODUCTS OF INFINITELY MANY POSETS

Def: Let $\langle \mathbb{P}_i \mid i \in I \rangle$ be a family of posets and let $\mathbb{1}_i$ be the max. el. of $\mathbb{P}_i$.

The finite support product of $\langle \mathbb{P}_i \mid i \in I \rangle$ is

$$\prod_{i \in I}^{FIN} \mathbb{P}_i := \{\langle p_i \mid i \in I \rangle \mid \forall i : p_i \in \mathbb{P}_i \;\&\; \{i \in I \mid p_i \neq \mathbb{1}_i\} \text{ is finite}\}$$

→ The ordering of $\prod_{i \in I}^{FIN} \mathbb{P}_i$ is coordinate-wise: $\langle p_i \rangle_{i \in I} \leq \langle q_i \rangle_{i \in I}$ iff $\forall i : p_i \leq q_i$

Theorem: If all $\mathbb{P}_i$ are Knaster, then $\prod_{i \in I}^{FIN} \mathbb{P}_i$ is ccc.

Corollary: Under MA_{ω_1}, if all $\mathbb{P}_i$ are ccc, then $\prod_{i \in I}^{FIN} \mathbb{P}_i$ is ccc.

↳ since then Knaster ⟺ ccc

Proof: Let $\{\langle p_i^\alpha \mid i \in I \rangle \mid \alpha < \omega_1\}$ be an uncountable subset of $\prod_{i \in I}^{FIN} \mathbb{P}_i$

↳ want two compatible elements ... with a common lower bound on each coordinate

→ look at the (finite) supports $S_\alpha := \{i \in I \mid p_i^\alpha \neq \mathbb{1}_i\}$ for $\alpha < \omega_1$

Δ-system lemma: $\exists A \in [\omega_1]^{\omega_1}$ and a finite root $R \subseteq I$ s.t.

$\forall \alpha \neq \beta$ in A: $S_\alpha \cap S_\beta = R$ ... they all have the same support

→ outside of R ... if $i \notin R$, then $p_i^\alpha = \mathbb{1}_i$ or $p_i^\beta = \mathbb{1}_i$ ... WLOG $p_i^\alpha = \mathbb{1}_i$

⇒ then $p_i^\beta \leq p_i^\alpha, p_i^\beta$ ... common lower bound

⇒ we only need to find the common lower bounds on the finite root R,

or in other words show that $\prod_{i \in R} \mathbb{P}_i$ is ccc

→ this can be done by iterating our proof that $\mathbb{P}$ ccc & $\mathbb{Q}$ Knaster ⇒ $\mathbb{P} \times \mathbb{Q}$ Knaster

↳ we always find the next uncountable $A' \subseteq A$ on which some $\mathbb{P}_i$, $i \in R$ is linked,

so everything has lower bounds

→ after we have this, we can easily get two compatible elements in $\langle p_i^\alpha \mid i \in I \rangle_{\alpha < \omega_1}$ ▨

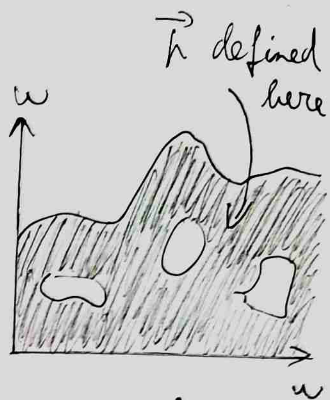
? Do we need finite support in the product?

→ consider an equivalent formulation of the Cohen forcing $\mathbb{C} := \{p; \omega \to \omega \mid |p| < \omega\}$

Fact: Every countable atomless poset $\mathbb{C}$ has the same forcing consequences.

Note: $\mathbb{C}$ is Knaster ... by the Δ-system lemma ... it in fact has precaliber ω_1

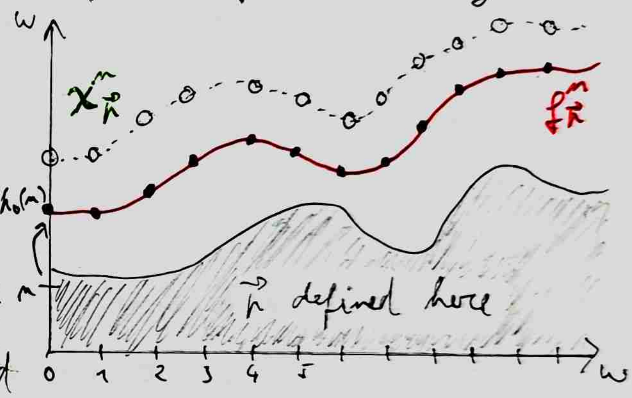
Proposition: $\mathbb{C}^\omega = \prod_{i<\omega} \mathbb{C}$ is not ccc.



Proof: We will show that forcing with $\mathbb{C}^\omega$ collapses 2^ω to ω

⇒ so it cannot be ccc, since ccc forcings preserve cardinals

- an element $\vec p = \langle p_i \mid i \in \omega \rangle$ can be seen as a partial function $\omega \times \omega \to \omega$, with only finitely many values in each column
- fix $\vec p \in \mathbb{C}$. For each $n \in \omega$, we may "try" to define a function $f^n_{\vec p} : \omega \to \omega$ by
  $f^n_{\vec p}(0) := p_0(n)$, $\quad f^n_{\vec p}(k+1) := p_{k+1}(f^n_{\vec p}(k))$
- we also "try" to define $X^n_{\vec p} : \omega \to \omega$ by
  $X^n_{\vec p}(0) := p_0(n+1)$, $\quad X^n_{\vec p}(k+1) := p_{k+1}(f^n_{\vec p}(k)+1)$



$\forall x : \omega \to \omega$, $D_x := \{\vec p \in \mathbb{C} \mid \exists n : X^n_{\vec p} = x\}$ is dense in

↳ given $\vec p \in \mathbb{C}^\omega$, we choose n large enough and add values to $\vec p$ so that the entire path for $f^n_{\vec p}$ is above the old "graph" of $\vec p$, but $f^n_{\vec p}$ is defined. We can now define the values of $\vec p$ above so that $X^n_{\vec p} = x$

- Let $G \subseteq \mathbb{C}^\omega$ be V-generic and put $\vec g := \bigcup G$, where we interpret $\vec p \in G$ as $p; \omega \times \omega \to \omega$

⇒ in the extension $V[G]$, every old function $x : \omega \to \omega$ from the ground model V is equal to $X^n_{\vec g}$ for some $n \in \omega$ ... since G intersects D_x

⇒ the set of functions $x \in \omega^\omega \cap V$ is countable in $V[G]$ ∎

Notation: $\prod^{FIN}_{\omega} \mathbb{P}$ denotes $\prod^{FIN}_{i<\omega} \mathbb{P}$ ... the ω finite support iteration of $\mathbb{P}$

SPECIALIZING A GIVEN TREE

We will present two approaches — low level different, high level similar — how to specialize any given tree of size $< 2^\omega$

<u>Theorem</u>: Let T be a tree of height ω_1 with no cofinal branches. Then:

$$MA_\kappa \implies \text{If } |T| \leq \kappa, \text{ then } T \text{ is special}$$

<u>Corollary</u>: $MA_{\omega_1} \implies$ Every Aronszajn tree is special.

- <u>Poset for adding a specializing function</u>

— denote by $s \| t$ that $s, t \in T$ lie on a common branch ... $s \leq_T t$ or $t \leq_T s$

<u>want</u>: $f : T \to \omega$ s.t. $s \| t \Rightarrow f(s) \neq f(t)$ ← finite approximations

$\mathbb{P}_T = \{ p : T \to \omega \mid |p| < \omega \ \& \ p(s) \neq p(t) \text{ if } s \| t \}$ ordered by $p \leq q$ iff $p \supseteq q$

<u>dense sets</u>: $\forall t \in T$ define

$D_t := \{ p \in \mathbb{P}_T \mid t \in \mathrm{Dom}(p) \}$ ... dense since $|p| < \omega$

$\Rightarrow$ if we show that $\mathbb{P}_T$ is ccc, then from MA_κ there $\exists$ filter $G \subseteq \mathbb{P}_T$ intersecting every D_t and $g := \bigcup G$ is the desired specializing function

- <u>Poset for adding an uncountable antichain</u> ← finite antichains

$A_T := \{ A \in [T]^{<\omega} \mid A \text{ is an *antichain} \}$ ordered by $A \leq A'$ iff $A \supseteq A'$

→ if we show that $\mathbb{P}$ is ccc, then $\prod^{FIN}_{\omega} A_T$ is also ccc since we assume MA_{ω_1}

<u>dense sets</u>: $\forall t \in T$ define

$D_t := \{ \vec{A} \in \prod^{FIN}_{\omega} A_T \mid \exists n \in \omega : t \in \vec{A}_n \}$ ... dense since finite support, we can always add the singleton *antichain $\{t\}$

<u>MA_κ</u>: $\exists$ filter $G \subseteq \prod^{FIN}_{\omega} A_T$ intersecting every D_t

$\forall n : A_n := \bigcup \{ \vec{A}_n \mid \vec{A} \in G \} \subseteq T$ is an *antichain

$T = \bigcup_{n<\omega} A_n$ since G intersects every D_t $\implies$ T is special

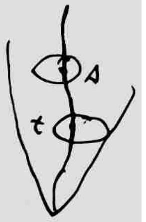
<u>Remark</u>: If S is a normalized Suslin tree and we force with A_S, then we can consider dense sets $D_\alpha := \{ A \in A_S \mid \exists x \in A : |x|_S \geq \alpha \}$

→ a V-generic filter then intersects every D_α, yielding an uncountable antichain

→ since ccc forcings preserve cardinals (specifically ω_1), this kills the Suslin tree

→ to prove that $\mathbb{P}_T$ and $\mathbb{A}_T$ are ccc, we will need the following lemma

<u>Lemma:</u> If T is a tree of height ω_1 and $\langle u_\alpha \mid \alpha < \omega_1 \rangle$ a family of disjoint sets, all of size $|u_\alpha| = n \geq 1$ s.t. $(\forall \alpha \neq \beta)(\exists s \in u_\alpha)(\exists t \in u_\beta): s \parallel t$, then T contains a cofinal branch.



<u>Proving that $\mathbb{P}_T$ is ccc:</u> Let $A \subseteq \mathbb{P}_T$ be uncountable

- Δ-system lemma for domains $\{\mathrm{Dom}(p) \mid p \in A\}$ ... $\exists R \subseteq T$ (finite) $\exists A_1 \in [A]^{\omega_1}$ s.t. $\forall p \neq q \in A_1: \mathrm{Dom}(p) \cap \mathrm{Dom}(q) = R$
- there are only countably many functions $r: R \to \omega$, so by Pigeonhole there $\exists$ uncountable $A_2 \subseteq A_1$ s.t. the restrictions $p \restriction R$ for $p \in A_2$ all coincide

- for $p \in A_2$ put $u_p := \mathrm{Dom}(p) \setminus R$
- WLOG $\exists n \in \omega$ s.t. $|u_p| = n$ for every $p \in A_2$ ... from the pigeonhole principle

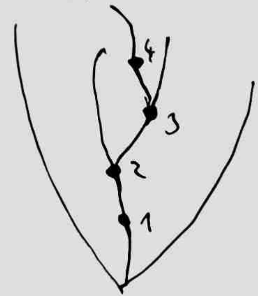
<u>want:</u> $p \neq q \in A_2$ s.t. $p \parallel q$ ... we can take $p \cup q$, but when $p \cup q \in \mathbb{P}_T$?
↳ what could go wrong? if $s \in u_p$ and $t \in u_q$ then we would like $s \not\parallel t$
→ suppose for contradiction that $(\forall p \neq q$ in $A_2)(\exists s \in u_p)(\exists t \in u_q): s \parallel t$
⇒ by Lemma there $\exists$ cofinal branch, contradicting our assumption on T

<u>Proving that $\mathbb{A}_T$ is ccc:</u> Let $M \subseteq \mathbb{A}_T$ be uncountable

- Δ-system lemma: $\exists M_1 \in [M]^{\omega_1}$ $\exists$ finite $\Delta \subseteq T$ s.t. $\forall A \neq A' \in M_1: A \cap A' = \Delta$
- WLOG $\Delta = \emptyset$ ... otherwise we delete it from all $A \in M_1$
- WLOG all $A \in M_1$ have the same size ... by pigeonhole

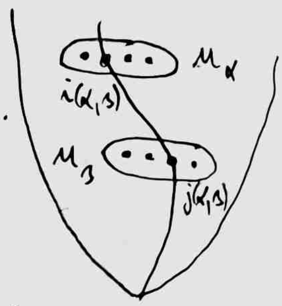
<u>want:</u> $A \neq A' \in M_1$ s.t. $A \parallel A'$ ... need $A \cup A'$ to be an *antichain
→ suppose for contradiction that $(\forall A \neq A'$ in $M_1)(\exists s \in A)(\exists t \in A'): s \parallel t$
⇒ we again get a cofinal branch ... contradiction

<u>Proof of lemma:</u> First note that this is easy if all u_α have size $|u_\alpha| = 1$
→ let $u_\alpha = \{t_\alpha\}$ ... we have $\forall \alpha < \beta: t_\alpha \parallel t_\beta$
👁 $\{t_\alpha \mid \alpha < \omega_1\} \subseteq T$ is a chain ... hence $\exists$ cofinal branch
↳ process t_α in order based on $|t_\alpha|_T$
<u>note</u> that $\alpha \neq \beta \Rightarrow |t_\alpha| \neq |t_\beta|$ since otherwise $t_\alpha \not\parallel t_\beta$



Suppose that all M_α have size $|M_\alpha| = n > 1$ and let $M_\alpha = \{t^\alpha_0, t^\alpha_1, \dots, t^\alpha_{n-1}\}$

- for $\alpha < \beta$ let $i(\alpha,\beta)$ and $j(\alpha,\beta)$ be s.t. $t^\alpha_{i(\alpha,\beta)} \parallel t^\beta_{j(\alpha,\beta)}$



We would like to say that $\exists$ uncountable $A \subseteq \omega_1$, $\exists\, i, j \in n$ s.t.

$\forall \alpha < \beta$ in A we have $i(\alpha,\beta) = i$ and $j(\alpha,\beta) = j$

and maybe recreate the argument for $n = 1$

→ but this does not quite work and moreover $\omega_1 \not\to (\omega_1)^2_{\text{finite}}$

! important trick: let $\mathcal{U} \subseteq P(\omega_1)$ be an uniform ultrafilter on ω_1

↳ all $U \in \mathcal{U}$ have size $|U| = \omega_1$

- Fix $\alpha \in \omega_1$ and for $i, j \in n$ define

$$S^j_i := \{\beta \in \omega_1 \mid \beta \neq \alpha \ \&\ t^\alpha_i \parallel t^\beta_j\}$$

$\exists\, i, j$ s.t. $S^j_i \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter and $\omega_1 = \{\alpha\} \cup \bigcup_{i,j<n} S^j_i$

↳ uncountable ∵ $\mathcal{U}$ is uniform

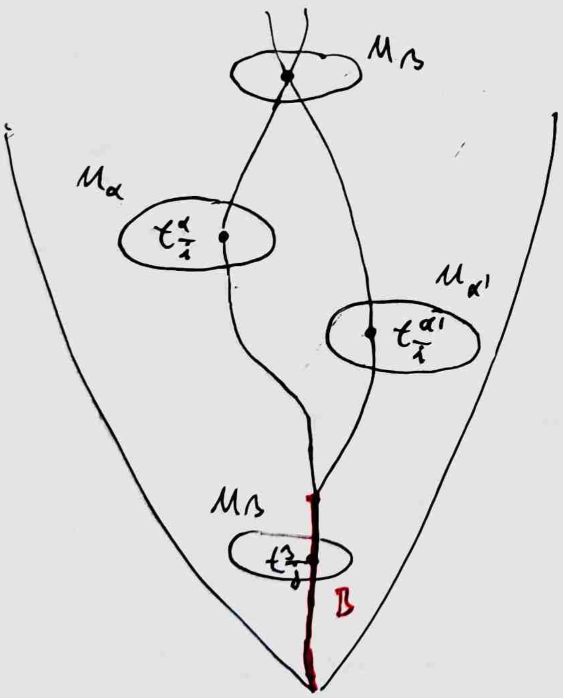
↳ denote this S^j_i by $S(\alpha)$ and put $i(\alpha) := i$, $j(\alpha) := j$

- Pigeonhole: $\exists$ uncountable $A \subseteq \omega_1$ $\exists\, \bar{i}, \bar{j}$ s.t. $\forall \alpha \in A$: $i(\alpha) = \bar{i}$ and $j(\alpha) = \bar{j}$

claim: $\forall \alpha \neq \alpha'$ in A: $t^\alpha_{\bar{i}} \parallel t^{\alpha'}_{\bar{i}}$, so by the same argument as for $n = 1$:

$\{t^\alpha_{\bar{i}} \mid \alpha \in A\}$ is a chain ⇒ hence $\exists$ cofinal branch

proof: Suppose that $t^\alpha_{\bar{i}}$ and $t^{\alpha'}_{\bar{i}}$ are not on a common branch



Consider $B := \{t \in T \mid t \leq t^\alpha_{\bar{i}} \ \&\ t \leq t^{\alpha'}_{\bar{i}}\}$

↳ $|B| \leq \omega$

size ω_1 since $\mathcal{U}$

Let $\beta \in S(\alpha) \cap S(\alpha')$ ... uncountably many

- there can be only countably many β for which

$$t^\beta_{\bar{j}} \leq t^\alpha_{\bar{i}} \ \&\ t^\beta_{\bar{j}} \leq t^{\alpha'}_{\bar{i}}$$

⇒ thus $\exists \beta \in S(\alpha) \cap S(\alpha')$ s.t.

$t^\beta_{\bar{j}}$ is above $t^\alpha_{\bar{i}}$ or $t^{\alpha'}_{\bar{i}}$, but

this is impossible since $t^\alpha_{\bar{i}} \not\parallel t^{\alpha'}_{\bar{i}}$ (see picture)

$\blacksquare$

STRENGTHENING MARTIN'S AXIOM

Let $MA^+_{\omega_1}$ be the following axiom:

MA_{ω_1} $\begin{cases} (\forall \mathbb{P} \text{ ccc poset})(\forall \mathcal{D} \text{ collection of dense sets in } \mathbb{P}, \text{ of size } |\mathcal{D}| = \omega_1) \\ \text{there exists a } \mathcal{D}\text{-generic filter } G \subseteq \mathbb{P} \text{ s.t.} \end{cases}$

$+$ $\begin{cases} \text{for every } \mathbb{P}\text{-name } \dot{S} \in V^{\mathbb{P}} \text{ for a stationary subset of } \omega_1 \text{ it holds that} \\ \dot{S}/G \in V[G] \ldots \text{"}\dot{S} \text{ evaluated over } G\text{"} \ldots \text{ is a stationary subset of } \omega_1 \in V[G] \end{cases}$

What does this mean?

→ since $\mathbb{P}$ is ccc, ω_1 stays ω_1

→ by "$\dot{S}$ being a $\mathbb{P}$-name for a stationary subset of ω_1" we mean that

$$\forall p \in \mathbb{P}: \ p \Vdash \text{"}\dot{S} \text{ is a stationary subset of } \omega_1\text{"}$$

→ so $\dot{S}$ always evaluates as a stationary set if we use a V-generic filter

$\Longrightarrow$ $MA^+_{\omega_1}$ says that the ground model already contains a "good enough"-generic filter

- every $\alpha \in \omega_1$ either ends up in $\dot{S}/G$ or it will not

$$S(\alpha) := \{p \in \mathbb{P} \mid p \Vdash \check{\alpha} \in \dot{S}\} \in V$$ ... recall the Forcing Theorem

$\Longrightarrow$ then clearly $\dot{S}/G := \{\alpha \in \omega_1 \mid G \cap S(\alpha) \neq \emptyset\}$

Theorem: $MA^+_{\omega_1} \iff MA_{\omega_1}$

Proof: Let $\mathbb{P}$ be ccc, $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ be dense in $\mathbb{P}$, and $\dot{S}$ be a $\mathbb{P}$ name ...

want: filter intersecting each D_α and such that $\dot{S}/G$ is stationary in ω_1

- consider $\prod^{FIN}_{i<\omega} \mathbb{P}$ ... since we have MA_{ω_1}, this is ccc

Define dense subsets of $\prod$:

- for $n \in \omega$, $\alpha < \omega_1$: $D_{n,\alpha} := \{\vec{p} \in \mathbb{P} \mid \vec{p}(n) \in D_\alpha\}$ ... dense ∵ D_α is dense

→ put $R := \{\alpha \in \omega_1 \mid S(\alpha) \neq \emptyset\}$ ... α that have a chance of getting into $\dot{S}/G$

R is stationary ... if not then $\exists$ club $C \subseteq \omega_1$ disjoint from R

→ C being a club is a Δ_0 formula so it is absolute

$\Longrightarrow$ C is still a club in $V[G]$ and no $\alpha \in C$ can end up in S/G

→ this contradicts $\dot{S}/G$ being stationary (it misses C in $V[G]$)

- for $\alpha \in R$ let $E_\alpha := \{\vec{p} \in \prod \mid \exists n \in \omega: \vec{p}(n) \in S(\alpha)\}$ ... dense ∵ $\vec{p}$ has finite support

Apply MA_{ω_1}: $\exists$ filter $G \subseteq \prod$ intersecting all $D_{n,\alpha}$ and E_α

→ let $G_n := \{\vec{p}(n) \mid \vec{p} \in G\}$ be the projection of G to its n-th coordinate

• $\forall n$: G_n is a filter ... because G was a filter

• $\forall n$: G_n intersects every D_α ... because G intersects every $D_{n,\alpha}$

→ for $\forall n \in \omega$ put $R_n := \dot{S}/G_n = \{\alpha \in \omega_1 \mid G_n \cap S(\alpha) \neq \emptyset\}$

$R = \bigcup_{n<\omega} R_n$ ... because G intersects every E_α

→ R is stationary $\Rightarrow \exists n$ s.t. $R_n = \dot{S}/G_n$ is stationary $\Rightarrow$ G_n is what we want ∎

BELL'S THEOREM

Theorem (Bell): $\forall \kappa$: $MA_\kappa(\sigma\text{-centered}) \iff \mathfrak{p} > \kappa$.

→ we already know "$\Rightarrow$"

→ we will now prove $\mathfrak{p} > \kappa \Rightarrow MA_\kappa(\sigma\text{-centered, size} \le \kappa)$ → $|\mathbb{P}| \le \kappa$ for posets $\mathbb{P}$

→ later on we will show that from MA_κ with poset size $\le \kappa$ we can get full MA_κ for posets of arbitrary size

Lemma [1] $\mathfrak{p} > \kappa \Rightarrow$ linked-$MA_\kappa(\sigma\text{-centered})$. That is: (downwards closed)

for $\forall$ σ-centered poset $\mathbb{P}$ and every family $\{D_\alpha \mid \alpha < \kappa\}$ of open dense sets in $\mathbb{P}$, there exists a linked set $L \subseteq \mathbb{P}$ intersecting every D_α

Lemma [2]: If $|\mathbb{P}| \le \kappa$, then $MA_\kappa(\{\mathbb{P}\}) \iff$ linked-$MA_\kappa(\{\mathbb{P}\})$

Corollary: $\mathfrak{p} > \kappa \Rightarrow MA_\kappa(\sigma\text{-centered, size} \le \kappa)$

↳ this almost prove Bell's theorem; we will finish it when we talk about poset embeddings

Proof of Lemma [2]: Clearly "$\Rightarrow$" since every filter is linked

For "$\Leftarrow$" fix a family $\{D_\alpha \mid \alpha < \kappa\}$ of dense subsets of $\mathbb{P}$ ... we want a filter intersecting them

• for $a, b \in \mathbb{P}$, let $D_{a,b} := \{p \in \mathbb{P} \mid p \perp a \vee p \perp b \vee (p \le a \ \& \ p \le b)\}$

• each $D_{a,b}$ is dense ... given $q \in \mathbb{P}$, if $q \perp a$ then $q \in D_{a,b}$. Otherwise there $\exists q' \le q, a$

in fact open dense ↳ if $q' \perp b$ then $q' \in D_{a,b}$. Otherwise $\exists q'' \le q', b$, so $q'' \le a, b$ and $q'' \in D_{a,b}$

The combined family $\{D_\alpha\} \cup \{D_{a,b}\}$ has size $\le \kappa + |\mathbb{P}|^2 = \kappa$. The sets $D_{a,b}$ are also open, as are the downward closures $\downarrow D_\alpha$. By linked-$MA_\kappa(\{\mathbb{P}\})$ there $\exists$ linked $L \subseteq \mathbb{P}$ intersecting each $\downarrow D_\alpha$ and $D_{a,b}$

The upward closure $\bar{L}$ intersects each D_α ... since L intersects $\downarrow D_\alpha$

claim: $\bar{L}$ is a filter ... for $(\forall p, q \in \bar{L})(\exists r \in \bar{L})\ r \le p, q$

→ given $p, q \in \bar{L}$, choose $r_p, r_q \in L$ s.t. $r_p \le p$ and $r_q \le q$, and let $r \in L \cap D_{r_p, r_q}$

→ since L is linked, $r \parallel r_p$ & $r \parallel r_q$, so by definition of $D_{a,b}$ we have $r \le r_p, r_q$, so $\le p, q$ ∎

Proof of Lemma $\boxed{1}$

Fix a σ-centered poset $\mathbb{P}$ with decomp. $\mathbb{P} = \bigcup_{n\in\omega} C_n$ into centered pieces and fix an open-dense family $\{D_\alpha \mid \alpha < \kappa\}$

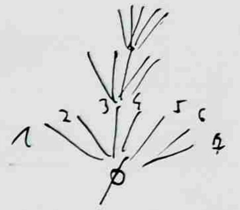
- WLOG $\mathbb{P}$ is atomless ... if $a \in \mathbb{P}$ is an atom, the principal filter $\{p \mid p \geq a\}$ is a linked set intersecting each D_α since $a \in$ all dense sets

We work in the tree $\omega^{<\omega}$. For $\emptyset \neq s \in \omega^{<\omega}$ we write $s = t^\frown(n)$ for its decomp into an initial segment t and last element $n \in \omega$

- For each $\alpha < \kappa$ we recursively define two maps:

$$S_\alpha : \omega^{<\omega} \to \mathbb{P} \quad \text{and} \quad A_\alpha : \omega^{<\omega} \to \mathcal{P}(\omega)$$

- Put $S_\alpha(\emptyset) := \mathbb{1}_{\mathbb{P}}$. Given $S_\alpha(t)$, define



$$A_\alpha(t) := \{n \in \omega \mid \exists q \in C_n \cap D_\alpha \ \text{s.t.}\ q \leq S_\alpha(t)\}$$

$$S_\alpha(t^\frown(n)) := \begin{cases} \mathbb{1}_{\mathbb{P}} & \text{if } n \notin A_\alpha, \\ \text{some } q \in C_n \cap D_\alpha \ \text{s.t.}\ q \leq S_\alpha(t) & \text{if } n \in A_\alpha. \end{cases}$$

Intuition: For each α, the map S_α traces decreasing chains through $\omega^{<\omega}$ that only use elements of D_α and the branching index n determines into which centered set C_n we descend

→ if descent into C_n is impossible (D_α is never the problem since its dense), we reset and start from $\mathbb{1}_{\mathbb{P}}$ again

→ the set $A_\alpha(t)$ records the non-restarting choices

Goal: Find a branch $b \in \omega^\omega$ along which every α eventually stops restarting

claim: Assuming $\kappa < \mathfrak{p}$, there $\exists\, b \in \omega^\omega$ s.t. $(\forall \alpha < \kappa)(\exists\, m_\alpha \in \omega)$ s.t.

$$\forall k \geq m_\alpha : \quad b(k) \in A_\alpha(b \restriction k)$$

→ $b \restriction k$ = initial segment up to but not including k

- Suppose we have such branch b and put

$$L := \{S_\alpha(b \restriction m_\alpha + 1) \mid \alpha < \kappa\} \subseteq \mathbb{P}$$

... we take the label of an element that is in the safe zone

- We verify that L is the desired linked set:
  - since $b(m_\alpha) \in A_\alpha(b \restriction m_\alpha)$, the recursion at step m_α does not restart, so $S_\alpha(b \restriction m_\alpha + 1) \in C_{b(m_\alpha)} \cap D_\alpha \subseteq D_\alpha$, so L intersects D_α
  - $\underline{L \text{ is linked (in fact centered)}}$: Given $E \in [\kappa]^{<\omega}$, we want to find a common lower bound of $X := \{S_\alpha(b \restriction m_\alpha + 1) \mid \alpha \in E\}$. Pick $m \geq \max\{m_\alpha \mid \alpha \in E\}$

→ for $\forall \alpha \in E$, the recursion has not restarted between m_α and m, so

$$S_\alpha(b \restriction m+1) \leq S_\alpha(b \restriction m_\alpha + 1) \quad \text{and} \quad S_\alpha(b \restriction m+1) \in C_{b(m)}$$

Since $C_{b(n)}$ is centered, the set $\{S_\alpha(b\restriction n+1) \mid \alpha \in E\} \subseteq C_{b(n)}$ has a common lower bound x, which is also a common lower bound of X.

Proof of the claim: We will first make two observations:

- finite intersections of open dense sets are open dense: $q \in \mathbb{P}$ $\leftarrow$ $p \in D_1$ $\leftarrow$ $p' \in D_2$ ... $p' \in D_1$ since open
- If $\mathbb{P}$ is atomless & separative then $\forall p \in \mathbb{P}\ \exists q, r \leq p$ s.t. $q \perp r$

Proof: $p \in \mathbb{P}$ is an atom $\iff \forall q, r \leq p$: $q \parallel r$

Clearly "$\Rightarrow$" since then $q, r = p$. For "$\Leftarrow$", let $q \leq p$. If $q < p$, then $p \nleq q$, so from separativity $\exists r \leq p$ s.t. $r \perp q$ $\lightning$

Corollary: We can build an infinite antichain $\{a_i \mid i \in \omega\}$ under any $p \in \mathbb{P}$

Proof: Given $p \in \mathbb{P}$, take $a_0, r_0 \leq p$ s.t. $a_0 \perp r_0$. Then iteratively take $a_{i+1}, r_{i+1} \leq r_i$ s.t. $a_{i+1} \perp r_{i+1}$. We are using $a \perp b$ & $c \leq a \Rightarrow c \perp b$

We prove the claim in 3 steps:

① The sets $A_\alpha(t)$ are always infinite and for $\forall t \in \omega^{<\omega}$, the set system $\{A_\alpha(t) \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ is centered ... intersecting finitely many $A_\alpha(t)$ is infinite

$\rightarrow$ fix t and $E \in [\kappa]^{<\omega}$. Want: $A := \bigcap_{\alpha \in E} A_\alpha(t)$ infinite

- we first get a common lower bound q of $\{S_\alpha(t) \mid \alpha \in E\}$
  - if $t = \emptyset$, all $S_\alpha(t) = \mathbb{1}$, so any $q \in \mathbb{P}$ works
  - otherwise every non-$\mathbb{1}$ value $S_\alpha(t) \in C_k$ where $t = s^\frown\langle k \rangle$ (k: last element of t)

  $\rightarrow$ since C_k is centered and E finite, there exists a common lower bound $q \in \mathbb{P}$
- since $D := \bigcap_{\alpha \in E} D_\alpha$ is open dense, there $\exists q' \in D$ s.t. $q' \leq q$.

$\rightarrow$ let $\{a_i \mid i \in \omega\}$ be an infinite antichain below q' ... so also below $\{S_\alpha(t) \mid \alpha \in E\}$

$\rightarrow$ since distinct a_i are incompatible, they must all lie in distinct sets C_n

$\rightarrow$ let n_i be such that $a_i \in C_{n_i}$. We will show $\{n_i \mid i \in \omega\} \subseteq A$, so $|A| = \omega$

$\rightarrow$ since D is open and $q' \in D$, we have $a_i \in D$, so for $\forall \alpha \in E$ we have $a_i \in C_{n_i} \cap D_\alpha$ and $a_i \leq S_\alpha(t)$, so $n_i \in A_\alpha(t)$

② Construct an infinite branch

$\rightarrow$ for any given t, the centered family $\{A_\alpha(t) \mid \alpha < \kappa\}$ has size $\leq \kappa < \mathfrak{p}$, and so it has a pseudointersection $A(t) \in [\omega]^\omega$ ... so $A(t) \subseteq^* A_\alpha(t)$

$\Rightarrow$ for $\forall \alpha < \kappa$ there exists a function $h_\alpha : \omega^{<\omega} \to \omega$ s.t. $A(t) \setminus h_\alpha(t) \subseteq A_\alpha(t)$

$\hookrightarrow$ we delete all the finitely many exceptions

$\rightarrow$ we would like to get a function $h : \omega^{<\omega} \to \omega$ that eventually dominates all h_α

→ identify $\omega^{<\omega}$ with ω via any bijection. Then the functions $h_\alpha : \omega^{<\omega} \to \omega$ are really functions $h_\alpha : \omega \to \omega$, so $h_\alpha \in \omega^\omega$

⇒ the family $\{h_\alpha \mid \alpha < \kappa\} \subseteq \omega^\omega$ has size $\leq \kappa < \mathfrak{p} \leq \mathfrak{b}$ (bounding number), so there is $h : \omega^{<\omega} \to \omega$ such that $h_\alpha \leq^* h$ for every $\alpha < \kappa$

- Construct a branch $b : \omega \to \omega$ recursively by choosing

$$b(k) \in A(b \restriction k) \setminus h(b \restriction k)$$

→ take a look at the elements of the pseudointersection of $\{A_\alpha(b \restriction k) \mid \alpha < \kappa\}$ and chose one that is beyond the value of h at $b \restriction k$

③ <u>Verify that this branch works</u>

→ fix $\alpha < \kappa$. We want n_α such that b does not reset after level n_α

- Since $h_\alpha \leq^* h$, past some finite n_α we have $h_\alpha(b \restriction k) \leq h(b \restriction k)$ for $\forall k \geq n_\alpha$

⇒ $b(k) \in A(b \restriction k) \setminus h(b \restriction k) \subseteq A(b \restriction k) \setminus h_\alpha(b \restriction k) \subseteq A_\alpha(b \restriction k)$ as we wanted ■

We will soon show how to get rid of the size $\leq \kappa$ requirement

POSET MORPHISMS

<u>Def</u>: Let $\mathbb{P}, \mathbb{Q}$ be posets. A map $i : \mathbb{P} \to \mathbb{Q}$ is a <u>poset morphism</u> if

$$\forall a, b \in \mathbb{P} : \quad a \leq b \implies i(a) \leq i(b)$$

upward closure is a filter ↯

👁 If $F \subseteq \mathbb{P}$ is a <u>filter</u> and i a morphism, then $i[F] \subseteq \mathbb{Q}$ is a <u>filter base</u>

↳ if $i(p), i(q) \in i[F]$, take $r \in F$ s.t. $r \leq p, q$. Then $i(r) \leq i(p), i(q)$

<u>Def</u>: $i : \mathbb{P} \to \mathbb{Q}$ is a <u>poset embedding</u> if

① $a \leq b \implies i(a) \leq i(b)$ ... poset morphism

② $a \perp b \implies i(a) \perp i(b)$

👁 "⇐" in ② also holds ... if $i(a) \perp i(b)$ but $a \| b$, then $p \leq a, b$ has $i(p) \leq i(a), i(b)$

"⇐" in ① holds if $\mathbb{P}$ is separative ... if $i(a) \leq i(b)$ but $a \not\leq b$ then $\exists p \leq a$, $p \perp b$ then $i(p) \leq i(a) \leq i(b)$ but $i(p) \perp i(b)$ ↯ ... $p \leq q \Rightarrow p \| q$

<u>Proposition</u>: If $\mathbb{P}$ is nice and $i : \mathbb{P} \to \mathbb{Q}$ a poset embedding, then <u>i is injective</u> & <u>$a \leq b \iff i(a) \leq i(b)$</u>

<u>Proof</u>: Using the previous observation we just need to show injectivity

→ if $i(a) = i(b)$ then $a \leq b$ & $b \leq a$, so from assymetry $a = b$ ■

If $i : \mathbb{P} \to \mathbb{Q}$ is a poset embedding and $\mathbb{Q}$ is ccc, then $\mathbb{P}$ is ccc.

↳ i maps antichains to antichains by ② and it is injective, so an uncountable antichain in $\mathbb{P}$ would induce an uncountable antichain in $\mathbb{Q}$

Proposition: Let $\mathbb{Q}$ be an infinite poset and $X \subseteq \mathbb{Q}$. Then there exists an infinite sub-poset $\mathbb{P} \subseteq \mathbb{Q}$ of size $|\mathbb{P}| = \max\{|X|, \omega\}$ s.t. $X \subseteq \mathbb{P}$ and the inclusion map $\mathbb{P} \hookrightarrow \mathbb{Q}$ is a poset embedding.

Moreover, if $\mathbb{Q}$ is σ-centered, then so is $\mathbb{P}$ restricted to the same decomp.

Proof: Put $X_0 := X$. Given X_n, fix a partial function $S_n ; [X_n]^{\leq n+1} \to \mathbb{Q}$ with the property that if $F \in [X_n]^{\leq n+1}$ has a common lower bound in $\mathbb{Q}$, then $S_n(F)$ is such a lower bound, and $S_n(F)$ is otherwise undefined. Put

$$X_{n+1} := X_n \cup \operatorname{Rng}(S_n) \quad \text{and} \quad \mathbb{P} := \bigcup_{n \in \omega} X_n$$

Intuition: Whenever a finite subset of $\mathbb{P}$ has a common lower bound in $\mathbb{Q}$, it will also have one in $\mathbb{P}$, added at some finite stage

- if X is infinite then $\max\{|X|, \omega\} = |X|$ and by induction $|\mathbb{P}| = X$
- if X is finite, then $\mathbb{P}$ as is could also be finite, so we will first select a countable $Y \subseteq \mathbb{Q}$ and start with $X_0 := X \cup Y$. Then $|\mathbb{P}| = \omega$.

→ the inclusion $\mathbb{P} \hookrightarrow \mathbb{Q}$ is trivially a poset morphism and since it preserves lower bounds, it is also a poset embedding

→ similarly it preserves σ-centeredness since the restriction of a centered set is centered ∎

Corollary: → size of the posets we consider

① $MA_\kappa \iff MA_\kappa(\text{size} \leq \kappa)$

② $MA \iff MA(\text{size} < 2^\omega)$ ... follows from ①

③ $MA_\kappa(\sigma\text{-centered}) \iff MA_\kappa(\sigma\text{-centered, size} \leq \kappa)$ ... finishing the proof of Bell's theorem

Remark: This tells us something about the consistency strength of Martin's axiom

→ since we consider only posets of size $< 2^\omega$, there are only set-many non-isomorphic posets we assert something about

→ compare this to the original formulation that asserted something about a proper class of posets

→ this hints that we do not need large cardinals to prove the consistency of MA

Proof: ① Let $\mathbb{Q}$ be a ccc poset of arbitrary size.

→ we will use the maximal antichain formulation of MA:

Let $\mathcal{A}$ be a family of $\leq \kappa$ many maximal antichains in $\mathbb{Q}$

↳ we want an $\mathcal{A}$-generic filter ← $\mathbb{Q}$ is ccc

Define $X := \bigcup \mathcal{A}$, so $|X| \leq \kappa \cdot \omega = \kappa$

⇒ by the proposition there exists a sub-poset $\mathbb{P} \subseteq \mathbb{Q}$ of size $\leq \kappa$ containing X such that the inclusion map $\mathbb{P} \hookrightarrow \mathbb{Q}$ is an embedding

$\mathbb{P}$ is ccc because it embeds into $\mathbb{Q}$ and has size $\leq \kappa$

⇒ by MA_κ (size $\leq \kappa$) there exists an $\mathcal{A}$-generic filter $G \subseteq \mathbb{P}$ in $\mathbb{P}$

⇒ its upward closure $\overline{G} \subseteq \mathbb{Q}$ is an $\mathcal{A}$-generic filter in $\mathbb{Q}$

③ By the same argument since if $\mathbb{Q}$ is σ-centered then $\mathbb{P}$ is as well ∎

REGULAR EMBEDDINGS

Def: A poset embedding $i : \mathbb{P} \to \mathbb{Q}$ is

- regular $\equiv$ $\forall$ predense $D \subseteq \mathbb{P}$, the image $i[D] \subseteq \mathbb{Q}$ is predense in $\mathbb{Q}$
- dense $\equiv$ $\forall$ dense $D \subseteq \mathbb{P}$, the image $i[D] \subseteq \mathbb{Q}$ is dense in $\mathbb{Q}$

Remark: Regular embeddings are also sometimes called complete in the literature

Observation: In the definition of regular embedding, we can replace "predense set" with "maximal antichain"

Proof: "⇒" Any maximal antichain is predense, so its image is predense. Moreover, the image of an antichain under an embedding is again an antichain. Finally, any predense antichain is maximal

"⇐" Let $D \subseteq \mathbb{P}$ be predense and take a maximal antichain $A \subseteq {\downarrow}D$ ← downward closure

A is max. antichain in ${\downarrow}D$ ⟺ A is max. antichain in $\mathbb{P}$

↳ if D is predense then ${\downarrow}D$ is dense and if we could add $p \in \mathbb{P}$ there is $q \leq p$ in ${\downarrow}D$ and we could add q

⇒ Hence $i[A] \subseteq \mathbb{Q}$ is a max. antichain in $\mathbb{Q}$ and so it is predense

want: $i[D]$ predense ... given $q \in \mathbb{Q}$ there is $a \in A$ s.t. $q \parallel i(a)$

since $a \in {\downarrow}D$ there is $d \in D$ with $a \leq d$ and $i(a) \leq i(d)$ ⇒ $q \parallel i(d)$ ∎

Observation: Every dense embedding is regular.

Pf: If $D \subseteq \mathbb{P}$ is predense then ${\downarrow}D$ is dense so $i[{\downarrow}D]$ is dense. Want: $i[D]$ predense

Given $q \in \mathbb{Q}$ there $\exists a \in {\downarrow}D$ s.t. $i(a) \leq q$. But $a \leq d$ for some $d \in D$ so $i(a) \leq i(d)$

↳ then $i(a)$ witnesses $q \parallel i(d)$ ∎

Observation: $i : \mathbb{P} \to \mathbb{Q}$ is dense $\iff$ there $\exists$ dense $D \subseteq \mathbb{P}$ s.t. $i[D]$ is dense in $\mathbb{Q}$

$\hookrightarrow$ it suffices for $i[\mathbb{P}]$ to be dense in $\mathbb{Q}$

Proof: Suppose $i[\mathbb{P}] \subseteq \mathbb{Q}$ is dense in $\mathbb{Q}$ and $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$. Want $i[D]$ dense

Given $q \in \mathbb{Q}$ there is $p \in \mathbb{P}$ s.t. $i(p) \leq q$. Since D is dense there $\exists d \in D : d \leq p$

$\Rightarrow i(d) \leq i(p) \leq q$ hence $i[D]$ is dense ■

Def: Let $i : \mathbb{P} \to \mathbb{Q}$ be an embedding. For $q \in \mathbb{Q}$, an element $p \in \mathbb{P}$ is a pseudoprojection of q if

$$\forall a \in \mathbb{P} : \quad a \parallel p \quad \Rightarrow \quad i(a) \parallel q$$

Intuition: A sufficient condition for $i(a)$ being compatible with q is a being compatible with its pseudoprojection

Equivalently:

$$\forall a \in \mathbb{P} : \quad a \leq p \quad \Rightarrow \quad i(a) \parallel q$$

$\hookrightarrow$ clearly $a \leq p \Rightarrow a \parallel p$, for "$\Leftarrow$" we take a lower bound

Proposition: $i : \mathbb{P} \to \mathbb{Q}$ is regular $\iff$ every $q \in \mathbb{Q}$ has a pseudoprojection

$\Rightarrow$ there $\exists$ map $PP : \mathbb{Q} \to \mathbb{P}$ where $PP(q)$ is a pseudoprojection of q

Proof: "$\Leftarrow$" If we have PP let $D \subseteq \mathbb{P}$ be predense ... we want $i[D]$ to be predense in $\mathbb{Q}$

Given $q \in \mathbb{Q}$ look at $PP(q)$ and take $d \in D$ s.t. $d \parallel PP(q)$. Then $i(d) \parallel q$

"$\Rightarrow$" Given $q \in \mathbb{Q}$, we want a pseudointersection. Consider

$R := \{ r \in \mathbb{P} \mid i(r) \perp q \}$ ... not predense in $\mathbb{P}$, otherwise $i[R]$ would be predense in $\mathbb{Q}$ so $i(r) \parallel q$ for some $r \in R$ ↯

Hence $\exists p \in \mathbb{P}$ s.t. $p \perp r$ for $\forall r \in R$

claim: p is a pseudoprojection of q ... if $a \parallel p$ then $a \notin R$ so $i(a) \parallel q$ ■

FORCING-EQUIVALENCE OF POSETS

Proposition: If $i: \mathbb{P} \to \mathbb{Q}$ is regular and $G \subseteq \mathbb{Q}$ a V-generic filter, then $i^{-1}[G] \subseteq \mathbb{P}$ is also a V-generic filter.

Proof: $i^{-1}[G]$ is upward closed ... if $p \in i^{-1}[G]$ and $p \leq p'$ then $i(p) \leq i(p')$ so $i(p') \in G$ since $i(p) \in G$, and thus $p' \in i^{-1}[G]$

- $i^{-1}[G]$ is linked: if $p, q \in i^{-1}[G]$ then $i(p) \parallel i(q)$ since G is a filter because i is an embedding also $p \parallel q$ (as $p \perp q \Leftrightarrow i(p) \perp i(q)$)
- $i^{-1}[G]$ is V-generic: it intersects every predense $D \subseteq \mathbb{P}$ (recall equivalent formulations of genericity we encountered when studying Martin's axiom)
  $\to$ $i(D)$ is predense in $\mathbb{Q}$ because i is regular
  $\Rightarrow$ G intersects $i[D]$ ... $g \in G$ has $i^{-1}(g) \in D$ $\Rightarrow$ $i^{-1}[G]$ intersects D
- $i^{-1}[G]$ is a filter: We use the same argument as when proving Bell's theorem: that a linked upward closed generic object is automatically a filter
  $\to$ let $p, q \in i^{-1}[G]$. We want $r \in i^{-1}[G]$ s.t. $r \leq p, q$. Consider
  $$D := \{ r \in \mathbb{P} \mid (r \leq p, q) \vee r \perp p \vee r \perp q \}$$
  $\to$ earlier we have shown that D is dense ... and thus predense
  $\Rightarrow \exists r \in D \cap i^{-1}[G]$ ... since $i^{-1}[G]$ is linked we have $r \parallel p$ and $r \parallel q$, so from the definition of D it must be $r \leq p, q$ ∎

Proposition: If $i: \mathbb{P} \to \mathbb{Q}$ is dense and $F \subseteq \mathbb{P}$ a V-generic filter, then the upward closure $\uparrow i[F] \subseteq \mathbb{Q}$ is also a V-generic filter

Remark: If $F \subseteq \mathbb{P}$ and $G \subseteq \mathbb{Q}$ are V-generic filters and i dense, then

$$i^{-1}[\uparrow i[F]] = F \quad \text{and} \quad \uparrow i[i^{-1}[G]] = G$$

so the image/preimage are filters

Proof: One of our first observations was that if $F \subseteq \mathbb{P}$ is a filter and $i: \mathbb{P} \to \mathbb{Q}$ a poset morphism then $i[F]$ is a filter base, so $\uparrow i[F]$ is a filter

- genericity: we will show that $\uparrow i[F]$ intersects every open dense $D \subseteq \mathbb{Q}$
  claim: $i^{-1}[D]$ is dense in $\mathbb{P}$ ... then $\exists x \in F \cap i^{-1}[D]$ and $i(x) \in i[F] \cap D$ ✓
  $\hookrightarrow$ let $p \in \mathbb{P}$. Since D is dense in $\mathbb{Q}$ there $\exists q \in D$ s.t. $q \leq i(p)$
  $\to$ i is dense $\Rightarrow$ $i[\mathbb{P}]$ is dense in $\mathbb{Q}$ $\Rightarrow$ $\exists q' \in i[\mathbb{P}]$ s.t. $q' \leq q \leq i(p)$
  $\to$ let $p' = i^{-1}(q')$. Since $i(p') \leq i(p)$ we have $p' \leq p$
  $\to$ since D is open and $q' \leq q$ also $q' \in D$ so $p' \in i^{-1}[D]$
  $\Big\}$ $i^{-1}[D]$ is dense ∎

EXAMPLE: THE COHEN FORCING

→ recall the Cohen forcing $\mathbb{C} = (\{p \mid \omega \to 2 \mid |p| < \omega\}, \supseteq)$ of finite partial functions $\omega \to 2$ ordered by reverse inclusion

→ we mentioned that it is equivalent to any other (nice) countable atomless poset

⇒ consider $\mathbb{C}' := (\omega^{<\omega}, \sqsupseteq)$ or $\mathbb{C}'' := (\{p \mid \omega \to \omega \mid |p| < \omega\}, \supseteq)$

<u>Theorem</u>: If $\mathbb{Q}$ is countable atomless then there $\exists$ dense embedding $i : \mathbb{C}' \to \mathbb{Q}$

⇒ thus all countable atomless posets have the same forcing consequences

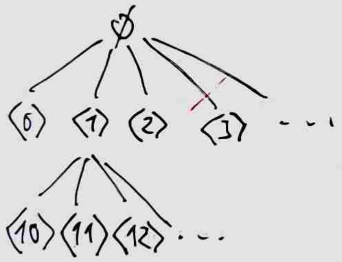
<u>Intuition</u>: Suppose $\mathbb{Q}$ is countable atomless and we want to show that it has the same consequences as $\mathbb{C}$... e.g. that it forces a Cohen real

→ why does $\mathbb{C}$ force a Cohen real? Because we get a get a filter intersecting some dense subsets of $\mathbb{C}$

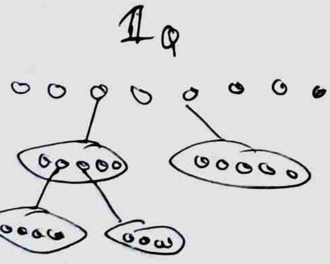
→ when we force with $\mathbb{Q}$, we get a V-generic filter on $\mathbb{Q}$ and the dense embedding into $\mathbb{C}$ gives us a V-generic filter on $\mathbb{C}$, so we can intersect those same dense sets and construct the Cohen real

<u>Proof</u>: We will use the fact (see proof of Bell's theorem) that if $\mathbb{Q}$ is atomless and separative then under any $q \in \mathbb{Q}$ there is an infinite antichain

→ this will allow us to replicate the tree structure of $\mathbb{C}'$ inside $\mathbb{Q}$

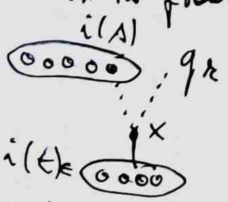


- map the root $\emptyset = \mathbb{1}_{\mathbb{C}'} \in \mathbb{C}'$ to $\mathbb{1}_{\mathbb{Q}}$
- take maximal countable antichain $a_0, a_1, \ldots$ below $\mathbb{1}_{\mathbb{Q}}$ and map nodes at level 1 to it as $\langle n \rangle \mapsto a_n$
- to extend the map from levels $0 - k$ to level $k+1$, for each $t \in \omega^k$ take a countable max antichain $a^t_0, a^t_1, \ldots$ below $i(t)$ and map $i(t^\frown\langle n \rangle) := a^t_n$



- to ensure density of this embedding we need $i[\mathbb{C}']$ to be dense in $\mathbb{Q}$. So before doing this construction, enumerate $\mathbb{Q} = \{q_n \mid n < \omega\}$ with $q_0 = \mathbb{1}_{\mathbb{Q}}$, and for $k \geq 1$, when mapping level $k+1$ of the tree, ensure that some $t \in \omega^{k+1}$ gets mapped below q_k

→ this is possible since every max. antichain is predense, so there exists $s \in \omega^k$



s.t. $i(s) \| q_k$, so there $\exists x \leq i(s), q_k$ and we can take the antichain below $i(s)$ to be below x. Every $t \in \omega^{k+1}$ mapped into this antichain will be below q_k

⇒ hence the mapping is dense and it is an embedding since we literally copied its structure ... one can formally check the definition ▨

Example: The diagonal $\Delta : \mathbb{P} \to \mathbb{P} \times \mathbb{P}$, $p \mapsto (p,p)$ is a poset embedding but it is in general not regular: if $\mathbb{P}$ contains orthogonal elements $p \perp q$ then $D := \{p,q\} \cup \{p,q\}^{\perp}$ is predense but $\Delta[D]$ is not predense

↳ $A^{\perp} := \{x \mid \forall a \in A : x \perp a\}$

so e.g. when $\mathbb{P}$ is atomless

because $(p,q) \in \mathbb{P} \times \mathbb{P}$ is incompatible with every $(x,x) \in \Delta[D]$

↳ ordered coordinate-wise so $(p,q) \parallel (p',q')$ iff $p \parallel p'$ & $q \parallel q'$

Example: The map $i : \mathbb{P} \to \mathbb{P} \times \mathbb{Q}$, $p \mapsto (p, \mathbb{1}_{\mathbb{Q}})$ is a regular embedding with pseudoprojection map $(p,q) \mapsto p$

→ it is regular since $\mathbb{1}_{\mathbb{Q}}$ is maximal element and for pseudoprojection:

if $a \le p$ then $i(a) = (a, \mathbb{1}_{\mathbb{Q}}) \parallel (p,q)$ since $\underset{a \le p}{a \parallel p}$ & $\underset{q \le \mathbb{1}_{\mathbb{Q}}}{\mathbb{1}_{\mathbb{Q}} \parallel q}$

SEPARATIVE QUOTIENT

→ as we have seen, separativity is an important property of forcing posets

→ it says that if p does not extend q, there should be some concreate reason why

↳ if $p \not\le q$, there should be $r \le p$ contradicting q ... $r \perp q$

→ if a poset is not separative, it contains conditions that look different and are not comparable but behave the same when it comes to compatibility with the rest of the poset

→ intuitively, compatibility is what forcing really cares about, so these conditions cary the same forcing information ... we can identify them

Def: For $x \in \mathbb{P}$ put $x^{\perp} := \{p \in \mathbb{P} \mid p \perp x\}$

Def: For $x, y \in \mathbb{P}$ write $x \sim y$ if $x^{\perp} = y^{\perp}$, or equiv. $\forall p \in \mathbb{P} : p \parallel x \iff p \parallel y$

The separative quotient of $\mathbb{P}$ is the poset

$\mathbb{P}/\sim = \{[x]_{\sim} \mid x \in \mathbb{P}\}$ ordered by $[x] \preccurlyeq [y]$ iff $x^{\perp} \supseteq y^{\perp}$,

so stronger conditions are more difficult to be compatible with

Equivalently $[x] \preccurlyeq [y] \iff \forall p \le x : p \parallel y$

"$\Rightarrow$" if $p \le x$ then $p \parallel x$ so $p \notin x^{\perp} \Rightarrow p \notin y^{\perp} \Rightarrow p \parallel y$

"$\Leftarrow$" if $p \perp y$ but $p \parallel x$ then $\exists q \le p, x$ and $q \parallel y$ so $\exists q' \le q, y$ but then $q' \le p, y$ so $p \parallel y$ ↯

Intuition: if $[x] \preccurlyeq [y]$ then $[x]$ forces everything $[y]$ forces

<u>Observation</u>: $\mathbb{P}/\sim$ is always separative ✓ using previous observation

<u>Proof</u>: If $[x] \not\leq [y]$ then $\exists p \leq x$ s.t. $x \perp y$, which is exactly separtivity

<u>Theorem</u>: The map $i: \mathbb{P} \to \mathbb{P}/\sim$, $x \mapsto [x]$ is a dense embedding.

<u>Corollary</u>: We can always WLOG assume that our forcing poset is separative, since otherwise we can use $\mathbb{P}/\sim$ which has the same forcing consequences.

<u>Proof</u>: We verify that it is a dense embedding

- $x \leq y \Rightarrow [x] \leq [y]$: if $x \leq y$ then $\forall p \leq x$ also $p \leq y$ so $p \| y$, thus $[x] \leq [y]$
- $x \perp y \Rightarrow [x] \perp [y]$: if $x \perp y$ but $[x] \| [y]$ then $\exists [z] \leq [x], [y]$, so $\forall p \leq z$: $p \| x$ & $p \| y$ ... in particular $z \| x$ ... $\exists q \leq z, x$
  $\to$ since $q \leq z$ we have $q \| y$ so $\exists q' \leq q, y$ ... now $q' \leq x, y$ so $x \| y$ ↯

- $i[\mathbb{P}]$ is dense in $\mathbb{P}/\sim$ ... trivially since $i[\mathbb{P}] = \mathbb{P}/\sim$ ∎

PROJECTIONS

→ another way to transfer generic filters

<u>Def</u>: A map $\pi: \mathbb{Q} \to \mathbb{P}$ is a projection if

i) $\pi(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$

ii) π is a poset morphism ... so $p \leq q \Rightarrow \pi(p) \leq \pi(q)$

iii) $(\forall q \in \mathbb{Q})(\forall p \in \mathbb{P})$ if $p \leq \pi(q)$ then $\exists q' \leq q$ s.t. $\pi(q') \leq p$

↳ if q maps to a condition weaker than p, we can find some q' stronger than q that maps to a condition stronger than p

<u>Observation</u>: If π is a projection, then the following also holds:

i') $\pi[\mathbb{Q}]$ is dense in $\mathbb{P}$

This is really the property we care about, so i) is often replaced by i')
↳ for example if our posets do not have maximal elements

<u>Proof</u>: Let $p \in \mathbb{P}$ ... then $p \leq \mathbb{1}_{\mathbb{P}} = \pi(\mathbb{1}_{\mathbb{Q}})$, so $\exists q$ s.t. $\pi(q) \leq p$. ∎

<u>Example</u>: i') is not that important either

→ let $\mathbb{Q}$ be a poset, $q \in \mathbb{Q}$ and consider $\mathbb{Q} \restriction q := \{p \in \mathbb{Q} \mid p \leq q\}$

→ then the inclusion map $\mathbb{Q} \restriction q \hookrightarrow \mathbb{Q}$ does transfer generic filters (just close the image of the filter upward)

→ it satisfies ii) and iii) but it fails both i) and i')

⇒ so i) and i') are not that important to transferring genericity

<u>Proposition</u>: If $\pi: \mathbb{Q} \to \mathbb{P}$ is a projection and $G \subseteq \mathbb{Q}$ is a V-generic filter, then the upward closure $\uparrow \pi[G]$ is a V-generic filter on $\mathbb{P}$.

<u>Proof</u>: $\uparrow \pi[G]$ is a filter $\because$ π is a poset morphism

- <u>genericity</u>: if $D \subseteq \mathbb{P}$ is open dense (closed downward), then $\pi^{-1}[D] \subseteq \mathbb{Q}$ is dense
  - ↳ given $q \in \mathbb{Q}$ we want $r \leq q$ s.t. $\pi(r) \in D$
  - ↳ look at $\pi(q)$ ... since D is dense there $\exists d \in D$ s.t. $d \leq \pi(q)$
  - $\Rightarrow$ by iii) there $\exists r \leq q$ s.t. $\pi(r) \leq d$ ... since D is open also $\pi(r) \in D$

Since $\pi^{-1}[D]$ is dense in $\mathbb{Q}$, it is intersected by G ... there $\exists q \in G$ s.t. $\pi(q) \in D$

$\Rightarrow$ so $\pi[G]$ intersects every open dense set D (recall that this is enough) ∎

<u>Remark</u>: We used neither i) nor i')

EXAMPLES: TRANSFERING FORCING PROPERTIES

<u>Strategy</u>: Construct a projection or a regular embedding

↳ dense embeddings are rare

<u>Theorem</u>: Hechler forcing adds a Cohen real

$$\mathbb{C} := (2^{<\omega}, \sqsupseteq) \text{ ... countable atomless poset}$$

$$\mathbb{H} := \{(f,n) \mid f: \omega \to \omega,\ n \in \omega\}$$

$$(f,n) \leq (f',n') \text{ iff } f \geq f' \ \&\ n \geq n' \ \&\ f \restriction n' = f' \restriction n'$$

↳ coordinate wise larger everywhere

↳ this is a slightly different formulation than what we have used before, but they are forcing-equivalent

<u>Want</u>: Transfer a V-generic from $\mathbb{H}$ to $\mathbb{C}$

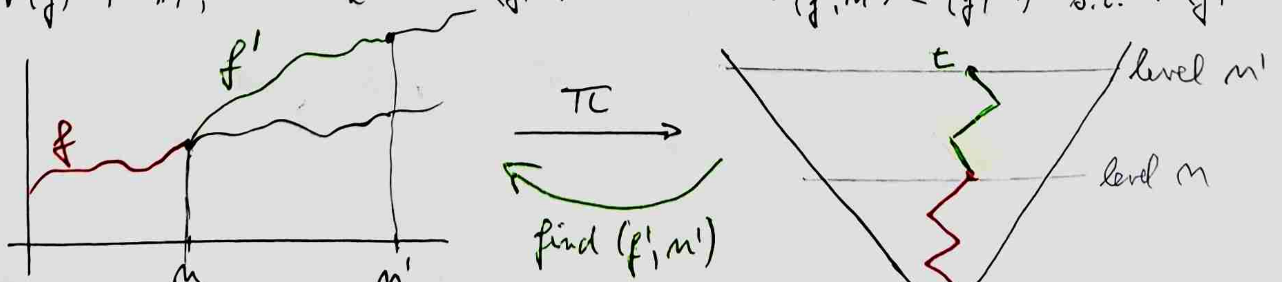
$\Rightarrow$ we define a projection $\pi: \mathbb{H} \to \mathbb{C}$ by $(f,n) \longmapsto f \restriction n$ modulo 2

<u>claim</u>: π is a projection

i) $\pi(\mathbb{1}_H) = \mathbb{1}_C$

ii) π is poset morphism ... $(f,n) \leq (f',n') \Rightarrow (f \restriction n \text{ mod } 2) \sqsupseteq (f' \restriction n' \text{ mod } 2)$ ↗ since $f \restriction n' = f' \restriction n'$

iii) $\forall (f,n) \in \mathbb{H}, \forall t \in \mathbb{C}: \pi(f,n) \geq t \Rightarrow \exists (f',n') \leq (f,n)$ s.t. $\pi(f',n') \leq t$



Hechler condition with fixed part & tail ∎

Example with Mathias forcing

$$\mathbb{M} := \{(a, X) \mid a \in [\omega]^{<\omega},\ X \in [\omega]^{\omega},\ \max(a) < \min(X)\}$$

$$(a, X) \leq (b, Y) \text{ iff } a \sqsupseteq b \ \&\ X \subseteq Y \ \&\ a \setminus b \subseteq Y$$

$\circledast$ ↳ 

... a end-extends b using elements from Y

Recall: Forcing with $\mathbb{M}$ adds a new set $X \in [\omega]^{\omega}$ with the reaping property

↳ X cannot be split into two infinite part using any ground model $A \in [\omega]^{\omega} \cap V$

Fact: $\mathbb{M}$ does not add a Cohen real

Theorem: $\mathbb{M} \times \mathbb{M}$ ordered coordinate-wise adds a Cohen real

Proof: We would like to define a projection $\pi : \mathbb{M} \times \mathbb{M} \to \mathbb{C}$

→ we first make $\mathbb{M} \times \mathbb{M}$ nicer ... only consider the set $D \subseteq \mathbb{M} \times \mathbb{M}$ of all pairs $(a, X), (b, Y)$ where $|a| = |b|$

D is dense (we can always extend the smaller one)

⇒ the inclusion map $i : D \hookrightarrow \mathbb{M} \times \mathbb{M}$ is a dense embedding

⇒ it suffices to define a projection $\pi : D \to \mathbb{C}$

Notation: for $a \subseteq \omega$ let $l_a : |a| \to a$ enumerate the elements of a in order

Given a pair $(a, X), (b, Y)$ where $|a| = |b|$, define their projection as the function $c : |a| \to 2$, where $c(i) = 0$ iff $l_a(i) < l_b(i)$ and 1 otherwise

↳ compare i^{th} elements of a and b

claim: π is a projection

i) $\pi(\mathbb{1}) = \mathbb{1}_{\mathbb{C}}$

ii) π is poset morphism because $\circledast$ requires end-extensions

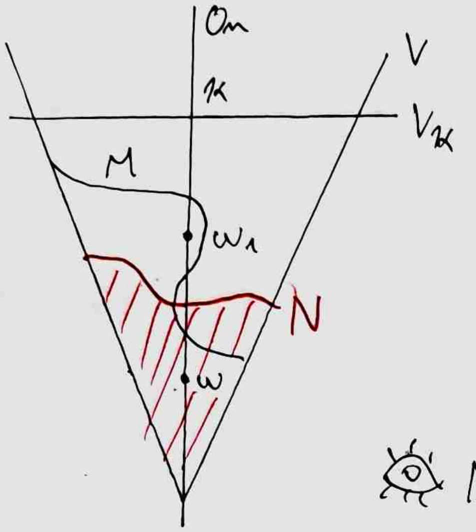
iii) if c' extends $c = \pi(\boxtimes)$, we can extend $\boxtimes$ to $\boxtimes'$ so that $\pi(\boxtimes') \leq c'$

↳ yes because X and Y are infinite (unbounded), we can extend a and b in a way so that they alternate which element on the i^{th} position is smaller according to c' ∎

MORE PRECISE IDEA OF FORCING

Where do V-generic filters come from?

→ we will start with an inaccessible cardinal κ, so $V_\kappa \models ZFC$



- Löwenheim-Skolem: there exists a countable elementary submodel $\langle M, \in \rangle$ of $\langle V_\kappa, \in \rangle$
  ↳ we need V_κ to be a set, we cannot work with V
- Mostowski collapse of M: there exists a (countable) transitive model $\langle N, \in \rangle \cong \langle M, \in \rangle$
  ↖ isomorphism

$N \cap On^V \in \omega_1^V$ ... On^N is a countable ordinal in V_κ
↳ since N is countable & transitive

We now work in N as in our ground model
↳ so when we do forcing, the ground model is always countable

Recall: Rasiowa-Sikorski: FA_ω(all posets) holds
↳ given any poset $\mathbb{P} \in N$, there can be only countably many dense sets of $\mathbb{P}$ in N
⇒ so there $\exists$ filter $G \subseteq \mathbb{P}$ in V_κ intersecting all these dense sets
→ if we also want $p \in G$ for some $p \in \mathbb{P}$, we can simply add it first and then construct the base of G under it ... $p \geq d_1 \geq d_2 \geq d_3 \geq \dots$, $d_n \in D_n$

The hard part is to show that $N[G]$ is a model (we will not do it)

Getting rid of the inaccessible κ

Recall the Compactness theorem of first order logic:
Theory T is consistent ⟺ every finite subtheory $T' \subseteq T$ is consistent

Idea: Any proof is always finite, so if T can prove a contradiction, then we can T' be the finite subtheory consisting of the finitely many axioms used, and we can prove the contradiction in T' via the same proof

Need: For every finite fragment ZFC^* of ZFC, there exists a set N such that $N[G] \models ZFC^*$

→ this can be achieved using the Reflection lemma, which is a consequence of Löwenheim - Skolem

<u>Reflection lemma</u>: Let φ be a formula, imagine a conjunction of finitely many axioms of ZFC. Then

$$\text{if } V \models \varphi, \text{ then } \exists \alpha \in On \text{ s.t. } V_\alpha \models \varphi$$

$\Rightarrow$ Suppose we want $N \in V$ so that $N[G] \models ZFC^*$

→ we can imagine that we did the construction with inaccessible κ and have N from that and an N-generic filter G

→ one can show that $N[G] \models ZFC^*$, but since the proof uses only finitely many axioms of $N \models ZFC$, we can consider a conjunction φ of these axioms

$\Rightarrow$ by the reflection lemma there is $\alpha \in On^V$ s.t. $V_\alpha \models \varphi$, so if $N := V_\alpha \models \varphi$, then we can show $N[G] \models ZFC^*$