BASICS OF GENERAL TOPOLOGY

Def: A topological space is a pair (X, σ) where $\sigma \subseteq P(X)$ s.t.

① $\emptyset, X \in \sigma$ ... open sets $\quad \hookrightarrow \sigma =$ topology

② $u, v \in \sigma \Rightarrow u \cap v \in \sigma$ ... finite intersections

③ $\mathcal{U} \subseteq \sigma \Rightarrow \bigcup \mathcal{U} \in \sigma$ ... infinite unions

Def: $U \subseteq X$ is closed if its complement is open

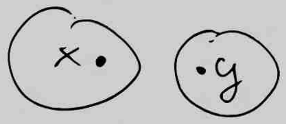
👁 arbitrary intersections & finite unions of closed sets are closed

Examples:

- metric space (M, d) with $\sigma = \{ U \subseteq M \mid \forall x \in U \, \exists \varepsilon > 0 : B(x, \varepsilon) \subseteq U \}$
  - ↳ in particular Euclidean space $\mathbb{R}^d$
- discrete topology on a set X has open sets $\sigma = P(X)$

Def: A topological space is Hausdorff if

$(\forall x, y \in X, \; x \neq y)(\exists U_x, U_y \in \sigma) : x \in U_x, \; y \in U_y \; \& \; U_x \cap U_y = \emptyset$



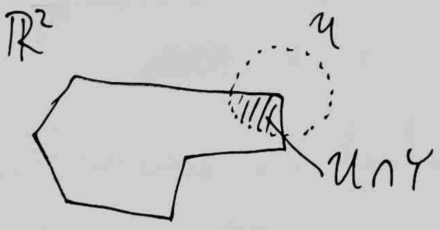
... distinct points can be separated by open sets

→ we will only consider Hausdorff spaces

Def: A subspace of a top. space (X, σ) is $(Y \subseteq X, \{ U \cap Y \mid U \in \sigma \})$

$X = \mathbb{R}^2$

$Y =$



👁 if $Y \notin \sigma$ then possibly $U \cap Y \notin \sigma$

Def: A mapping $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is

- open if $U \in \sigma_X \Rightarrow f[U] \in \sigma_Y$
- closed if U closed $\Rightarrow f[U]$ closed
- continuous if $U \in \sigma_Y \Rightarrow f^{-1}[U] \in \sigma_X$ ... preimages of open sets are open

$f: X \to Y$ is continuous $\Leftrightarrow$ $\forall A \subseteq Y$ closed is $f^{-1}[A] \subseteq X$ closed

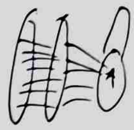
"$\Rightarrow$" $A \subseteq Y$ closed, then A^c open, so $f^{-1}[A^c] = (f^{-1}[A])^c$ open, so $f^{-1}[A]$ closed

"$\Leftarrow$" analogously

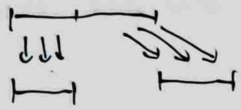
composition

$f: X \to Y$ & $g: Y \to Z$ continuous $\Rightarrow$ $f \circ g : X \to Z$ continuous

↳ $A \subseteq Z$ open, then $g^{-1}[A]$ open, so $f^{-1}[g^{-1}[A]] = (f \circ g)^{-1}[A]$ open.



$f: X \to Y$ open $\not\Rightarrow$ f continuous

↳  $f: (0,2) \to (0,1) \cup [2,3)$ where $(0,2)$ and $(0,1) \cup [2,3)$ are subspaces of $\mathbb{R}$

$f^{-1}[[2,3)] = [1,2)$ is not open in $(0,2)$

Note: In metric spaces, $f: (M,d) \to (M',d')$ is continuous $\Leftrightarrow$

$(\forall x \in M)(\forall \varepsilon > 0)(\exists \delta > 0): f[B(x,\delta)] \subseteq B(f(x),\varepsilon)$

Def: $f: X \to Y$ is a homeomorphism $\equiv$ f is bijective & f, f^{-1} are continuous.

↳ (X, O_X) and (Y, O_Y) are homeomorphic $(X \cong Y)$ if $\exists$ hom. $X \to Y$

f is a homeomorphism $\equiv$ f is bijective, continuous, and open/closed

↳ open from basic definition, closed from the observation above

Def: A basis of a space (X,O) is a set $\mathcal{B} \subseteq O$ s.t. $\forall U \in O$ $\exists \mathcal{B}' \subseteq \mathcal{B}$ s.t. $U = \bigcup \mathcal{B}'$.

Example: The set of all open intervals of $\mathbb{R}$ is a basis for the Euclidean topology

Def: Let (X, O_X) and (Y, O_Y) be top. spaces. The product topology on $X \times Y$ has the collection $\{U \times V \mid U \in O_X, V \in O_Y\}$ as a basis.

Ex: The spaces $(\mathbb{R}^2, O_{Euclid})$ and $(\mathbb{R} \times \mathbb{R}, O_{product})$ are homeomorphic

Def: The closure of $Y \subseteq (X,O)$ is $\overline{Y}$ or $cl(Y) := \bigcap \{C \subseteq X \mid Y \subseteq C, C \text{ is closed}\}$

- boundary of Y is $\partial Y := \overline{Y} \cap \overline{X \setminus Y}$
- interior of Y is $int(Y) := \overline{Y} \setminus \partial Y$

Equiv: $int(Y) = \{y \in Y \mid \exists U \in O : y \in U \subseteq Y\}$

In metric spaces: $cl(Y) = \{x \mid dist(x,Y) = 0\}$ where $dist(x,Y) = \inf\{d(x,y) \mid y \in Y\}$

Def: $K \subseteq (X,\sigma)$ is compact $\equiv$ $\forall$ open cover of K $\exists$ finite subcover $\searrow$ finite $\mathcal{U}' \subseteq \mathcal{U}$

- open cover of K is $\mathcal{U} \subseteq \sigma$ s.t. $K \subseteq \bigcup \mathcal{U}$

Theorem (Heine-Borel): $X \subseteq \mathbb{R}^d$ is compact $\Leftrightarrow$ it is closed & bounded

Theorem: A metric space (M,d) is compact $\Leftrightarrow$ it is sequentially compact

That is: every infinite sequence has a convergent subsequence

Theorem: If X is a compact top. space and $f: X \to \mathbb{R}$ is continuous, then f attains a minimum and a maximum.

Fact: A continuous function on a compact metric space is uniformly continuous: $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall x : f[B(x,\delta)] \subseteq B(f(x),\varepsilon)$

Proposition: Continuous mappings preserve compactness.

$f: (X,\sigma_X) \to (Y,\sigma_Y)$ continuous & $K \subseteq X$ compact $\Rightarrow f[K]$ compact

Proof: Let $\mathcal{U} \subseteq \sigma_Y$ be an open cover of $f[K]$

$\to \mathcal{V} := \{ f^{-1}[u] \mid u \in \mathcal{U} \}$ is an open cover of K ... f is continuous

$\Rightarrow$ let $\mathcal{V}_0 \subseteq \mathcal{V}$ be a finite subcover ... K is compact

$\mathcal{V}_0 = \{ f^{-1}[u] \mid u \in \mathcal{U}_0 \}$ ... $\mathcal{U}_0$ is a finite subcover of $f[K]$ ▨

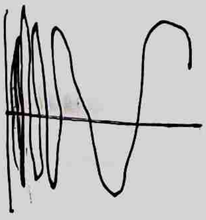
Def: $K \subseteq (X,\sigma)$ is connected if it cannot be written as a union of two disjoint nonempty open sets

$\to$ continuous $f: [0,1] \to X$ s.t. $f(0) = x$, $f(1) = y$

Def: $K \subseteq (X,\sigma)$ is path-connected $\equiv$ $\forall x,y \in X \; \exists$ path connecting x to y

! path connected is strictly stronger than connected

Example:



$$M = \{(0,y) \mid y \in [1,-1]\} \cup \{(x, \sin(\tfrac{1}{x})) \mid x \in (0,1]\}$$

is connected but not path-connected

HOMOTOPY EQUIVALENCE

Def: Let X be a topological space. $Y \subseteq X$ is called a (strong) deformation retract of X if ∃ continuous $F: X \times [0,1] \to X$ s.t.

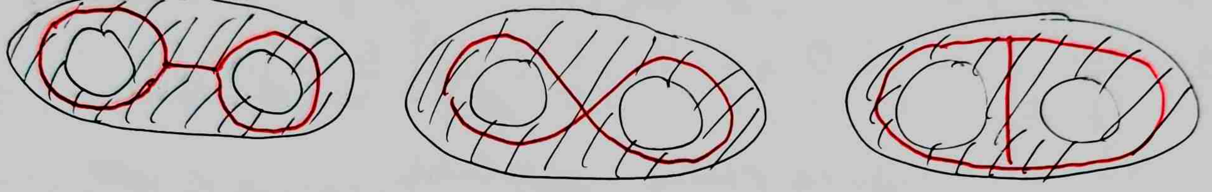
① $f_0 = id_X$

② $\forall t \, \forall y \in Y: f_t(y) = y$

③ $f_1[X] = Y$

$f_t : X \to X$, $x \mapsto F(x,t)$
↳ $t \in [0,1]$

Def: X and Y are homotopy equivalent ($X \simeq Y$) if ∃ space Z s.t. both X and Y are deformation retracts of Z



Example: The Möbius strip is not homeomorphic to a circle, but it is homotopic

Equivalent definition of homotopy:

Def: Two continuous maps $f, g: X \to Y$ are homotopic ($f \sim g$) if ∃ continuous $H: X \times [0,1] \to X$ s.t. $h_0 = f$ & $h_1 = g$.

Def: $f: X \to Y$ is null-homotopic if $f \sim$ constant function $g: x \mapsto y_0^{\in Y}$

Fact: X and Y are homotopy equivalent ⟺ ∃ continuous maps $g: X \to Y$ and $h: Y \to X$ s.t. $g \circ h \sim id_X$, $h \circ g \sim id_Y$

Proof sketch:

👁 if Y is a deformation retract of X by mapping F, then $f_0 \sim f_1$

→ take $h: Y \to X$ as the inclusion map $f(y) = y$ $\forall y \in Y$.

→ take $g: X \to Y$ as f_1, noting $g \upharpoonright Y = id_Y$

- $h \circ g = id_Y$, so $h \circ g \sim id_Y$
- $g \circ h = f_1$ and $f_1 \sim f_0 = id_X$

"⇒": if $X \simeq Y$, ∃ Z s.t. X, Y are def. r. of Z

→ by 👁, $X \simeq Z$ and $Y \simeq Z$ according to the above definition

↗ just check transitivity

"⇐" is very difficult and requires deep results ▨

Homeomorphic ($X \cong Y$) $\Rightarrow$ Homotopy equivalent ($X \simeq Y$)

↳ given homeomorphism $f: X \to Y$ we have $f \circ f^{-1} = id_X$ and $f^{-1} \circ f = id_Y$

BASIC GEOMETRY

- d-dimensional ball $B^d := \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$
- (d-1)-dimensional sphere $S^{d-1} := \{x \in \mathbb{R}^d \mid \|x\| = 1\}$

$$\partial B^d = S^{d-1}$$

- hyperplane in $\mathbb{R}^d$ is a (d-1)-dimensional affine subspace
  $h = \{x \in \mathbb{R}^d \mid \langle \bar{v} | x \rangle = b\}$ for some $\underset{\overset{\neq}{0}}{\bar{v}} \in \mathbb{R}^d$, $b \in \mathbb{R}$
  $h^+ := \{x \in \mathbb{R}^d \mid \langle \bar{v} | x \rangle < b\}$ ... open half-space
- $C \subseteq \mathbb{R}^d$ is convex if $\forall x, y \in C \ \forall t \in [0,1]: \ xt + y(1-t) \in C$
- convex combination of points $x_1, \dots, x_n \in \mathbb{R}^d$ is $\sum_{i=1}^{n} x_i \alpha_i$, $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$
- convex hull of $X \subseteq \mathbb{R}^d$ is $\mathrm{conv}(X) :=$ set of all convex comb. of points in X
  ↳ each $x \in \mathrm{conv}(X)$ can be written as a conv. comb of $\leq d+1$ points in X
- affine combination of points $x_1, \dots, x_n \in \mathbb{R}^d$ is $\sum_{i=1}^{n} x_i \alpha_i$, $\alpha_i \in \mathbb{R}$, $\sum \alpha_i = 1$
- $x_1, \dots, x_n$ are affinely dependent if one is an affine combination of the others; otherwise they are affinely independent
  → $n=2$: $x_1 \neq x_2$, $n=3$: x_1, x_2, x_3 not on a common line, $n=4$: plane

- convex polytope = convex hull of a finite point set $A \in \mathbb{R}^d$
  ↳ alternatively, a bounded intersection of finitely many half-spaces
  → face of P is either P itself, or $P \cap h$ where h is a hyperplane s.t. at most one of the open half-spaces of h intersect P

GEOMETRIC SIMPLICIAL COMPLEXES

Def: A k-simplex σ is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$. Its dimension is $\dim \sigma := k$

0-simplex • , 1-simplex •—• , 2-simplex ▲ , 3-simplex 

Def: The convex hull of an arbitrary set of vertices of σ is a face of σ (a special case of a face of a convex polytope).

👁 every face of a simplex is itself a simplex

Ex: faces of Δ: ▲, /, \, —, $3\times$•, $\emptyset$ ... 8 faces

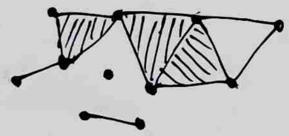
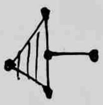
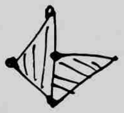
Def: 0-dim face = vertex, 1-dim face = edge, $(d-1)$-dim face = facet

Def: A (geometric) simplicial complex is a family Δ of simplicies s.t.

① $\forall$ face of a simplex $\sigma \in \Delta$ also belongs to Δ

② $\forall \sigma_1, \sigma_2 \in \Delta$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2

→ more complex shape created by gluing together simplicies

 good , but  or  badd !

Def: The union of all simplices $\sigma \in \Delta$ is $\|\Delta\| \subseteq \mathbb{R}^d$ ... polyhedron of Δ

- $\dim(\Delta) := \max\{\dim \sigma \mid \sigma \in \Delta\}$
- $V(\Delta) :=$ union of the vertex sets of all $\sigma \in \Delta$ ... vertices of Δ

👁 The set of all faces of a simplex is a simplicial complex

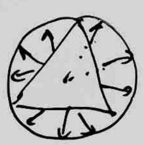
Def: A subcomplex of Δ is any $\Delta' \subseteq \Delta$ that is itself a GSC.

Def: σ^k is the simplicial complex consisting of faces of a k-simplex σ

The boundary of σ^k is the subcomplex $\sigma^k \setminus \{\sigma\}$ ... $\blacktriangle \rightsquigarrow \triangle$

Def: A triangulation of a space X is a simplicial complex Δ s.t. $X \cong \|\Delta\|$.

- triangulation of S^{d-1} ... boundary of σ^d
  ↳ using the central projection

$\bigcirc \cong \triangle$ by 

Exercise: The diameter of a simplex σ is attained by a pair of its vertices.

ABSTRACT SIMPLICIAL COMPLEXES

Def: An abstract simplicial complex is a pair (V, K) where $K \subseteq P(V)$

- $\forall A \in K : \quad B \subseteq A \Rightarrow B \in K$ ... hereditary set system
- sets in K are called (abstract) simplicies
- $\dim(K) := \max \{|F|-1 \mid F \in K\}$

Note: Usually $V = \bigcup K$, so we will often write just K

👁 If Δ is a geom. simplicial complex then $(V(\Delta), \{V(\sigma) \mid \sigma \in \Delta\})$ is abstract s. compl.

Def: Δ is a geometric realization of K if the abstract s.c. of Δ is K

👁 Every K has a geom. realization in $\mathbb{R}^{|V|-1}$ (in fact in $\mathbb{R}^{2\dim(K)-1}$)

↳ we define a subcomplex of $\sigma^{|V|-1}$ by $\Delta := \{\operatorname{conv}(F) \mid F \in K\}$, where we identify the vertices of $\sigma^{|V|-1}$ with the set V

Def: If Δ realizaises K, then $\|\Delta\|$ is called a polyhedron of K.

Fact: The polyhedron of K is unique up to homeomorphism ... denote $\|K\|$

Def: $f : V(K) \to V(L)$ is a simplicial mapping of K into L if

$F \in K \Rightarrow f[F] \in L$ ... maps simplicies to (possibly smaller) simplicies

Def: Isomorphism of K, L = bijection $f : V(K) \to V(L)$ s.t. f, f^{-1} are simplicial.

↳ if isomorphism exists, we write $K \cong L$

Geometric realization of a simplicial mapping

Def: Let Δ_K and Δ_L be realizations of K and L and $f : V_K \to V_L$ simplicial.

We define the mapping

$$\|f\| : \|\Delta_K\| \to \|\Delta_L\|$$

by affinely extending f to the relative interiors of simplicies.

If $\sigma \in \Delta_K$ and has vertices $v_1, \dots, v_k$ and

$x = \sum v_i \alpha_i$, $\alpha_i \geq 0$, $\sum \alpha_i = 1$, put $\|f\|(x) := \sum \alpha_i f(v_i)$

Fact:

f simplicial $\Rightarrow$ $\|f\|$ continuous

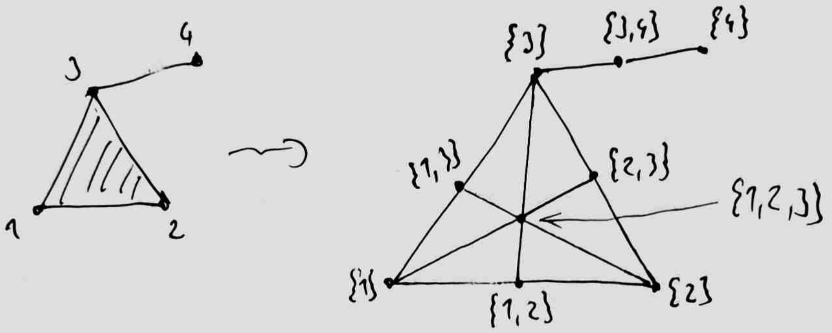
f injective $\Rightarrow$ $\|f\|$ injective

f isomorphism $\Rightarrow$ $\|f\|$ homeomorphism

BARYCENTRIC SUBDIVISION

Def: The barycentric subdivision of an abstract simplicial complex (V,K) is

$$sd(K) := (K \setminus \{\emptyset\}, \{A \subseteq K \mid A \text{ is } \subseteq\text{-chain in } K \setminus \{\emptyset\}\})$$



chains for example:

$1 \subseteq 12 \subseteq 123$

$3 \subseteq 23$

$4 \subseteq 34$

2

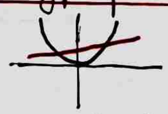
Exercise: $\|K\| \cong \|sd(K)\|$

Idea: By doing iterated barycentric subdivisions, one can construct arbitrarily fine triangulations of a given polyhedron.

THE MOMENT CURVE

Def: The moment curve in $\mathbb{R}^d$ is $\{\gamma(t) \mid t \in \mathbb{R}\}$ where $\gamma(t) := (t, t^2, \dots, t^d)$

Lemma: No hyperplane intersects γ in $\mathbb{R}^d$ in more than d points

Example: $\mathbb{R}^2$:  γ = parabole

Corollary: Every set of $d+1$ points on γ is affinely independent.
Moreover, if γ crosses a hyperplane h at d distinct points, then at each intersection, it crosses from one half-space to the other.

Proof: Let h be a hyperplane given by $a_1x_1 + a_2x_2 + \dots + a_dx_d = b$, $\vec{a} \neq 0$

- $\gamma(t) \in h \Rightarrow a_1t + a_2t^2 + \dots + a_dt^d = b$

$\Rightarrow$ intersections with h = roots of $p(t) := -b + \sum a_i t^i$

- but degree$(p) \leq d$, so there are at most d real roots

$\rightarrow$ if there are d distinct roots (intersections), then p changes sign at each root, so γ passes from one open half-space of h to the other ▨

Theorem: Every finite d-dimensional simplicial complex K has a geometric realization in $\mathbb{R}^{2d+1}$

Proof: Choose $f: V(K) \to \mathbb{R}^{2d+1}$ mapping $V(K)$ to distinct vertices of γ in $\mathbb{R}^{2d+1}$

- if $A, B \in K$, then $|A \cup B| \leq (d+1) + (d+1) = 2d+2$, so the corresponding points in $f[A \cup B]$ are affinely independent claim: $g: K \to \mathbb{R}^{2d+1}$, $A \mapsto \sigma_A := \mathrm{conv}(f[A])$ realizes K as Δ

proof: For $A, B \in K$ consider the simplex $\sigma_{A \cup B}$ with vertex set $f[A \cup B]$
Since it is affinely independent, σ_A and σ_B are faces of $\sigma_{A \cup B}$
Hence $\sigma_A \cap \sigma_B = \sigma_{A \cap B}$ since faces of a simplex form a simplicial complex ▨

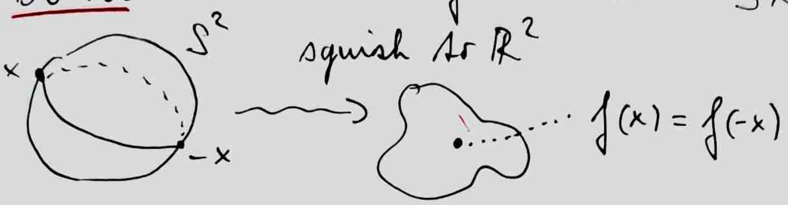
THE BORSOK-ULAM THEOREM

$\partial B^n = S^{n-1} \subseteq \mathbb{R}^n$

→ we give 7 equivalent formulations

Def: $f: X \to \mathbb{R}^d$ is antipodal $\equiv$ f is continuous & $\forall x \in X: f(-x) = -f(x)$

① BU1a: $\forall$ continuous $f: S^n \to \mathbb{R}^n$ $\exists x \in S^n$ s.t. $f(x) = f(-x)$



② BU1b: $\forall$ antipodal $f: S^n \to \mathbb{R}^n$ $\exists x \in S^n$ s.t. $f(x) = 0$

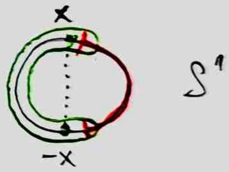
③ BU2a: $\nexists$ antipodal $f: S^n \to S^{n-1}$

④ BU2b: $\nexists$ continuous $f: B^n \to S^{n-1}$ that is antipodal on $\partial B^n = S^{n-1}$

⑤ LS-c: $\forall$ closed cover $F_1, \dots, F_{n+1}$ of S^n $\exists i \in [n+1]$ s.t. $x, -x \in F_i$

⑥ LS-o: $\forall$ open cover —॥—

⑦ LS-co: $\forall$ cover by closed or open sets —॥—



History: Conjectured by Ulam, proved by Borsuk in 1933
The earliest reference is by Lyusternik and Shnirelman in 1930

Example: At any point in time, there are always two opposite points on the Earth that have the same temperature and air pressure
↳ use BU1a with $S^2 \ni x \mapsto (\text{temp}, \text{pressure}) \in \mathbb{R}^2$

Theorem (Brouwer fixed point theorem): $\forall$ continuous $f: B^n \to B^n$ $\exists x$ s.t. $f(x) = x$.

Proof from BU2b: Suppose that $f: B^n \to B^n$ is continuous & $\forall x: f(x) \neq x$.
→ define $g: B^n \to S^{n-1}$ by $x \mapsto$ intersection of ∂B^n with the ray emitted from $f(x)$ in the direction of x
→ intuitively, g is continuous (small wiggles of x mean small wiggles of $f(x)$ ∵ f is continuous, so small wiggles of $g(x)$)
→ g is antipodal on $\partial B^n = S^{n-1}$ since $g \upharpoonright S^{n-1} = id_{S^{n-1}}$ ↯

▨

PROOVING THE EQUIVALENCES

$$\begin{tikzcd}
BU2b \arrow[r, leftrightarrow] & BU2a \arrow[r, leftrightarrow] & BU1b \arrow[r, leftrightarrow] & BU1a \arrow[d] & LS\text{-}o \arrow[d] \\
 & & & LS\text{-}c \arrow[ull, bend left] \arrow[ur] & LS\text{-}co \arrow[l]
\end{tikzcd}$$

BU1a ⟺ BU1b ⟺ BU2a

1) **BU1a ⇒ BU1b**: Let $f: S^n \to \mathbb{R}^n$ be antipodal

since it is continuous, by BU1a $\exists x$ s.t. $f(x) = f(-x) = -f(x) \Rightarrow f(x) = 0$

2) **BU1b ⇒ BU1a**: Let $f: S^n \to \mathbb{R}^n$ be continuous and define

$g: S^n \to \mathbb{R}^n$ by $g(x) := f(x) - f(-x)$

→ g is antipodal: $g(-x) = f(-x) - f(x) = -g(x)$

$\Rightarrow \exists x$ s.t. $g(x) = 0$, so $f(x) = f(-x)$

3) **BU1b ⇒ BU2a**: For contradiction $\exists$ antipodal $f: S^n \to S^{n-1} \subseteq \mathbb{R}^n$

→ by BU1b $\exists x \in S^n$ s.t. $f(x) = 0$ ↯ $0 \notin S^{n-1}$

4) **BU2a ⇒ BU1b**: For contradiction $\exists$ antipodal $f: S^n \to \mathbb{R}^n$ s.t. $\forall x: f(x) \neq 0$

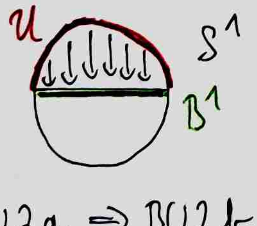
→ nothing maps to the origin, so we can rescale f:

$g: S^n \to S^{n-1}$ by $g(x) := \frac{f(x)}{\|f(x)\|}$ contradicts BU2a ↯

👁 g is antipodal since f is antipodal $g(-x) = \frac{f(-x)}{\|f(-x)\|} = \frac{-f(x)}{\|-f(x)\|} = -g(x)$

BU2a ⟺ BU2b

👁$_1$ The projection $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$ is a homeomorphism of the upper hemisphere $U := \{x \in S^n \mid x_{n+1} \geq 0\}$ of S^n with $\mathbb{B}^n$



👁$_2$ π is antipodal on the equator of S^n

$$\pi(-x_1, \dots, -x_n, 0) = -(x_1, \dots, x_n) = -\pi(x_1, \dots, x_n, 0)$$

5) **BU2a ⇒ BU2b**: For contradiction $\exists$ continuous $f: B^n \to S^{n-1}$, antipodal on $\partial B^n = S^{n-1}$

→ define $g: S^n \to S^{n-1}$ by $g(x) := \begin{cases} f(\pi(x)), & x \in U \\ -f(\pi(-x)), & x \notin U \end{cases}$ ... $g(-x) = -g(x)$

→ g is well-defined from 👁 2

→ g is continuous since it is continuous on both the closed lower and upper hemispheres of S^n, so g is antipodal $S^n \to S^{n-1}$ ↯

↳ claim: If $X = A_1 \cup \dots \cup A_n$ where $\forall A_i$ is closed, and $f: X \to Y$ is continuous on each A_i, then f is continuous on X.

proof: $f_i := f \restriction A_i$ continuous ... $B \subseteq Y \cap f[A_i]$ closed $\Rightarrow f_i^{-1}[B]$ closed

let $B \subseteq Y$ be closed, want $f^{-1}[B]$ closed

→ $B \cap f[X] = B_1 \cup \dots \cup B_n$ where $B_i = B \cap f[A_i]$ ↗ finite union of closed = closed

$\Rightarrow f_i^{-1}[B_i] = f^{-1}[B_i]$ closed $\Rightarrow f^{-1}[B] = f^{-1}[B_1] \cup \dots \cup f^{-1}[B_n]$ closed ∎

6) BU2b ⇒ BU2a: For contradiction $\exists$ antipodal $f: S^n \to S^{n-1}$

→ define $g: B^n \to S^{n-1}$ by $g(x) := f(\pi^{-1}(x))$ ... we use 👁 1

- g is antipodal on $\partial B^n = S^{n-1}$ by 👁 2 because composition of antipodal maps is antipodal

BU1a ⇒ LS-c ⇒ BU2a

7) LS-c ⇒ BU2a: For contradiction $\exists$ antipodal $f: S^n \to S^{n-1}$

claim: There exists a closed cover $F_1, \dots, F_{n+1}$ of S^{n-1} s.t. $\forall i: \underbrace{F_i \cap (-F_i) = \emptyset}_{\nexists x:\, x, -x \in F_i}$

↳ then $f^{-1}[F_1], \dots, f^{-1}[F_{n+1}]$ would contradict LS-c ... if $x, -x \in f^{-1}[F_i]$ $f(x), f(-x) = -f(x) \in F_i$

proof: Take the regular n-simplex, it has $\binom{n+1}{n} = n+1$ facets

$F_i :=$ projection of the i-th facet onto S^{n-1}

- $\forall i: F_i \cap (-F_i) = \emptyset$ since there is always a hyperplane separating F_i from the origin, so also from $-F_i$

8) BU1a ⇒ LS-c: Let $F_1, \dots, F_{n+1}$ be a closed cover of S^n; want $x, -x \in F_i$

→ define $f: S^n \to \mathbb{R}^n$ by $f(x) := (\operatorname{dist}(x, F_1), \dots, \operatorname{dist}(x, F_n))$

→ distances with the Euclidean metric are continuous, so f is continuous

- BU1a $\Rightarrow \exists x \in S^n$ s.t. $f(x) = f(-x) =: y = (y_1, \dots, y_n)$

a) if $\exists i \in [n]$ s.t. $y_i = 0$, then $\operatorname{dist}(x, F_i) = \operatorname{dist}(-x, F_i) = 0$

↳ since F_i is closed, $x, -x \in F_i$ ✓

b) otherwise $\forall i \in [n]: y_i \neq 0$, so $x \notin F_1 \cup \dots \cup F_n$, $-x \notin F_1 \cup \dots \cup F_n$ ⟶ $\exists\, i \in [n]: \operatorname{dist}(x, F_i) > 0$

$\Rightarrow x, -x \in F_{n+1}$ ✓

LS-c ⇒ LS-o ⇒ LS-co ... clearly LS-co ⇒ LS-c and LS-o

9) LS-c ⇒ LS-o: Let $U_1, \dots, U_{n+1}$ be an open cover of S^n

- $\forall x \in S^n$ let V_x be an open neighborhood of x s.t. $\operatorname{cl}(V_x) \subseteq U_i$ $\forall i: x \in U_i$

$\Rightarrow (V_x)_{x \in S^n}$ is an open cover $\xRightarrow{S^n \text{ compact}} \exists$ finite subcover $(V_x)_{x \in A}$

→ put $F_i := \bigcup \{\operatorname{cl}(V_x) \mid x \in A \cap U_i\}$ ... closed cover of S^n s.t. $\forall i: F_i \subseteq U_i$

- LS-c: $\exists i: x, -x \in F_i \subseteq U_i$, we get LS-o for U_i

10, $LS\text{-}o \Rightarrow LS\text{-}co$: For contradiction $\exists$ cover $F_1, \dots, F_{n+1}$ of S^n by closed or open sets s.t. $\forall i: F_i \cap (-F_i) = \emptyset$

→ WLOG $F_1, \dots, F_k$ closed, the rest open

$$\varepsilon := \min_{i=1,\dots,k} \operatorname{dist}(F_i, -F_i) > 0$$

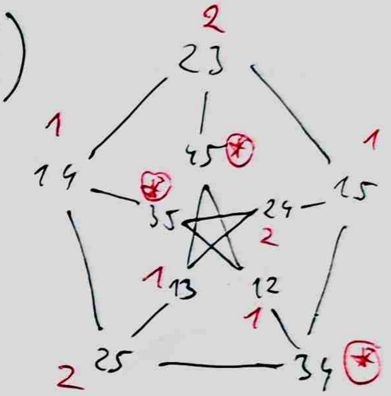
⇒ put $D_i := \{x \in S^n \mid \operatorname{dist}(x, F_i) < \frac{\varepsilon}{4}\}$ … $\frac{\varepsilon}{4}$-neighborhood of F_i

→ then $D_1, \dots, D_k, F_{k+1}, \dots, F_{n+1}$ is a bad open cover ↯ ∎

LOVÁSZ–KNESER THEOREM

Def (Kneser graph): Let $k > 0$, $n \geq 2k-1$.

$$KG_{n,k} := \left(V = [n]^k, \; E = \{AB \in [V]^2 \mid A \cap B = \emptyset\}\right)$$



Lemma: $\chi(KG_{n,k}) \leq n - 2k + 2$

Proof: First assign to each $A \in [n]^k$ the color $\min(A)$

→ this would use $n - k + 1$ colors

→ but some of the final colors are redundant:

if $|M| = 2k-1$ and $A, B \subseteq M$ have size k, then $AB \notin E$ … $A \cap B \neq \emptyset$

⇒ if $\min(A) \leq n - 2k + 1$, color $\chi(A) := \min(A)$

→ and give a special color ⊛ to all other vertices

} $n - 2k + 2$ colors ∎

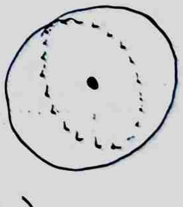
Theorem: $\chi(KG_{n,k}) = n - 2k + 2$

Remark: Kneser conjectured it, Lovász proved it

Proof: Let $d = n - 2k + 1$, for contradiction let χ be a proper d-coloring of $KG_{n,k}$

→ take $X \subseteq S^d$ … n points in general position

⇒ $\forall$ great circle cutting S^d into two hemispheres contains at most d points of X



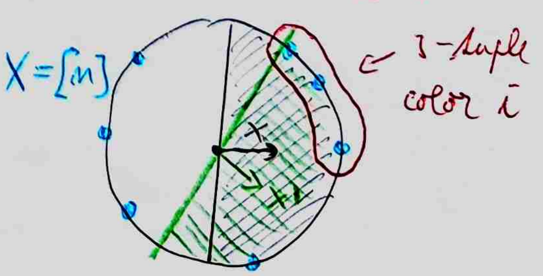
$S^d \subseteq \mathbb{R}^{d+1}$

→ identify X with $[n]$ (formally we take a bijection)

so we image $KG_{n,k}$ being drawn on S^d

→ construct sets $A_1, \dots, A_d \subseteq S^d$ as follows:

- put $x \in S^d$ into A_i if the open half-space $H(x) := \{y \in \mathbb{R}^{d+1} \mid \langle x, y \rangle > 0\}$ contains a k-tuple (vertex of $KG_{n,k}$) that has color i w.r.t. χ

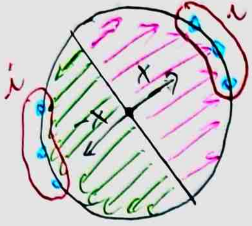


each A_i is open ∵ we consider open half-spaces … if we wiggle x, the k-tuple is still there

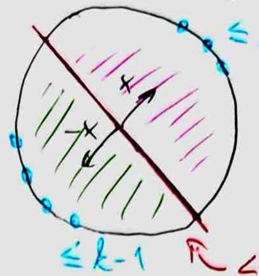
- define $A_{d+1} := S^d \setminus (A_1 \cup \dots \cup A_d)$ … closed set

$\Rightarrow$ by LS-co there $\exists i \in [d+1]$ and $x \in S^d$ s.t. $x, -x \in A_i$

a) $\underline{i \in [d]}$: the two k-tuples corresponding to x and $-x$ are disjoined, so they are joined by an edge, but they share color i ↯



b) $\underline{i = d+1}$: $x \in A_{d+1} \Leftrightarrow |X \cap H(x)| < k$



$\le k-1$ → on the bounding hyperplane there are at most d points (by general position)

$|X| \le d + 2(k-1) \le n-1$ ↯ ▨

DOL'NIKOV'S THEOREM

Def: Hypergraph is $(X, \mathcal{F})$ where X is finite and $\mathcal{F} \subseteq P(X)$

$\chi(\mathcal{F}) :=$ min k s.t. $\exists$ coloring $f: X \to k$ s.t. no $e \in \mathcal{F}$ is monochromatic

Def: The colorability deficit captures how many vertices do we need to remove for the hypergraph to become k-colorable

$\underline{cd_k(\mathcal{F})} := \min \{ |Y| \mid Y \subseteq X \ \& \ (Y, \mathcal{F} \cap P(Y)) \text{ is } k\text{-colorable} \}$

Def: The Kneser graph for a set system $\mathcal{F} \subseteq P(X)$

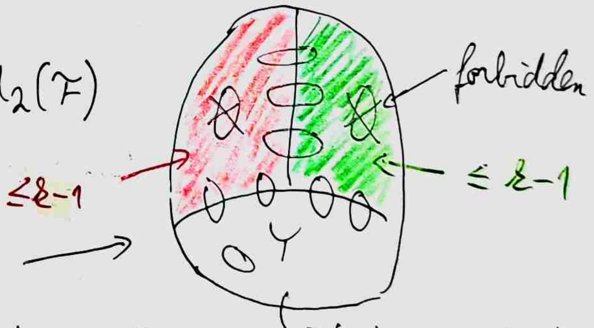
$\underline{KG(\mathcal{F})} := \left(V = \mathcal{F}, \ E = \{ AB \in [V]^2 \mid A \cap B = \emptyset \} \right)$

👁 $KG_{n,k} = KG([n]^k)$

Theorem (Dolnikov): $\chi(KG(\mathcal{F})) \ge cd_2(\mathcal{F})$

👁 Dolnikov $\Rightarrow$ Lovász-Kneser

↳ because $cd([n]^k) \ge n - 2k + 2$ →



↳ $|Y| \ge n - 2(k-1) = n - 2k + 2$

Proof: Let $d = \chi(KG(\mathcal{F}))$ and fix a proper coloring

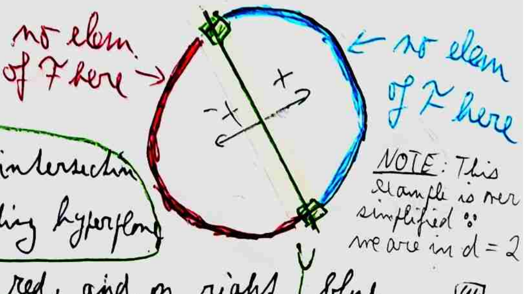
→ define sets $A_1, \dots, A_d, A_{d+1} \subseteq S^d$ as in the previous proof

- $\underline{A_i, i \le d}$: put $x \in S^d$ into A_i if $H(x)$ contains an element of $\mathcal{F}$ of color i where we again chose $X \subseteq S^d$ in general position → ↳ $\mathcal{F} \subseteq P(x)$
- $\underline{A_{d+1}} := S^d \setminus (A_1 \cup \dots \cup A_d)$

$\Rightarrow$ by LS-co $\exists i \, \exists x \in S^d$ s.t. $x, -x \in A_i$

a) $\underline{i \in [d]}$: we get the same contradiction as before ↯

b) $\underline{i = d+1}$: Take $Y :=$ intersection of S^d with the bounding hyperplane

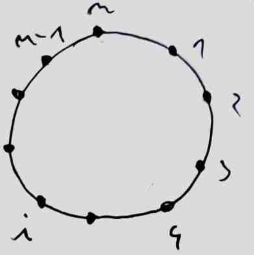


NOTE: This example is over simplified ∵ we are in $d = 2$

→ color points on left red, and on right blue

$\Rightarrow cd_2(\mathcal{F}) \le |Y| \le d$ ∵ general position ▨

SCHRIJVER'S THEOREM



no two consecutive points

Def: We call $S \in [n]^k$ **stable** if $\forall i \in S$: $(i+1) \bmod n \notin S$

Notation: $[n]^k_{stab}$:= collection of all stable $S \in [n]^k$

Def: **Schrijver's graph** $SG_{n,k} := KG([n]^k_{stab})$

Theorem (Schrijver): For $n \geq 2k-1$: $\chi(SG_{n,k}) = \chi(KG_{n,k}) = n-2k+2$

↳ even though we removed many edges, χ stays the same

Exercise: Schrijver graphs are **critical**: removing any vertex reduces χ

→ we will prove it using Gale's lemma:

Lemma (Gale's lemma): $\forall d \geq 0$ $\forall k \geq 1$ $\exists X \subseteq S^d$ containing $2k+d$ points s.t. every open hemisphere of S^d contains $\geq k$ points of X

Moreover, under a suitable identification of X with $[n]$,

⊛ every open hemisphere contains k points of X that form a stable set.

Proof of Schrijver's theorem: Consider $SG_{n,k}$ and let $d := n-2k$

- → let $X \subseteq S^d$ be a set from Gale's lemma, noting $|X| = n$
- → identify X with $[n]$ so that ⊛ holds
- → for contradiction, suppose that a proper $(d+1)$-coloring of $SG_{n,k}$ exists
- → define $A_1, \ldots, A_{d+1} \subseteq S^d$ similar to before
  - put $x \in S^d$ to A_i if the open half-plane $H(x)$ contains a ⊛ stable k-tuple of color i

$A_1, \ldots, A_{d+1}$ form an open cover of S^d

- → they are open as before, and
- → $\forall x \in S^n$ $\exists i$ s.t. $x \in A_i$ since by Gale's lemma, $H(x)$ contains ⊛ or a stable k-tuple $\geq k$ points of X

→ by LS-o there $\exists i$ $\exists x$ s.t. $x, -x \in A_i$

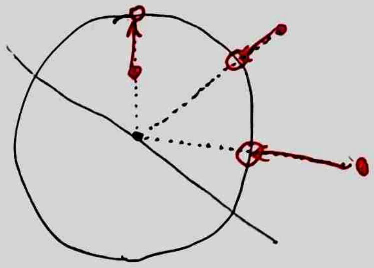
→ as before, this leads to a contradiction because the two k-tuples corresponding to x and $-x$ are disjoint, and thus form a monochr. edge ↯

→ if we do not use ⊛, we prove the Lovász–Kneser theorem

→ using ⊛ yields Schrijver's theorem since we can choose the k-tuples to be stable

Proof of Gale's lemma: Reformulation: $\exists v_1, v_2, \dots, v_{2k+d} \in \mathbb{R}^{d+1} \setminus \{0\}$ s.t.

$\forall$ open half-space defined by a hyperplane passing through the origin contains $\geq k$ points of X (where v_i have distinct projections to S^d)

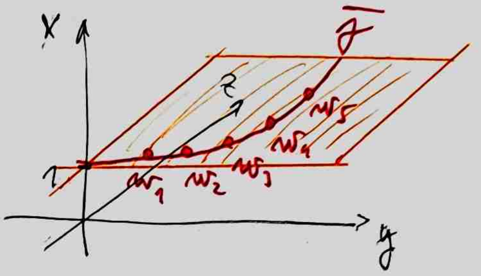


→ recall that $\gamma(t)$ denotes the moment curve in $\mathbb{R}^d$

↳ $\gamma = \{(t, t^2, \dots, t^d) \in \mathbb{R}^d \mid t \in \mathbb{R}\}$

↳ we lift it one dimension higher to $\mathbb{R}^{d+1}$, into the hyperplane $x_1 = 1$

$\bar{\gamma} := \{(1, t, t^2, \dots, t^d) \in \mathbb{R}^{d+1} \mid t \in \mathbb{R}\}$



→ take $2k+d$ distinct points on $\bar{\gamma}$

↳ for instance $w_i := \bar{\gamma}(i)$, $i \in [2k+d]$

- even points : $w_2, w_4, w_6, \dots$
- odd points : $w_1, w_3, w_5, \dots$ • define $\underline{v_i := (-1)^i w_i} = \begin{cases} w_i, & \text{even } w_i \\ -w_i, & \text{odd } w_i \end{cases}$

→ let h be a hyperplane passing through the origin

↳ want: the two open half-spaces h^+ and h^- both contain $\geq k$ points v_i

→ we will show it for h^+ ... h^- similar

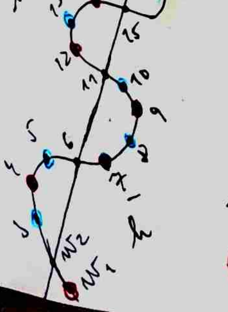
◎ <u>want: # even points $w_i \in h^+$ + # odd points $w_i \in h^-$ $\geq k$</u> ⊠

→ by the same proof as for the moment curve, $\bar{\gamma}$ and h intersect in at most d points. Moreover, if there are d intersections, then $\bar{\gamma}$ crosses from h^+ to h^- (or h^- to h^+) at each intersection

⇒ rotate h about the origin so that it contains exactly d points of $W := \{w_1, \dots, w_{d+2k}\}$, while no point of W crosses from one side of h to the other during the motion

→ this is possible: if h already contains $j < d$ points of W, then they + the origin determine a subspace of dimension $j < d$, and we can rotate h about it until we hit another point of W

⇒ after doing this, let W_{ON} be the d points of W on h, and $W_{OFF} := W - W_{ON}$



- at every $w \in W_{ON}$, $\bar{\gamma}$ crosses from one side to the other

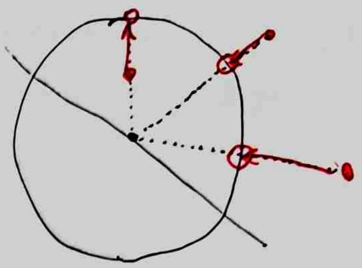
⊠ color $w \in W_{OFF}$ <u>red</u> if w is even $\in h^+$ or w is odd $\in h^-$ and blue otherwise

◎ as we follow $\bar{\gamma}$, the points of W_{OFF} alternate colors

⇒ # red points $\geq \lfloor \frac{1}{2} |W_{OFF}| \rfloor = \lfloor \frac{1}{2} 2k \rfloor = k$ ... we are done

⊕ Moreover, the red points form a stable set (when numbered along $\bar{\gamma}$) ▨

Proof of Gale's lemma: Reformulation: $\exists\, v_1, v_2, \dots, v_{2k+d} \in \mathbb{R}^{d+1} \setminus \{0\}$ s.t.



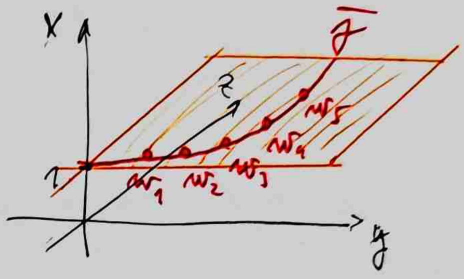
$\forall$ open half-space defined by a hyperplane passing through the origin contains $\geq k$ points of X (where v_i have distinct projections to S^d)

→ recall that $\gamma(t)$ denotes the moment curve in $\mathbb{R}^d$

↳ $\gamma = \{(t, t^2, \dots, t^d) \in \mathbb{R}^d \mid t \in \mathbb{R}\}$

↳ we lift it one dimension higher to $\mathbb{R}^{d+1}$, into the hyperplane $x_1 = 1$

$$\bar{\gamma} := \{(1, t, t^2, \dots, t^d) \in \mathbb{R}^{d+1} \mid t \in \mathbb{R}\}$$



→ take $2k+d$ distinct points on $\bar{\gamma}$

↳ for instance $w_i := \bar{\gamma}(i)$, $i \in [2k+d]$

- even points: $w_2, w_4, w_6, \dots$
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→ let h be a hyperplane passing through the origin

↳ want: the two open half-spaces h^+ and h^- both contain $\geq k$ points v_i

→ we will show it for h^+ ... h^- similar

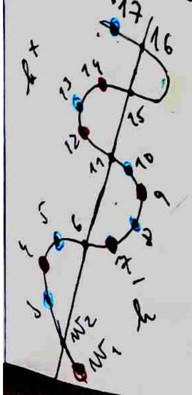
want: # even points $w_i \in h^+$ + # odd points $w_i \in h^-$ $\geq k$ ▨

→ by the same proof as for the moment curve, $\bar{\gamma}$ and h intersect in at most d points. Moreover, if there are d intersections, then $\bar{\gamma}$ crosses from h^+ to h^- (or h^- to h^+) at each intersection

⇒ rotate h about the origin so that it contains exactly d points of $W := \{w_1, \dots, w_{d+2k}\}$, while no point of W crosses from one side of h to the other during the motion

→ this is possible: if h already contains $j < d$ points of W, then they + the origin determine a subspace of dimension $j < d$, and we can rotate h about it until we hit another point of W

⇒ after doing this, let W_{ON} be the d points of W on h, and $W_{OFF} := W \setminus W_{ON}$



- at every $w \in W_{ON}$, $\bar{\gamma}$ crosses from one side to the other

▨ color $w \in W_{OFF}$ red if w is even $\in h^+$ or w is odd $\in h^-$ and blue otherwise

as we follow $\bar{\gamma}$, the points of W_{OFF} alternate colors

⇒ # red points $\geq \lfloor \frac{1}{2} |W_{OFF}| \rfloor = \lfloor \frac{1}{2} 2k \rfloor = k$ ... we are done

Moreover, the red points form a stable set (when numbered along $\bar{\gamma}$) ▨

HAM-SANDWICH THEOREM

Def: A σ-algebra on a set X is a $\Sigma \subseteq P(X)$ closed under complements, and countable unions and intersections

Def: Given a set X and a σ-algebra Σ on X, a function

$\mu : \Sigma \to \mathbb{R} \cup \{+\infty\}$ ↳ specifying measurable sets

is called a measure on X if

① $\mu(\emptyset) = 0$

② $\mu(E) \geq 0$ for $\forall E \in \Sigma$

③ countable additivity: $(E_k)_{k \in \omega}$ pairwise disjoint $\Rightarrow \mu\left(\bigcup E_k\right) = \sum \mu(E_k)$

Def: A finite Borel measure μ on $\mathbb{R}^d$ is a measure on $\mathbb{R}^d$ s.t. all open subsets of $\mathbb{R}^d$ are measurable and $0 < \mu(\mathbb{R}^d) < \infty$

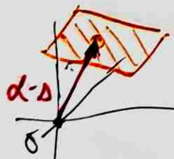
Remark: It is enough for open "intervals" to be measurable since every open set is a countable union of open "intervals" (in higher dimensions, boxes).

Example: Given a compact set $A \subseteq \mathbb{R}^d$ with $\lambda(A) > 0$, where λ is the Lebesgue measure, we may define a finite Borel measure

$\mu(X) := \lambda(X \cap A)$ for $\forall$ Lebesgue measurable $X \subseteq \mathbb{R}^d$.

Proposition: If A_1, A_2, A_3 are compact sets in $\mathbb{R}^3$, representing ham, cheese, and bread, then there exists a plane P in $\mathbb{R}^3$ that simultaneously bisects (divides into two parts of equal Lebesgue-measure) all three sets.

Proof: Assign to each $x \in S^2$ three numbers $(\alpha_1, \alpha_2, \alpha_3)$ such that the hyperplane h_i with normal vector s, shifted by $\alpha_i x$ from the origin, bisects A_i ... such α_i exists because A_i is compact



$\Rightarrow$ define $f : S^2 \to \mathbb{R}^2$ by $x \mapsto (\alpha_1 - \alpha_2, \alpha_2 - \alpha_3)$

f is continuous, so by BU1a: $\exists x \in S^2$ s.t. $f(x) = f(-x)$

→ since $-x$ is assigned $(-\alpha_1, -\alpha_2, -\alpha_3)$, we have

$f(x) = (\alpha_1 - \alpha_2, \alpha_2 - \alpha_3) = f(-x) = (-\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3)$

$\Rightarrow 2\alpha_1 = 2\alpha_2$ & $2\alpha_2 = 2\alpha_3 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3$

→ hence the three bisecting planes for x coinside ▨

Def: A finite Borel measure μ on $\mathbb{R}^d$ is a mass distribution if every hyperplane has measure zero.

Theorem (Ham sandwich for mass distributions): If $\mu_1, \dots, \mu_d$ are mass distributions in $\mathbb{R}^d$, then there $\exists$ hyperplane h s.t.

$$\mu_i(h^+) = \frac{1}{2} \mu_i(\mathbb{R}^d) \quad \text{for } \forall i, \quad \leftarrow \text{each } \mu_i \text{ is bisected by } h$$

where h^+ denotes one of the half-spaces defined by h

Note: Given a compact set $A \subseteq \mathbb{R}^d$, and μ defined by $\mu(X) := \lambda(X \cap A)$, then $\mu(h^+) = \frac{1}{2}\mu(\mathbb{R}^d)$ if $\lambda(h^+ \cap A) = \frac{1}{2}\lambda(A)$ ... h bisects A

Proof: For every $u = (u_0, u_1, \dots, u_d) \in S^d \subseteq \mathbb{R}^{d+1}$ define

$$h^+(u) := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid u_1 x_1 + \dots + u_d x_d \leq u_0\}$$

- $h^+(u)$ is a half-space $\iff (u_1, \dots, u_d) \neq (0, \dots, 0)$
- if $u = (u_0, 0, \dots, 0)$, then necessarily $u_0 = \pm 1$, where $h^+(1, 0, \dots, 0) = \mathbb{R}^d$ and $h^+(-1, 0, \dots, 0) = \emptyset$

→ define $f: S^d \to \mathbb{R}^d$ by $f_i(u) := \mu_i(h^+(u))$ ... and $f(u) = (f_1(u), \dots, f_d(u))$

claim: f is continuous (intuitively for sure)

$\Rightarrow$ BU1a: $\exists u^* \in S^d$ s.t. $f(u^*) = f(-u^*)$ ... 👁 $u^* \neq (u_0, 0, \dots, 0)$

→ since $h(u)$ and $h(-u)$ are opposite half-spaces, $h^+(u^*)$ is the desired half-space ... $f_i(u^*) = \mu_i(h^+(u^*)) = f(-u^*) = \mu_i(h^+(-u^*)) = \mu_i(h^-(u^*))$

proof of claim: to show continuity, we use Heine's definition of continuity

→ let $(u_n)_{n \in \omega}$ be a sequence of points of S^d such that $\lim_{n \to \infty} u_n = u$

$\Rightarrow$ want: $\lim_{n \to \infty} \mu_i(h^+(u_n)) = \mu_i(h^+(u))$

→ for $v \in S^n$ define char. function of $h^+(v)$ by $g_v : \mathbb{R}^d \to \{0,1\}$, $x \mapsto \begin{cases} 1, & x \in h^+(v) \\ 0, & x \notin h^+(v) \end{cases}$

→ put $g_n := g_{u_n}$ and $g := g_u$

👁 if $x \notin \partial h^+(u)$ ... not on the boundary... then for all sufficiently large n we have $x \in h^+(u_n) \iff x \in h^+(u)$

$\Rightarrow$ so if $x \notin \partial h^+(u)$, then $g_n(x) \to g(x)$

→ recall that $\partial h^+(u)$ has μ_i-measure zero, so

$$\mu_i(h^+(u_n)) = \int g_n \, d\mu_i \to \int g \, d\mu_i = \mu_i(h^+(u))$$

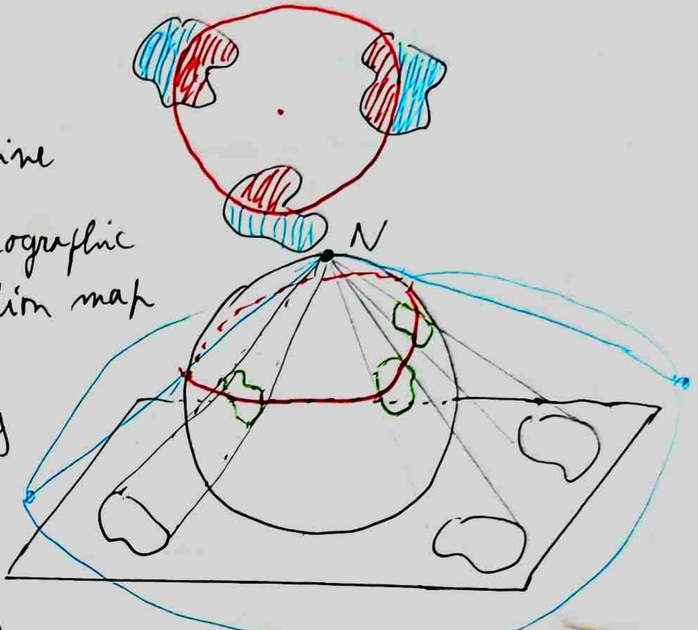
formally, we invoke the Lebesgue dominated convergence theorem

since the g_n converge to g μ_i-almost everywhere ▨

Proposition: Let μ_1, μ_2, μ_3 be mass distributions in $\mathbb{R}^2$ that, moreover, assign measure zero to each circle. Then μ_1, μ_2, μ_3 can be silmutaneously halved by a circle or by a straight line

Intuition: If $A_1, A_2, A_3 \subseteq \mathbb{R}^2$ are compact sets, then the asociated measures $\mu_i(X) := \lambda(A_i \cap X)$ satisfy the circle condition, so if A_1, A_2, A_3 cannot be silmutaneously bisected by a line, they can be halved by a circle

→ Ham-sandwich bisects A_1, A_2 by a line



Proof: Let $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ be the stereographic projection map

$\uparrow$ North pole

Define mass distributions ν_i on $\mathbb{R}^3$ by

$$\nu_i(X) := \mu_i(\pi(X \cap S^2 \setminus \{N\}))$$

By the ham-sandwich, there exists hyperplane bisecting ν_1, ν_2, ν_3 ... h

👁 $h \cap S^2$ is a circle C

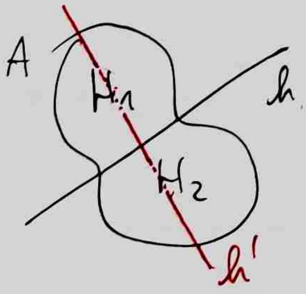
→ if $N \in C$, then $\pi(C \setminus \{N\})$ = line
→ if $N \notin C$, then $\pi(C)$ = circle
} either way, halving μ_1, μ_2, μ_3 ▨

EQUIPARTITION THEOREMS

Proposition: Any mass distribution μ in $\mathbb{R}^2$ can be dissected into 4 equal parts by two lines

Proof: First bisect μ (if ham-sandwich can bisect two μs, it can do one)

$\Rightarrow \exists$ line h defining half-spaces H_1 and H_2 s.t. $\mu(H_1) = \mu(H_2)$



→ define mass distributions

$$\mu_1(X) := \mu(H_1 \cap X)$$
$$\mu_2(X) := \mu(H_2 \cap X)$$

→ using Ham-sandwich again, we find h' bisecting μ_1, μ_2 ▨

Theorem: Any mass distribution μ in $\mathbb{R}^3$ can be dissected into 8 equal parts by three hyperplanes

Proof: Similar idea, but more complicated ; skipped ▨

Observation: There is a mass-distribution in $\mathbb{R}^5$ that cannot be dissected into 32 equal parts using 5 hyperplanes

Proof: 5 hyperplanes determine 32 orthants

→ recall: every hyperplane intersects the moment curve γ in $\mathbb{R}^5$ in at most 5 points ... 5 hyperplane ⇒ 25 points

⇒ take our mass distribution as a piece of the moment curve

↳ any 5 hyperplanes divide it into at most $26 < 32$ segments ▨

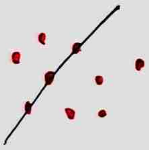
more precisely use a 1-dim measure along the moment curve

Open problem: $\mathbb{R}^4$ and 16 equal parts

HAM-SANDWICH FOR POINT SETS

Theorem (HS for point sets): Let $A_1, \dots, A_d \subseteq \mathbb{R}^d$ be finite sets of points. Then there $\exists$ hyperplane h that silmultaneously "bisects" $A_1, \dots, A_d$

Def: h bisects A if both of the open half-spaces h^+ and h^- defined by h contain at most $\lfloor \frac{1}{2}|A| \rfloor$ points of A. $8 \to 4$, $7 \to 3$



→ if $|A| = 2k+1$, then $|h \cap A| \geq 1$

↳ at least 1 point lies on h

Proof: Idea: replace points of A_i by tiny balls and apply ham-sandwich for measures

- we first prove a limited version: assume that

① $\forall i$: $|A_i|$ is odd ② $A_1, \dots, A_d$ are disjoint ③ $A_1 \cup \dots \cup A_d$ is in general position

↳ no $d+1$ points lie on a common hyperplane

→ $A_i^{\varepsilon} := \{ B(x, \varepsilon) \mid x \in A_i \}$ ... replace points by balls

→ choose ε so small s.t. no $d+1$ balls of $A_1^{\varepsilon} \cup \dots \cup A_d^{\varepsilon}$ can be intersected by a common hyperplane ... this is possible due to ③

→ let h be a hyperplane bisecting the sets A_i^{ε}

↳ we use ham-sandwich for compact sets – finite union of compact sets is compact

since all A_i are odd, h has to intersect at least one ball from each A_i^{ε}, and since it can intersect at most d balls, exactly 1 from each A_i^{ε} is intersected

→ moreover, h splits this ball in half, so it passes through its center

⇒ h bisects each A_i

→ next, we drop condition ② and ③, and then finally ①

- <u>dropping condition on position</u> - A_i still have to be odd

→ for $\forall \eta > 0$, let $A_{i,\eta}$ arise from A_i by moving each point by at most η s.t. $A_{1,\eta} \dot\cup A_{2,\eta} \dot\cup \dots \dot\cup A_{d,\eta}$ is in general position

⇒ let h_η bisect the $A_{i,\eta}$ ... sequence of hyperplanes

↳ $h_\eta = \{ x \in \mathbb{R}^d \mid \langle a_\eta | x \rangle = b_\eta \}$

→ since there are only finitely many points, the sets $A_{i,\eta}$ are bounded, so the points $(\vec{a}_\eta, b_\eta) \in \mathbb{R}^{d+1}$ are also bounded (enclosed in a ball)

⇒ by compactness, there ∃ convergent subsequence ... $\eta_1 > \eta_2 > \eta_3 > \dots$ s.t. $(\vec{a}_{\eta_i}, b_{\eta_i})_{i \in \omega} \to (\vec{a}, b)$ for some $\vec{a} \in \mathbb{R}^d$, $b \in \mathbb{R}$

→ let h be the hyperplane determined by $\langle a | x \rangle = b$

<u>claim</u>: h bisects all the A_i

if $x \notin h$, say $\mathrm{dist}(x, h) = \delta$, then for all sufficiently large j we have $\mathrm{dist}(x, h_j) \geq \frac{1}{2}\delta$

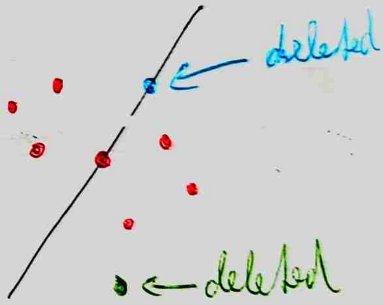
⇒ if h^+ contains k points of A, then h_j^+ contain $\geq k$ points of A

→ so h is bisecting since h_j is bisecting

- <u>dropping condition on odd size</u>

→ if some A_i are even, we delete 1 point from them so that all A_i are odd

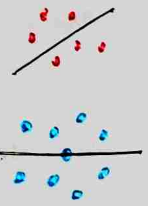
⇒ bisect the resulting odd sets and add back in the deleted points ... this is still a bisection (see the definition) ▨



<u>Theorem</u> (<u>Ham-sandwich for points in general position</u>): Let $A_1, \dots, A_d \subseteq \mathbb{R}^d$ be disjoint finite sets of points in general position (no more than d points of $A_1 \dot\cup \dots \dot\cup A_d$ are contained in any hyperplane).

→ Then there ∃ hyperplane h that bisects each A_i such that

⊛
- $|A_i|$ even ⇒ each half-plane contains exactly $\frac{1}{2}|A_i|$ points
- $|A_i|$ odd ⇒ h contains exactly 1 point of A_i



<u>Proof</u>: Let h be a hyperplane given by the usual ham sandwich

→ it could contain many points (up to d) from each A_i, but at most d in total

↳ ∵ general position

⇒ we will modify it

→ h is uniquely determined by any d affinely independent points on h

⊛ if $|A_i|$ is odd, h already contains at least one point from A_i

→ for each excess point $x \in A_i$ s.t. $x \in h$, decide whether we should move it to h^+ or to h^- so that ⊛ is satisfied

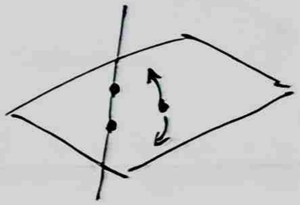
→ we cannot move the points, but we can move the plane!

→ let B be the points of A_i on h we want to get rid of

↳ $|B| \leq d$, let $C \supseteq B$ have size $|C| = d$ ... we add extra affinely independent points of h s.t. C determines h

→ for each $x \in B$, let x' be either in h^+ or h^- at distance ε from x, depending on to which side should the hyperplane tilt

→ for sufficiently small ε, the set $\{x' \mid x \in B\} \cup (C \setminus B)$ is still affinely independent, and the hyperplane it determines has the desired property ■



<u>Intuition</u>: A d-dimensional hyperplane has d degrees of freedom, so if there are $\leq d$ bad points, we can rotate it so that each bad point goes to the side we want

→ plane in $\mathbb{R}^3$ ... given two points on it that should stay, and one that should "move", imagine rotating the plane about the line determined by the two fixed points

APPLICATIONS OF THE HAM-SANDWICH THEOREM

Theorem (Akiyama, Alon): Let $A_1, \dots, A_d \subseteq \mathbb{R}^d$ be disjoint finite sets of points in general position s.t. $|A_i| = n$ for $\forall i$.

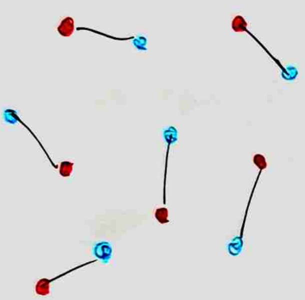
→ imagine that A_i are different color classes

Then there are rainbow d-tuples $R_1, \dots, R_n$, $|R_j| = d$ for $\forall j$

↳ $R_1 \dot\cup \dots \dot\cup R_n = A_1 \dot\cup \dots \dot\cup A_d$ & $\forall i \forall j: |A_i \cap R_j| = 1$

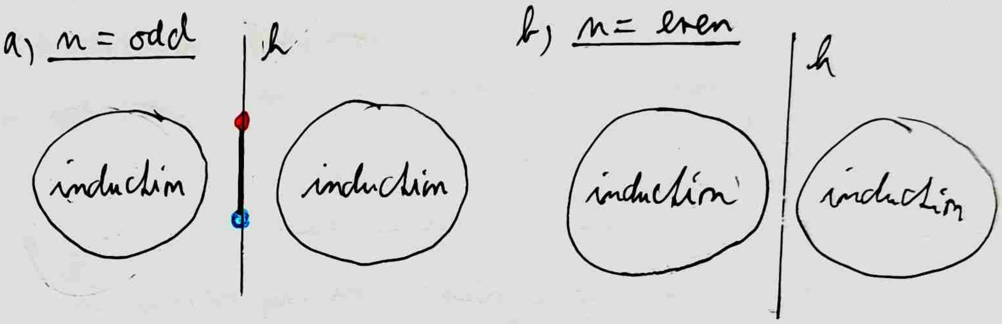
s.t. the convex hulls $\text{conv}(R_1), \dots, \text{conv}(R_n)$ are pair-wise disjoint.

Picture for $d = 2$



Proof: By induction on n. If $n = 1$, nothing to do.

- for $n > 1$, use ham-sandwich for points in general position



a) n = odd

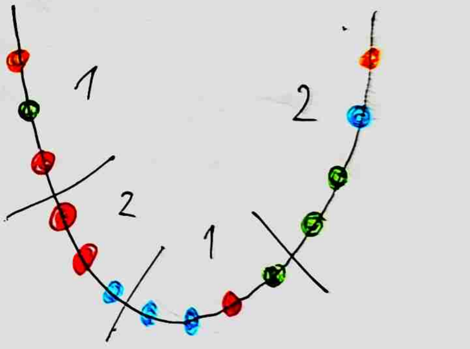
b) n = even

∎

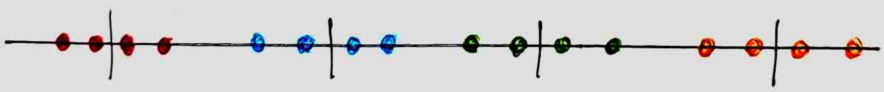
The Necklace Theorem

→ two thieves have stolen a necklace with d types of gems on it, but they do not know their value

⇒ they want to divide the gems of each kind evenly, while making as little cuts as possible



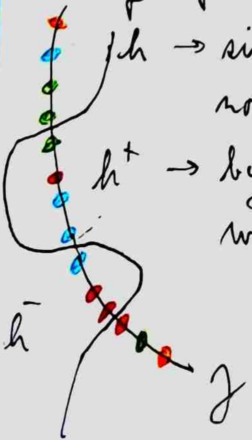
👁 there are examples where d cuts are needed



gems of each type = even!

Theorem: Every open-ended necklace with d kinds of gems can be divided between two thieves using no more than d cuts.

Proof: Lift the necklace to the momentum curve γ in $\mathbb{R}^d$



→ since no hyperplane intersects γ in $> d$ points, the gems are now in general position

→ by ham-sandwich for points in general position, there $\exists$ hyperplane h which divides the necklace how we want

↳ no gem lies on h since all # gems are even!

→ since h intersects γ in at most d points, d cuts are enough ∎

→ there is also a continuous version of the Necklace theorem

Def: A continuous pobobility measure on $[0,1]$ is a Borel measure μ on $[0,1]$ s.t.

① $\text{Range}(\mu) \subseteq [0,1]$, $\mu(\emptyset) = 0$, $\mu(X) = 1$

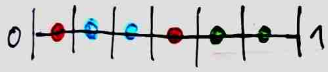
② $\int_0^x d\mu = \mu([0,x])$ is continuous in $x \in [0,1]$

Theorem (Hobby-Rice theorem): Let $\mu_1, \dots, \mu_d$ be cont. prob. measures on $[0,1]$. Then there $\exists$ partition of $[0,1]$ into $d+1$ intervals $I_0, I_1, \dots, I_d$ (using d cut points) and signs $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d \in \{-1,+1\}$ s.t.

$$\sum_{j=0}^{d} \varepsilon_j \mu_i(I_j) = 0 \quad \text{for } \forall i = 1, \dots, d$$

Intuition: first thief gets + sign and second − sign intervals

this implies the neck-lace theorem... if there are n stones in total,

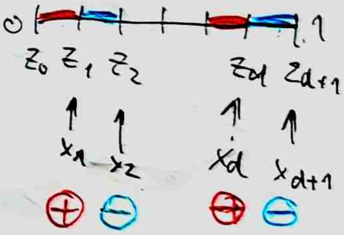


we can imagine them dividing $[0,1]$ into n equal parts

→ we define μ_i so that all of its mass is on the part corresponding to ith stone

Remark: This version is more general: imagine that a metal stick is an alloy of d types of prescious metals, and we want to split them

Proof: For each $x = (x_1, x_2, \dots, x_d, x_{d+1}) \in S^d$ let



$z_j := x_1^2 + \dots + x_j^2$, so $0 = z_0 \le z_1 \le \dots \le z_d \le z_{d+1} = 1$

- $I_j := [z_j, z_{j+1}]$ for $j = 0, 1, \dots, d$
- $\varepsilon_j := \text{sign}(x_{j+1})$ for $j = 0, 1, \dots, d$

⇒ we get a continuous map $g: S^d \to \mathbb{R}^d$, $g(x) = (g_1(x), \dots, g_d(x))$ where

$$g_i(x) := \sum_{j=0}^{d} \varepsilon_j \mu(I_j)$$

... amount of i-stone given to the first thief minus the amount allocated to the second one

more like $\varepsilon_j(x)$ and $I_j(x)$

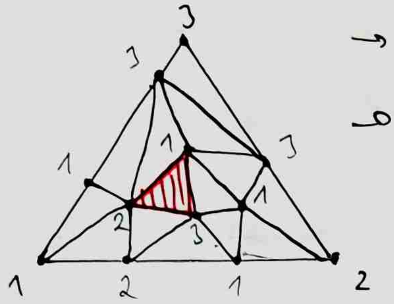
g is antipodal ... $g(-x) = -g(x)$ since z_j stays the same and the sign swaps

⇒ BU1b: $\exists x \in S^d$ s.t. $g(x) = 0$.

↳ $\varepsilon_j(x)$ and $I_j(x)$ are as desired ▨

TUCKER'S LEMMA – towards proving Borsuk-Ulam

Motivation: Sperner's lemma in $\mathbb{R}^2$



→ triangulation of  with labeled vertices

→ on outer edges, we may only use the labels of the original vertices of that edge

↳ we can label the inner vertices arbitrarily

Sperner's lemma: Always ∃ rainbow triangle – using all 3 colors

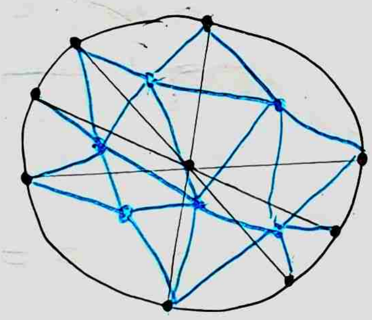
↳ generalizes to higher dimensions & is related to Brower's fixed point thm.

Tucker's lemma: Similar statement related to the Borsuk-Ulam theorem

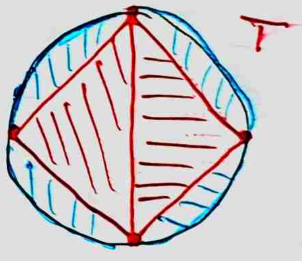
Recall: BU2b: ∄ continuous $f: B^n \to S^{n-1}$ antipodal on $\partial B^n = S^{n-1}$

Def: A triangulation T of B^n is antipodally symmetric on ∂B^n if the set of simplicies of T whose verticies lie on $\partial B^n = S^{n-1}$ is a triangulation of S^{n-1}, and it is antipodally symetric:

- if $\sigma \subseteq S^{n-1}$ is a simplex of T, then $-\sigma$ is also a simplex of T



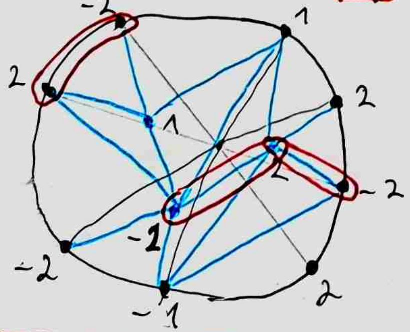
! we identify the outer faces of T with the regions of B^n that they correspond to

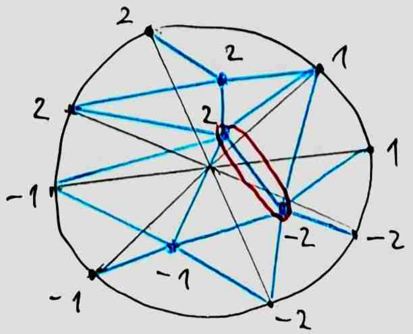


Theorem (Tucker's lemma): Let T be a triangulation of B^n, antipodally symmetric on ∂B^n. Let $\lambda : V(T) \to \{1,-1,2,-2,\dots,n,-n\}$ be a labeling of its vertices s.t. $\lambda(-v) = -\lambda(v)$ for $\forall v \in \partial B^n \cap V(T)$.

↳ so λ is antipodal on $\partial B^n = S^{n-1}$

Then there ∃ edge of T with vertices labeled i and $-i$ for some $i \in [n]$.



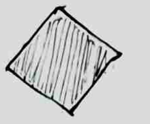


← only 1 such edge in this example

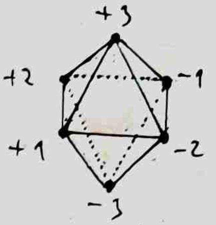
→ we first reformulate it using crosspolytopes ↱ $e_i = (0,\ldots,0,\underset{\text{i-th index}}{1},0,\ldots,0)$

<u>Def</u>: The d-dimensional crosspolytope is $\hat{B}^n := \mathrm{conv}\{e_1,-e_1,\ldots,e_d,-e_d\} \subseteq \mathbb{R}^d$

↪ unit ball under the Manhattan norm

$d=1$: 
$d=2$: 
$d=3$: 

$n=3$:
facets: $\{\pm1,\pm2,\pm3\}$



<u>Def</u>: $\Diamond^{n-1}$ is the abstract simplicial complex with vertex set $V=\{1,-1,2,-2,\ldots,n,-n\}$ and with faces $F \subseteq V$: $F \in \Diamond^{n-1}$ if $\nexists i: i,-i \in F$

👁 $\Diamond^{n-1}$ is the abstract complex corresponding to the boundary of $\hat{B}^n$

↳ if we turn $\hat{B}^n$ into a geometric simplicial complex, remove the full face $\hat{B}^n$ and turn it into an abstract complex, we get $\Diamond^{n-1}$

👁 $\|\Diamond^{n-1}\| \cong S^{n-1}$ … $\Diamond^{n-1}$ is an antipodally symmetric triangulation of S^{n-1}

<u>Theorem</u> (<u>Rephrased Tucker's lemma</u>): Let T be a triangulation of B^n, antipodally symmetric on ∂B^n. Then there is no map $\lambda: V(T) \to V(\Diamond^{n-1})$ that is a simplicial map of T into $\Diamond^{n-1}$ and is antipodal on $\partial B^n = S^{n-1}$

<u>Lemma</u>: BU2b ⇒ Tucker's lemma.

<u>Proof</u>: For contradiction assume that T and λ contradict TL as above.

→ since λ is a simplicial map, $\|\lambda\|: \|T\| \to \|\Diamond^{n-1}\|$ is continuous

→ since $\|T\| \cong B^n$ and $\|\Diamond^{n-1}\| \cong S^{n-1}$, it defines a continuous map $f: B^n \to S^{n-1}$ (we use the projection 

→ since λ is antipodal on the boundary, f is antipodal on $\partial B^n = S^{n-1}$

⇒ BU2b: no such map can exist ↯

▨

<u>Lemma</u>: TL ⇒ BU2b.

<u>Proof</u>: For contradiction let $f: B^n \to S^{n-1}$ be continuous, antipodal on $\partial B^n = S^{n-1}$

→ <u>want</u>: construct T and λ contradicting Tucker's lemma

- let T be any triangulation of B^n, antipodally symmetric on ∂B^n, with simplex diameter at most δ (only small triangles)
  ↳ to be specified later

  ↳ such T can be obtained by repeated barycentric subdivisions of $\Diamond^{n-1}$ and then we fill in the interior with small enough simplicies

→ note that if $\Diamond$ is antipodally sym. then $sd(\Diamond)$ is also

→ set $\varepsilon := \frac{1}{\sqrt{n}}$ ... then for $\forall y = (y_1, \dots, y_n) \in \partial B^n = S^{n-1}$

$\exists i$ s.t. $|y_i| \geq \varepsilon$ ... otherwise $\sum y_i^2 < 1$

→ f is continuous on B^n,
and B^n is compact, so f is actually uniformly continuous

$\Rightarrow \exists \delta > 0$ s.t. $\forall x, x' \in B^n : \|x - x'\| < \delta \Rightarrow \|f(x) - f(x')\| < 2\varepsilon$

→ use this δ to construct T

→ define $\lambda(V(T)) \to \{1, -1, 2, -2, \dots, n, -n\}$ as follows:

such i exists

- for $v \in V(T)$, set $k(v) :=$ min i s.t. $|f(v)_i| \geq \varepsilon$
  ↳ i-th coordinate

$$\lambda(v) := \text{sign}(f(v)_{k(v)}) \cdot k(v) = \begin{cases} +k(v), & f(v)_{k(v)} > 0 \\ -k(v), & f(v)_{k(v)} < 0 \end{cases}$$

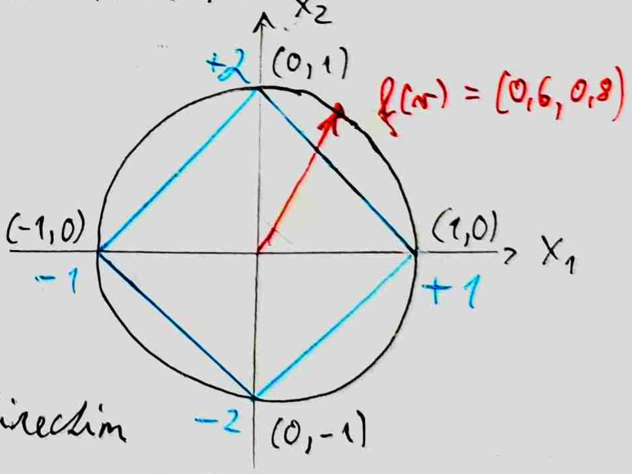
👁 since f is antipodal, we have
$k(-v) = k(v)$ and $\lambda(-v) = \text{sign}(-f(v)_{k(v)})\, k(v) = -\lambda(v)$

→ Tucker's lemma: $\exists$ edge $\{v, v'\}$ with
complementary colors ... WLOG $i := \lambda(v) = -\lambda(v') > 0$

$\Rightarrow f(v)_i \geq \varepsilon$ and $f(v')_i \leq -\varepsilon$, so $\|f(v) - f(v')\| \geq 2\varepsilon$ ↯ ▨

Intuition: Think of $S^{n-1} \cong \|\Diamond^{n-1}\|$ to understand λ

- suppose that $f : v \mapsto (0.6, 0.8) \in S^1$
  $\varepsilon = \frac{1}{\sqrt{2}} \approx 0.7$



→ $k(v) = 2$ says that $f(v)$ points mostly in the direction of the 2nd axis

→ $\text{sign}(f(v)_{k(v)}) = +1$ says in the positive direction

$\Rightarrow \lambda(v) = +2$ is essentially the closest vertex of $\Diamond^{n-1}$ to $f(v)$, but the formalism with $k(v)$ breaks ties

PROOF OF TUCKER'S LEMMA – and thus of Borsuk-Ulam

→ we will in fact prove a slightly weaker version – only for some special triangulations T

↳ but our proof "TL ⇒ BU2b" only requires the validity of TL of a sequence of triangulations with simplex diameter tending to zero

⇒ after we have BU2b, we obtain full Tucker's lemma since BU2b ⇒ TL

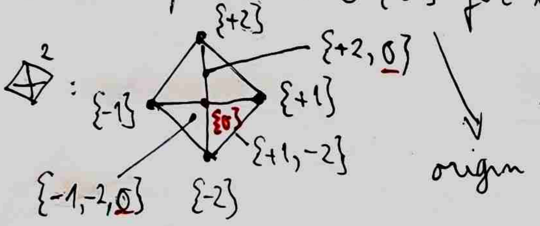
Assumptions about T:

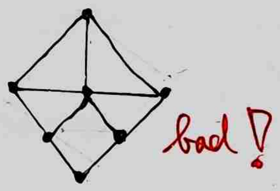
Special TL ⇒ BU2b ⇒ TL

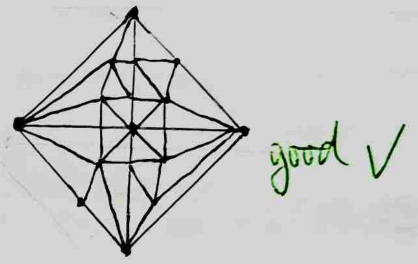
→ we first replace B^n by the cross-polytope $\hat{B}^n$ ... $B^n \cong \hat{B}^n$

→ let $\oplus^n$ be the natural triangulation of $\hat{B}^n$ viewed as a geometric complex

↳ each simplex $\sigma \in \oplus^n$ either lies in $\Diamond^{n-1}$ (the boundary of $\hat{B}^n$) or equals $\tau \cup \{0\}$ for some $\tau \in \Diamond^{n-1}$







Def: A triangulation T of $\hat{B}^n$ refines $\oplus^n$ if $\forall \sigma \in T \ \exists \tau \in \oplus^n$ s.t. $\sigma \subseteq \tau$

↳ viewed as a geometric simp. complex

Def: T is a special triangulation if it is a triangulation of $\hat{B}^n$ that refines $\oplus^n$ and is antipodally symmetric on the boundary $\partial \hat{B}^n = \|\Diamond^{n-1}\|$

one can construct arbitrarily fine special triangulations by repeatedly taking barycentric subdivisions of $\oplus^n$

Proof of Tucker's lemma for special triangulations:

→ let T be a special triangulation of $\hat{B}^n$ and $\lambda: V(T) \to \{\pm 1, \pm 2, \dots, \pm n\}$ labeling

→ plan:

- we will describe certain interesting simplicies ... happy simplicies
- define a graph on happy simplicies
- assume for contradiction that no edge of T is complementary → has vertices i, -i
- show that then the graph has exactly one vertex with odd degree

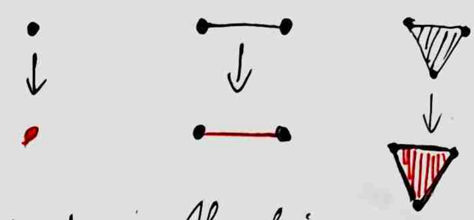
⇒ contradiction since $\sum_{v \in V} \deg(v) = 2 \cdot |E|$

→ for $\sigma \in T$, write $\lambda(\sigma) := \{\lambda(v) \mid v \in \mathbb{R}^n$ is a vertex of $\sigma \subseteq \mathbb{R}^n\}$

→ for $\sigma \in T$, choose $x = (x_1, \dots, x_n)$ in the relative interior of σ and set

$$S(\sigma) := \{+i \mid x_i > 0\} \cup \{-i \mid x_i < 0\}$$

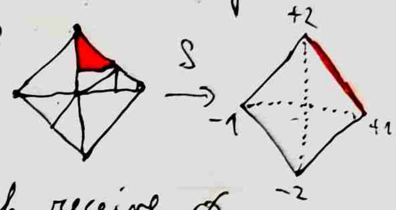
dim σ = 0 , dim σ = 1, dim σ = 2



<u>Note</u>: relative interior = interior
from the point of view of σ

👁 since T refines $\diamondsuit^n$ which is determined by the coordinate hyperplanes, $S(\sigma)$ does not depend on the choice of x

<u>Intuition</u>: $S(\sigma)$ is the vertex set of the simplex of $\diamondsuit^{n-1}$ where σ is mapped by the central projection from 0



↳ the exception are the simplicies ∅ and {0}, which receive ∅

<u>Def</u>: $\sigma \in T$ is <u>happy</u> if $S(\sigma) \subseteq \lambda(\sigma)$! $\sigma \neq \emptyset$

↳ $S(\sigma)$ ~ the labels that σ likes

! $S(\sigma)$ does not depend on λ in any way

👁 $\{0\} \in T$ is always happy

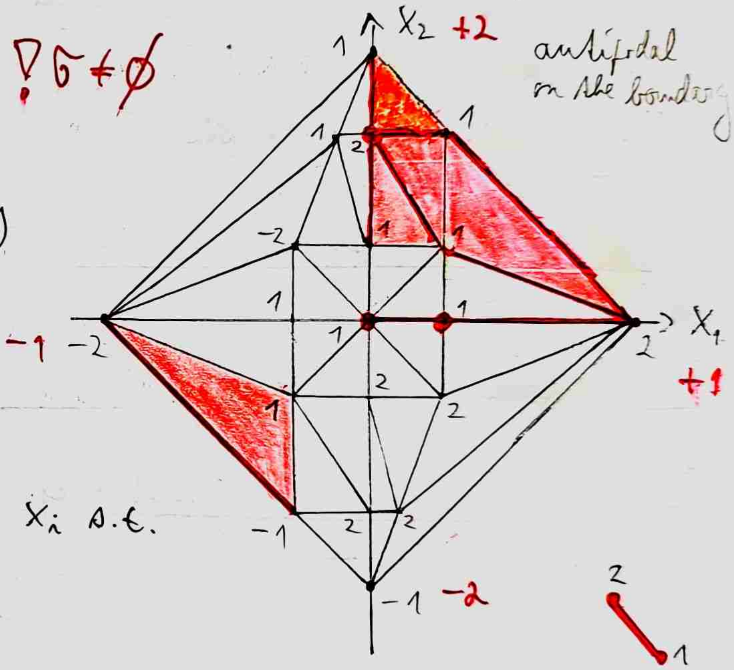
→ fix happy $\sigma \neq \emptyset$, $k := |S(\sigma)|$

👁 $k-1 \leq \dim \sigma \leq k$

⊛

↳ σ lies in the subspace L_σ of the axes x_i s.t. i or $-i \in S(\sigma)$, so $\dim(\sigma) \leq k$

↳ but $|\sigma| \geq k$, so $\dim \sigma \geq k-1$



antipodal on the boundary

<u>Def</u>: σ is <u>tight</u> if $\dim \sigma = k-1$ ... we need to provide all labels to make σ happy

<u>loose</u> if $\dim \sigma = k$ ... either one label occurs twice, or there is an extra label not appearing in $S(\sigma)$

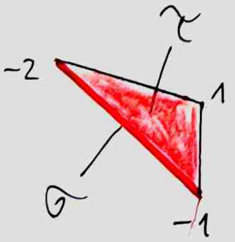
👁 happy boundary simplicies are always tight

→ we define a graph G

- vertices of G = happy simplicies

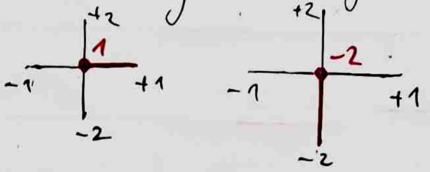
↱ $\sigma = -\tau \subseteq \partial \hat{B}^n$

- $\sigma\tau \in E(G)$ if a) σ and τ are antipodal boundary simplicies, or
  b) σ is a facet of τ with $\lambda(\sigma) = S(\tau)$
  ↳ the labels of σ alone already make τ happy



σ is a facet of τ and $S(\tau) = \lambda(\sigma) = \{-2, -1\}$

$\deg_G(\{o\}) = 1$ since it adjacent (by b1) to exactly the edge of the triangulation made happy by $\underline{\lambda(\sigma)}$



<u>claim</u>: If there is no $\{i,-i\}$ edge in T, then all other vertices of G have degree 2 } This will finish the proof

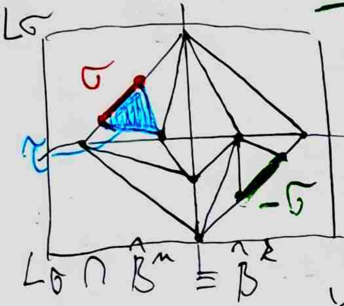
<u>proof</u>: Let $\sigma \neq \{o\}$ be a happy simplex

① <u>σ is tight</u> ... if $\sigma\tau \in E(G)$, then either $\tau = -\sigma$ or σ is a facet of τ

↳ ∵ σ cannot be made happy by the labels of a facet

1a) <u>σ is a boundary simplex</u>

⇒ then $\underline{-\sigma}$ is a neighbour of $\underline{\sigma}$ → ∵ λ is antipodal on the boundary



→ the other neighbours are simplicies $\underline{\tau}$ of $\dim \tau = \dim \sigma + 1$ s.t. σ is a facet of τ and $\lambda(\sigma)$ make τ happy

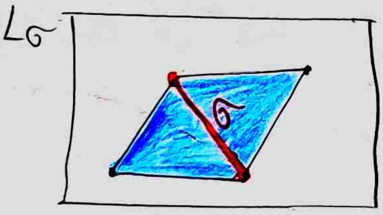
→ σ lies in the previously defined ⊛ subspace $L\sigma$ of dimension $k := \dim \sigma + 1$

$L\sigma \cap \hat{B}^n \equiv \hat{B}^k$ ... the intersection is a crosspolytop

↳ the simplicies of T in $L\sigma$ triangulate $\hat{B}^k$

⇒ σ is a boundary $(k-1)$-dim. simplex in a triangulation of $\hat{B}^k$, and is thus a facet of precisely one k-simplex ⇒ <u>τ is unique</u>

2b) <u>σ is not on $\partial\hat{B}^n$</u>



→ similar to before, σ lives in $\hat{B}^k = L_\sigma \cap \hat{B}^n$ and the two neighbours of σ are the two unique simplicies of dimension $\dim \sigma + 1$ containing it as a facet

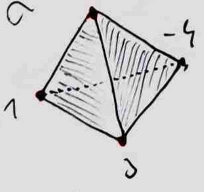
② <u>σ is loose</u> $k := \dim k = |S(\sigma)|$

1a) <u>$S(\sigma) = \lambda(\sigma)$</u> ... one of the labels occurs twice on σ

→ σ is loose so $\sigma \notin \partial\hat{B}^n$, and σ cannot be a facet of a happy simplex

⇒ σ is adjacent to two of its facets : labels of σ : $\underline{1, -1}, 2, \underline{2, 3, 4, -5}$

2b) <u>$\exists!\, i \in \lambda(\sigma) \setminus S(\sigma)$</u> ... note $-i \notin S(\sigma)$ as otherwise, there would be a $\{i,-i\}$ edge



$S(\sigma) = \{1,2,3\}$

→ one neighbour of σ is the facet τ of σ with vertex set $S(\sigma)$

→ the other one is a unique loose simplex σ' s.t.

- σ is a facet of σ', and $\lambda(\sigma) = S(\sigma') = S(\sigma) \cup \{i\}$ makes σ' happy

⇐ we reach σ' by going from the relative int. of σ in the $\mathrm{sign}(i)$ direction of the $X_{|i|}$ axis

∎

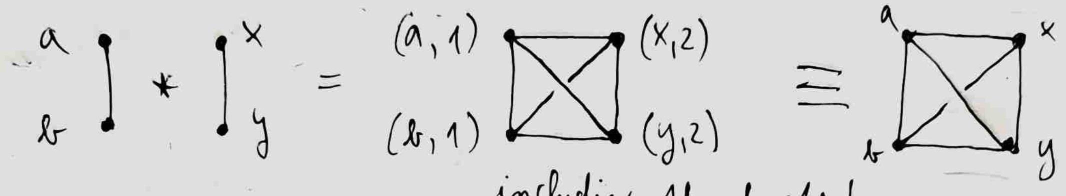
JOINS OF SIMPLICIAL COMPLEXES

Notation: $A_1 \uplus A_2 \uplus \cdots \uplus A_n := (A_1 \times \{1\}) \cup (A_2 \times \{2\}) \cup \cdots \cup (A_n \times \{n\})$

Def: The join $K * L$ of two abstract simplicial complexes K and L is the simplicial complex with vertex set $V(K) \uplus V(L)$ and simplicies

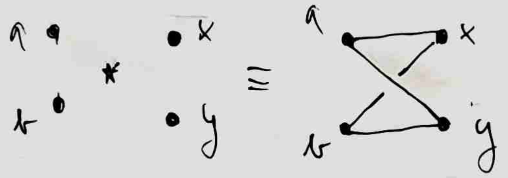
$$\{A \uplus B \mid A \in K, B \in L\} \quad \ldots \text{ note: } A \uplus \emptyset = A \times \{1\}$$

Example:



$= \sigma^3$

including the facets!



... geom. real. homeomorphic to S^1

$K*(L*M) = (K*L)*M$ up to renaming of vertices

Def: $K_1 * K_2 * \cdots * K_n := \{A_1 \uplus A_2 \uplus \cdots \uplus A_n \mid \forall i: A_i \in K_i\}$

$K^{*n} := K * K * \cdots * K$ ... n-fold join of K

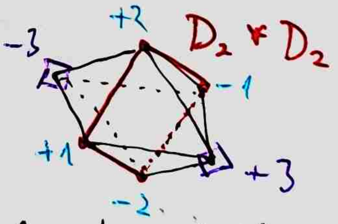
Notation: D_n := discrete set with n points ... $\|D_2 * D_2\| \cong S_1$

→ abstract simplicial complex

Lemma: $\underbrace{\|D_2 * \cdots * D_2\|}_{n \text{ times}} \cong S^{n-1}$

$D_2 = \{\emptyset, \{1\}, \{2\}\}$

Proof:



By induction → $D_2 * D_2 * D_2$:

$\|(D_2)^{*n}\| = \|\Diamond^{n-1}\| \cong S^{n-1}$ ... boundary of the n-dim crosspolytope

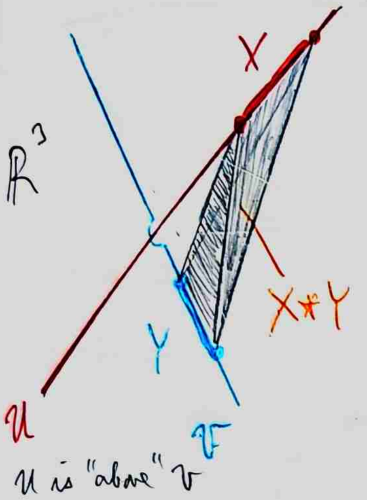
→ adding an extra D_2 corresponds to adding two new labels: $(n+1), -(n+1)$ ▨

→ we will show:

$\|K_1\| \cong \|K_2\|$ & $\|L_1\| \cong \|L_2\| \Rightarrow \|K_1 * L_1\| \cong \|K_2 * L_2\|$

GEOMETRIC JOINS

Def: Let X, Y be bounded top. subspaces of $\mathbb{R}^n$ s.t. $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are skew affine subspaces of $\mathbb{R}^n$. That is:



formally: $\mathcal{U} \cap \mathcal{V} = \emptyset$ and the affine hull of $\mathcal{U} \cup \mathcal{V}$ has dimension $\dim \mathcal{U} + \dim \mathcal{V} + 1$

in practice: if $\dim \mathcal{U} = d_1$ and $\dim \mathcal{V} = d_2$ and

$A \subseteq \mathcal{U}$ ... $d_1 + 1$ affinely ind. points of $\mathcal{U}$

$B \subseteq \mathcal{V}$ ... $d_2 + 1$ affinely ind. points of $\mathcal{V}$

$\Rightarrow$ then $A \dot{\cup} \mathcal{V}$ is $d_1 + d_2 + 2$ affinely ind. points in $\mathbb{R}^n$

Then the geometric join of X and Y is

$$X * Y := \{ tx + (1-t)y \mid t \in [0,1],\ x \in X,\ y \in Y \} \subseteq \mathbb{R}^n$$

if X and Y are convex, then $X * Y$ is the convex hull of $X \cup Y$

Exercise: $\|K * L\| = \|K\| * \|L\|$ ⊛ $X * Y \cong Y * X$, $(X*Y)*Z \cong X*(Y*Z)$

Corollary: Since $\|D_2^{*n}\| \cong S^{n-1}$, we have $S^k * S^l \cong S^{k+l+1}$

$$\|D_2\| \cong S^0 \Rightarrow (S^0)^{*n} \cong S^{n-1}$$

Lemma: Let $X_1, X_2 \subseteq \mathbb{R}^{n_1}$ and $Y_1, Y_2 \subseteq \mathbb{R}^{n_2}$ ... $\mathbb{R}^{n_1}, \mathbb{R}^{n_2}$ skew subs. of $\mathbb{R}^{n_1+n_2+1}$.

If $X_1 \cong X_2$ and $Y_1 \cong Y_2$, then $X_1 * Y_1 \cong X_2 * Y_2$

Proof: Let $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ be homeomorphisms.

- $\to$ define $h : X_1 * Y_1 \to X_2 * Y_2$ by $\underbrace{tx + (1-t)y}_{z \in X_1 * Y_1} \longmapsto t\,h_X(x) + (1-t)\,h_Y(y)$ $\qquad h = h_X * h_Y$
- $\to$ since $x \mapsto h_X(x)$ and $y \mapsto h_Y(y)$ are bijective, h is also bijective
- $\to$ and the continuity of h and h^{-1} follows similarly
  - $\hookrightarrow$ small wiggles of z ~ small wiggles of x and y $\nearrow$ and so also of $h(z)$
  - ~ small wiggles of $h_X(x)$ and $h_Y(y)$ ▨

Proposition: $\|K_1\| \cong \|K_2\|$ & $\|L_1\| \cong \|L_2\|$ $\Rightarrow$ $\|K_1 * L_1\| \cong \|K_2 * L_2\|$

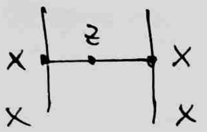
Proof: $\underset{⊛}{\|K_1 * L_1\| \cong} \|K_1\| * \|L_1\| \underset{\text{lemma}}{\overset{}{\cong}} \|K_2\| * \|L_2\| \underset{⊛}{\cong} \|K_2 * L_2\|$ ▨

JOINS OF MAPS

Notation: We write the point $z \in X * Y$, where
$z = tx + (1-t)y$, $x \in X$, $y \in Y$ as $tx \oplus (1-t)y \in X * Y$

! $tx \oplus (1-t)y \neq (1-t)y \oplus tx$

Example:



$z = \frac{1}{2}x \oplus \frac{1}{2}x \in X * X$

→ similarly $t_1 x_1 \oplus t_2 x_2 \oplus \cdots \oplus t_n x_n \in X_1 * X_2 * \cdots * X_n$ $t_i \in [0,1]$, $\sum t_i = 1$

Def: Given continuous maps $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$, define
$f * g : X_1 * Y_1 \to X_2 * Y_2$ by $tx \oplus (1-t)y \mapsto tf(x) \oplus (1-t)g(y)$

Def: Given abstract simplicial complexes K_1, K_2, L_1, L_2 and simplicial maps $f : V(K_1) \to V(K_2)$ and $g : V(L_1) \to V(L_2)$, define
$f * g : V(K_1 * L_1) \to V(K_2 * L_2)$ by
$$\begin{cases} (v,1) \mapsto (f(v),1) & v \in K_1 \\ (v,2) \mapsto (g(v),2) & v \in L_1 \end{cases}$$

Fact: $\|f * g\| = \|f\| * \|g\|$

$\mathbb{Z}_2$-SPACES & $\mathbb{Z}_2$-MAPS

Def: A $\mathbb{Z}_2$-space is a pair (X, γ) where X is a top. space and $\gamma : X \to X$ is a continuous map s.t. $\gamma^2 = \gamma \circ \gamma = id_X$. ⇒ γ is a homeomorphism

- We say that γ is a $\mathbb{Z}_2$-action on X.
- γ is free if $\forall x \in X : \gamma(x) \neq x$ … no fixed points
  ⇒ then (X, γ) is a free $\mathbb{Z}_2$-space

Def: Let (X, γ) and (Y, μ) be $\mathbb{Z}_2$-spaces. Then a $\mathbb{Z}_2$-map between them is a continuous map $f : X \to Y$ s.t. $f \circ \mu = \gamma \circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\mu"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

← this diagram commutes

Examples: → not free since $-0 = 0$

- $(\mathbb{R}^n, -)$ where $- : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto -x$ is minus
- $(S^{n-1}, -)$ … free $\mathbb{Z}_2$-space

Observations: Alternative formulation of BU2a

- BU2a: $\nexists$ antipodal $f : S^n \to S^{n-1}$ … $f(-x) = -f(x)$
- There is no $\mathbb{Z}_2$-map from $(S^n, -)$ to $(S^{n-1}, -)$.

The transposition $\mathbb{Z}_2$-action: → not free: $\tau(\frac{1}{2}x \oplus \frac{1}{2}x) = \frac{1}{2}x \oplus \frac{1}{2}x$

- $(X * X, \tau)$ where $\tau(tx \oplus (1-t)y) := (1-t)y \oplus tx$

$X \subseteq \mathbb{R}^n$

Joins of $\mathbb{Z}_2$-spaces:

If (X, γ) and (Y, μ) are $\mathbb{Z}_2$-spaces, then $(X * Y, \gamma * \mu)$ is a $\mathbb{Z}_2$-space.

↳ $\gamma * \mu : tx \oplus (1-t)\mu \mapsto t\gamma(x) \oplus (1-t)\mu(y)$

If γ and μ are free, then $\gamma * \mu$ is free

Def: A simplicial $\mathbb{Z}_2$-complex is a pair (K, γ), where K is an abstract simplicial complex and $\gamma : V(K) \to V(K)$ is a simplicial map s.t. $\gamma \circ \gamma = id_{V(K)}$

γ is free if $\|\gamma\|$ is free for $\|K\|$

(K, γ) is $\mathbb{Z}_2$-complex $\iff$ $(\|K\|, \|\gamma\|)$ is $\mathbb{Z}_2$-space

Joins of simplicial $\mathbb{Z}_2$-complexes:

$(K, \gamma), (L, \mu)$ $\mathbb{Z}_2$-compl. $\Rightarrow$ $(K * L, \gamma * \mu)$ $\mathbb{Z}_2$-compl.

$(\|K * L\|, \|\gamma * \mu\|) \cong (\|K\| * \|L\|, \|\gamma\| * \|\mu\|)$

↳ example: $\|D_2 * \cdots * D_2\| = \|D_2\| * \cdots * \|D_2\|$, so:

$$\left(\|D_2^{*n}\|, \|\tau^{*n}\|\right) \cong \left(\|D_2\|^{*n}, \|\tau\|^{*n}\right) \cong (S^{n-1}, -)$$

↓

$D_2 =$ 

τ swaps the two points

DELETED JOINS

↗ transposition

<u>Recall</u>: $X \subseteq \mathbb{R}^d$, $(X * X, \tau)$ where $\tau(tx \oplus (1-t)y) = (1-t)y \oplus tx$ is a non-free $\mathbb{Z}_2$-space.

<u>Def</u>: The <u>deleted join</u> of $X \subseteq \mathbb{R}^d$ is

$$X^{*2}_{\Delta} := X * X \setminus \{ \tfrac{1}{2}x \oplus \tfrac{1}{2}x \mid x \in X \}$$

similarly X^{*n}_{Δ} is n-fold deleted join

👁 (X^{*2}_{Δ}, τ) is a free $\mathbb{Z}_2$-space

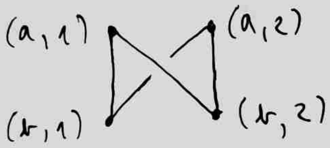
<u>Def</u>: The <u>deleted join</u> of an abstract <u>simplicial complex</u> K is

$$K^{*2}_{\Delta} := \{ A \uplus B \mid A, B \in K \ \& \ A \cap B = \emptyset \} \subseteq K * K = K^{*2}$$

👁 <u>(K^{*2}_{Δ}, τ)</u> is a free simplicial $\mathbb{Z}_2$-complex

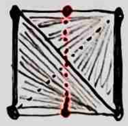
$\hookrightarrow$ $\tau : V(K^{*2}_{\Delta}) \to V(K^{*2}_{\Delta})$ where $\begin{cases} (v,1) \mapsto (v,2) \\ (v,2) \mapsto (v,1) \end{cases}$ $v \in V(K)$

<u>Example</u>: $K = \sigma^1 =$ 

K^{*2}_{Δ}  $\cong D_2 * D_2$ … $\|(\sigma^1)^{*2}_{\Delta}\| \cong S^1$

! $\|K^{*2}_{\Delta}\| \not\cong \|K\|^{*2}_{\Delta}$ in general

↗ $\|K\|^{*2} \cong \|\sigma^3\| =$ 

$\hookrightarrow$ take $K = \sigma^1$, then $\|K\|^{*2}_{\Delta}$ is 

in the deleted join, we remove this part → we "drill a hole"

<u>Lemma</u>: If K is an abstract complex, then there exist $\mathbb{Z}_2$-maps <u>$\|K^{*2}_{\Delta}\| \to \|K\|^{*2}_{\Delta}$</u> and <u>$\|K\|^{*2}_{\Delta} \to \|K^{*2}_{\Delta}\|$</u>.

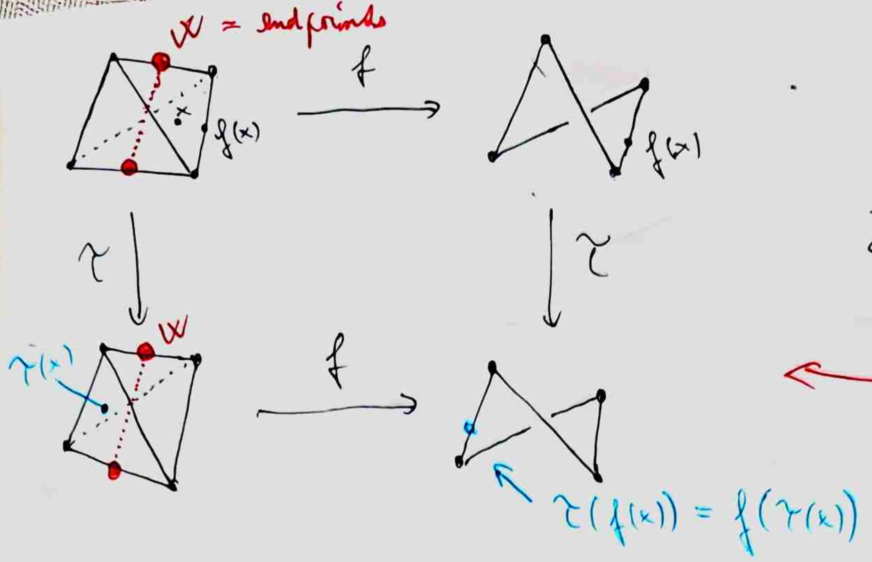
$\hookrightarrow$ the $\mathbb{Z}_2$-actions are transpositions τ

<u>Proof sketch</u>:

- $\|K^{*2}_{\Delta}\| \to \|K\|^{*2}_{\Delta}$ is just the inclusion map since $\|K^{*2}_{\Delta}\| \subseteq \|K\|^{*2}_{\Delta}$
- $\|K\|^{*2}_{\Delta} \to \|K^{*2}_{\Delta}\|$ … it is sufficient to show for σ^n since K is made from σ^n

→ let V be the vertex set of σ^n and put $W := \{ \tfrac{1}{2}v \oplus \tfrac{1}{2}v \mid v \in V \} \subseteq \|\sigma^n\|^{*2}$ where we interpret v as a vertex in $\|\sigma^n\|$

→ for $\forall x \in \|\sigma^n\|^{*2}_{\Delta}$ there $\exists f(x) \in \|(\sigma^n)^{*2}_{\Delta}\|$ s.t. x can be expressed as a unique convex combination of points in W and $f(x)$



$x \mapsto f(x)$ is the desired $\mathbb{Z}_2$-map $(\|\sigma^n\|^{*2}_\Delta, \tau) \to (\|(\sigma^n)^{*2}_\Delta\|, \tau)$

← diagram commutes

∎

$\underline{\text{Lemma}}$: If K and L are simplicial complexes, then $\underline{(K*L)^{*2}_\Delta \cong K^{*2}_\Delta * L^{*2}_\Delta}$.
Moreover, this isomorphism commutes with the respective translation actions τ_{K*L} and $\tau_K * \tau_L$ on these complexes.

$\underline{\text{Proof}}$: We need to show that $(K*L)^{*2}_\Delta$ and $K^{*2}_\Delta * L^{*2}_\Delta$ are the same up to formalism

- $\underline{\text{simplicies of } (K*L)^{*2}_\Delta}$ $\ni (\alpha_1 \uplus \beta_1) \uplus (\alpha_2 \uplus \beta_2)$, $\alpha_1, \alpha_2 \in K$ and $\beta_1, \beta_2 \in L$ and $\underline{(\alpha_1 \uplus \beta_1) \cap (\alpha_2 \uplus \beta_2) = \emptyset}$ ⊛
- $\underline{\text{simplicies of } K^{*2}_\Delta * L^{*2}_\Delta}$ $\ni (\alpha_1 \uplus \alpha_2) \uplus (\beta_1 \uplus \beta_2)$, $\alpha_1, \alpha_2 \in K$ and $\beta_1, \beta_2 \in L$ and $\underline{\alpha_1 \cap \alpha_2 = \emptyset \;\&\; \beta_1 \cap \beta_2 = \emptyset}$ ⊛⊛

☼ ⊛ ⟺ ⊛⊛ .... take isomorphism $\pi : V((K*L)^{*2}_\Delta) \to V(K^{*2}_\Delta * L^{*2}_\Delta)$
$((v,i),j) \mapsto ((v,j),i)$

↳ $i \sim$ is v from L or K? $j \sim$ from which copy is v?

- $\underline{\text{action on } (K*L)^{*2}_\Delta}$

$\tau_{K*L} : ((v,1),1/2) \mapsto ((v,1),2/1)$ ... $v \in V(K)$ encoded by $(v,1)$
$((u,2),1/2) \mapsto ((u,2),2/1)$ ... $u \in V(L)$ encoded by $(u,2)$

- $\underline{\text{action on } K^{*2}_\Delta * L^{*2}_\Delta}$ is $\tau_K * \tau_L$, where

$\tau_K : (v, 1/2) \mapsto (v, 2/1)$ ... $v \in V(K)$
$\tau_L : (u, 1/2) \mapsto (u, 2/1)$ ... $u \in V(L)$

$\tau_K * \tau_L : ((v,1/2),1) \mapsto (\tau_K((v,1/2)),1) = ((v,2/1),1)$ ... $v \in V(K)$
$((u,1/2),2) \mapsto (\tau_L((u,1/2)),2) = ((u,2/1),2)$ ... $u \in V(L)$

→ clearly $\pi \circ (\tau_K * \tau_L) = \tau_{K*L} \circ \pi$

∎

Corollary: $\|(\sigma^n)^{*2}_\Delta\| \cong S^n$ ... recall that we saw $\|(\sigma^1)^{*2}_\Delta\| \cong S^1$

Proof:

→ note that $\sigma^n \cong (\sigma^0)^{*(n+1)}$

σ^0 = single point

$$(\sigma^n)^{*2}_\Delta \cong \left((\sigma^0)^{*(n+1)}\right)^{*2}_\Delta \underset{\uparrow}{\cong} \left((\sigma^0)^{*2}_\Delta\right)^{*(n+1)} \underset{\uparrow}{\cong} (D_2)^{*(n+1)}$$

⇒ recall: $\|(D_2)^{*n}\| \cong S^{n-1}$. previous lemma

$(\sigma^0)^{*2}_\Delta$ is

▨

Remark: Moreover $(\|(\sigma^n)^{*2}_\Delta\|, \tau) \cong (S^n, -)$

↳ $((\sigma^n)^{*2}_\Delta, \tau) \cong ((D_2)^{*(n+1)}, \tau_2^{*(n+1)})$, where

↳ and recall: $(\|(D_2)^{*n}\|, \tau_2^{*n}) \cong (S^{n-1}, -)$

NON-EMBEDDABILITY THEOREM

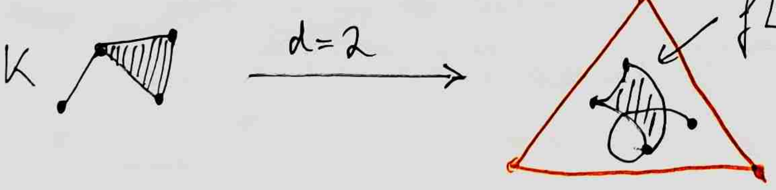
Theorem: Let K be an abstract simplicial complex such that there exists a continuous map $f : \|K\| \to \mathbb{R}^d$ s.t. $f[\alpha] \cap f[\beta] = \emptyset$ for every two disjoint faces α, β of the geometric realization $\|K\|$ of K.

⇒ Then there is a map $\|K^{*2}_\Delta\| \to \|\sigma^d\|^{*2}_\Delta$ that is a $\mathbb{Z}_2$-map w.r.t. the respective transposition actions.

Proof: Note that $\|K\|$ is compact (closed & bounded).

→ since f is continuous, $f[\|K\|] \subseteq \mathbb{R}^d$ is also compact

⇒ enclose $f[\|K\|]$ in a large enough d-simplex $\|\sigma^d\|$



WLOG  $= \|\sigma^d\|$

- $f * f : \|K\|^{*2} \to \|\sigma^d\|^{*2}$ ... $tx \oplus (1-t)y \longmapsto tf(x) \oplus (1-t)f(y)$

claim: This is a $\mathbb{Z}_2$-map w.r.t. the transposition actions on $\|K\|^{*2}$ and $\|\sigma^d\|^{*2}$

$$\left[\begin{array}{l} tx \oplus (1-t)y \xrightarrow{f} tf(x) \oplus (1-t)f(y) \xrightarrow{\tau} (1-t)f(y) \oplus tf(x) \\ tx \oplus (1-t)y \xrightarrow{\tau} (1-t)y \oplus tx \xrightarrow{\tau} (1-t)f(y) \oplus tf(x) \end{array}\right.$$

- if $A \uplus B \in K^{*2}_\Delta$, then by definition A and B are disjoint

⇒ the geometric interpretations α, β of these faces are also disjoint

$\Rightarrow$ by assumption $\forall x \in \alpha\ \forall y \in \beta : f(x) \neq f(y)$ $\quad z \in \|\sigma^d\|$

$\Rightarrow$ so $(f * f)(tx \oplus (1-t)y) = tf(x) \oplus (1-t) f(y) \neq \frac{1}{2}z + \frac{1}{2}z$

Hence $f*f$ restricted to $\|K^{*2}_\Delta\|$ maps into $\|\sigma^d\|^{2*}_\Delta$ $\quad$ ↖ it is never of this form ▨

↳ we implicitly use that $\|K\| * \|K\| \cong \|K * K\|$

THE TOPOLOGICAL RADON THEOREM

Theorem $\boxed{1}$ (Topological Radon theorem): Let $f : \|\sigma^{d+1}\| \to \mathbb{R}^d$ be continuous. $\subseteq \mathbb{R}^{d+1}$

Then there $\exists$ two disjoint faces $A, B \in \sigma^{d+1}$ s.t. $f[\|A\|] \cap f[\|B\|] \neq \emptyset$.

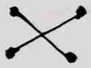
Theorem $\boxed{2}$ (Radon's theorem for affine maps): Let $f : \|\sigma^{d+1}\| \to \mathbb{R}^d$ be an affine map.

That is: $f(\alpha_1 x_1 + \dots + \alpha_{d+1} x_{d+1}) = \alpha_1 f(x_1) + \dots + \alpha_{d+1} f(x_{d+1})$ if $\sum \alpha_i = 0, \alpha_i \in \mathbb{R}$

Then there $\exists$ two disjoint faces $A, B \in \sigma^{d+1}$ s.t. $f[\|A\|] \cap f[\|B\|] \neq \emptyset$

$\boxed{1} \Rightarrow \boxed{2}$ Since every affine map is determined by the images of the $d+2$ vertices of σ^{d+1}, and is thus continuous

Theorem $\boxed{3}$ (Radon's theorem): Every set $X = \{x_1, \dots, x_{d+2}\} \subseteq \mathbb{R}^d$ can be divided into two disjoint subsets whose convex hulls intersect



Observation: $\boxed{2} \iff \boxed{3}$

Proof: Each f from $\boxed{2}$ is determined by the images of the $d+2$ vertices of σ^{d+1}. The image of a face is the convex hull of the images of its vertices ▨

Proof of Theorem $\boxed{1}$: Suppose there $\exists f : \|\sigma^{d+1}\| \to \mathbb{R}^d$ s.t. $f[\|A\|] \cap f[\|B\|] = \emptyset$ for $\forall A \cap B = \emptyset$

→ by the non-emb. thm. there $\exists$ $\mathbb{Z}_2$-map

$$(\|(\sigma^{d+1})^{*2}_\Delta\|, \tau) \longrightarrow (\|\sigma^d\|^{*2}_\Delta, \tau)$$

→ by a pevious lemma there $\exists$ $\mathbb{Z}_2$-map $(\|\sigma^d\|^{*2}_\Delta, \tau) \longrightarrow (\|(\sigma^d)^{*2}_\Delta\|, \tau)$

$$\Rightarrow (\|(\sigma^{d+1})^{*2}_\Delta\|, \tau) \longrightarrow (\|(\sigma^d)^{*2}_\Delta, \tau)$$

→ since $\|(\sigma^d)^{*2}_\Delta\| \cong S^d$, there $\exists$ $\mathbb{Z}_2$-map $(S^{d+1}, -) \to (S^d, -)$

$\Rightarrow$ contradicting BU2a (recall that we can formulate it like this) ▨

Direct proof of Theorem $\boxed{3}$: Any d+2 points $x_1, \dots, x_{d+2}$ in $\mathbb{R}^d$ are affinely dependent

⇒ fix an affine dependence $\alpha_1 x_1 + \dots + \alpha_{d+2} x_{d+2} = 0$ for some $(\alpha_1, \dots, \alpha_{d+2}) \neq 0$, $\sum \alpha_i = 0$

→ define $I^+ := \{ i \in [d+2] \mid \alpha_i > 0 \}$ and $I^- := [d+2] \setminus I_1$

⇒ then $S := \sum_{i \in I^+} \alpha_i = - \sum_{j \in I^-} \alpha_j$

$x := \sum_{i \in I^+} \frac{\alpha_i}{S} x_i = - \sum_{j \in I^-} \frac{\alpha_j}{S} x_j$ is a convex combination of points in $X^+ := \{ x_i \mid i \in I^+ \}$ as well as $X^- := X \setminus X^+$ ∎

REMARKS ON NON-EMBEDIBILITY THEOREMS

→ recall that we can view a graph $G = (V, E)$ as an abstract simplicial complex of dimension 1, and a planar drawing of G can be viewed as a continuous map $f: \|G\| \to \mathbb{R}^2$ s.t. $\forall \underset{x \neq y}{x, y} \in \|G\|$ we have $f(x) \neq f(y)$

→ recall that the non-planarity of $K_{3,3}$ and K_5 is usually argued using the Jordan curve theorem

note: $K_{3,3} \cong D_3 * D_3$

⊛

→ in fact, something stronger is true:

For ∀ continuous map $f: \|K_{3,3}\| \to \mathbb{R}^2$, there are disjoint simplicies α, β of $\|K_{3,3}\|$ such that $f[\alpha] \cap f[\beta] \neq \emptyset$

⇒ the problem is two disjoint edges intersecting

→ loops are not a problem to planarity

Theorem (Van Kampen–Flores theorem, version for $K_{3,3}$): For ∀ continuous map $f: \| (D_3)^{*(d+1)} \| \to \mathbb{R}^{2d}$, there ∃ α, β disjoint simplicies s.t. $f[\alpha] \cap f[\beta] \neq \emptyset$

⊛

→ one can always embed a graph into $\mathbb{R}^3$ by placing vertices in general position and connecting edges by straight lines

→ similarly, a d-dimensional complex always embeds into $\mathbb{R}^{2d+1}$

⇒ but this theorem says that they do not (in general) embed into $\mathbb{R}^{2d}$

G-SPACES & G-MAPS

G group with operation $g \cdot h$

Def: Let G be a finite group and X be a top. space. A *G-action* on X is a family $\Phi = (\varphi_g)_{g \in G}$ of homeomorphisms $\varphi_g : X \to X$ s.t. $\varphi_h \circ \varphi_g = \varphi_{h \cdot g}$ for $\forall h, g \in G$. We call (X, Φ) a *G-space*.

→ for $\mathbb{Z}_2$ we had only 1 homeomorphism corresponding to the non-trivial element, and clearly the hom for the trivial el. has to be the identity

Def: If (X, Φ) and (Y, Ψ) are two G-spaces, then a continuous map $f : X \to Y$ is a *G-map* if $f \circ \psi_g = \varphi_g \circ f$ for $\forall g \in G$.

$\varphi_e = id_X$
↓
trivial or neutral el.

Def: Let $k \geq -1$ be an integer. A topological space is *k-connected* if $\forall \, l \in \{-1, 0, 1, \dots, k\}$, every continuous map $f : S^l \to X$ can be extended to a continuous map $\bar{f} : B^{l+1} \to X$.

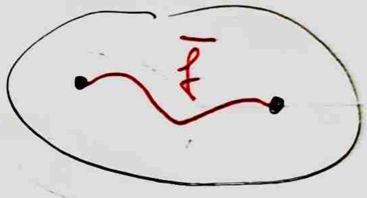
Remark: $S^{-1} := \emptyset$, $B^0 := \{0\}$ … just a point

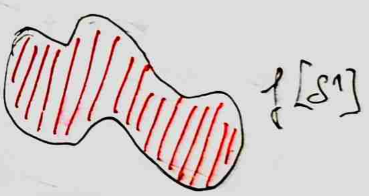
Examples:

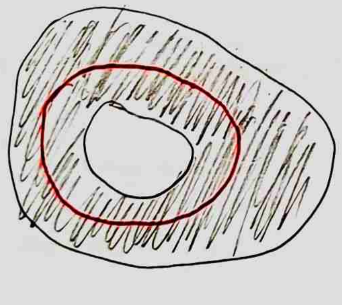
(-1)-connected ≡ non-empty

0-connected ≡ non-empty & path-connected

1-connected ≡ 0-connected & *simply-connected*







← this is not simply connected

Def: (X, Φ) is a *free G-space*, and Φ is a *free G-action* if

$$\forall g \in G : \; g \neq e \implies \varphi_g \text{ has no fixed points}$$

↳ neutral element of the group

Def: A *simplicial G-complex* is a simplicial complex $\|K\|$ made into a G-space by homeomorphisms φ_g of $\|K\|$ so that the abstract maps φ'_g are simplicial maps $K \to K$

$\hookrightarrow \|\varphi'_g\| = \varphi_g$

→ we think about $\|K\|$, φ_g and K, φ'_g simultaneously

$E_n G$-SPACES

Def: Let G be a finite group, $|G| > 1$, $n \geq 0$.

A G-space (X, Φ) is an $E_n G$-space if

① a finite simplicial G-complex ... so $\Phi = (\varphi_g)_{g \in G}$ are geometric realizations of simplicial maps

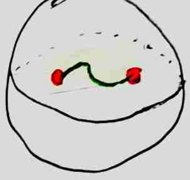
② the complex is n-dimensional

→ and X is a polyhedron realizing said simplicial complex

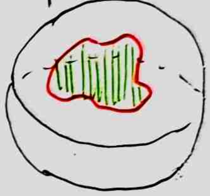
③ X is $(n-1)$-connected

④ (X, Φ) is a free G-space

$f: S^0 \to S^2$



$f: S^1 \to S^2$



Examples:

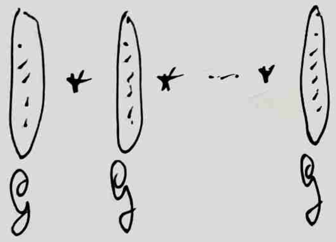
① $(\|D_2^{*(n+1)}\|, \nu) \cong (S^n, -)$

→ this is an $E_n \mathbb{Z}_2$-space

→ we should show that S^n is $(n-1)$-connected

↳ intuitively, we can map the inside of S^l, $l < n$ to a "regim" separated by $f[S^l]$

② $G^{*(n+1)}$ ... $(n+1)$-fold join of G viewed as the discrete space $G \cong D_{|G|}$



- actions on a single copy of G

$\varphi_g : G \to G$, $h \mapsto g \cdot h$

- actions on $G^{*(n+1)}$: join of maps $\overbrace{\varphi_g * \cdots * \varphi_g}^{n+1}$

↳ vertices of $G^{*(n+1)}$ are (g, i) where $g \in G$ and $i \in [n+1]$

↳ encodes from which copy is g from

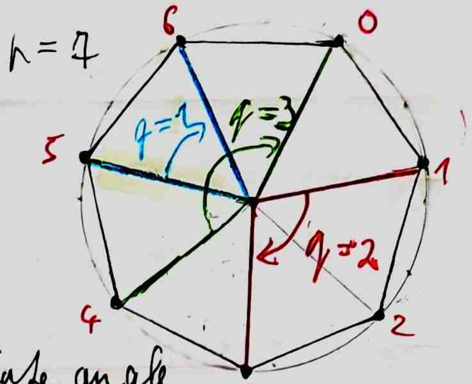
$\Rightarrow (\varphi_g * \cdots * \varphi_g)(g, i) = (g \cdot h, i)$

Claim: $G^{*(n+1)}$ with $(\varphi_g)_{g \in G}$ is a $E_n G$-space

→ clearly simplicial G-complex, n-dimensional, actions are free

→ $G^{*(n+1)}$ is $(n-1)$-connected (proof skipped)

③ $G = \mathbb{Z}_p$, p = prime, n = odd integer



- first describe $n=1$

→ let P_p be the regular p-gon ... $\|P_p\| \cong S^1$

→ actions φ_q for $q \in \mathbb{Z}_p$ ... rotation by appropriate angle

⟹ φ_q is the geom. real. of $V(P_p) \to V(P_p)$ ↙ ... P_p is an abstract simplicial complex

- general odd $n = 2k-1$

→ take k copies $P_p * \cdots * P_p =: K$

$$\|K\| \cong (S^1)^{*k} \cong (S^0 * S^0)^{*k} = (S^0)^{*k} * (S^0)^{*k} = S^{k-1} * S^{k-1} = S^{2k-1} = S^n$$

⟹ K is a triangulation of S^n

- action on $\|K\|$: $\varphi_q * \cdots * \varphi_q$ ... k-fold join, $q \in \mathbb{Z}_p$

claim: $\|K\|$ with this action is an $E_n\mathbb{Z}_p$-space

↳ connectedness follows from the fact that $\|K\| \cong S^n$ is $(n-1)$-connected

Connection with Borsuk-Ulam:

→ recall that BU2a can be restated as:

There is no $\mathbb{Z}_2$-map from $(S^n, -)$ to $(S^{n-1}, -)$

→ according to example ①, $(S^n, -)$ is an $E_n\mathbb{Z}_2$-space

Theorem (BU for E_nG-spaces): There is no G-map from an E_nG-space to an $E_{n-1}G$-space.

Theorem (Dold's theorem): Let G be a finite group with $|G| > 1$.
Let X be an n-connected G-space and let Y be a free G-space of dimension at most n (for us, Y is homeomorphic to a polyhedron).
Then there is no G-map $X \to Y$.

DELETED JOINS FOR G-SPACES

↗ 2-wise = pair-wise

Def: Let $n \geq k \geq 2$. An n-tuple $(x_1,\dots,x_n)$ is k-wise distinct if no k among x_i are equal (less than k may be).

Def: The n-fold k-wise deleted join of a space $X \subseteq \mathbb{R}^d$ is the space

$$X^{*n}_{\Delta(k)} := X^{*n} \setminus \left\{ \tfrac{1}{n}x_1 \oplus \cdots \oplus \tfrac{1}{n}x_n \;\middle|\; \begin{array}{l}(x_1,\dots,x_n) \text{ is not} \\ k\text{-wise distinct}\end{array} \right\}$$

→ in $X^{*n}_{\Delta(k)}$, all points on the "diagonal" are k-wise distinct

Def: The n-fold k-wise deleted join of a simplicial complex K is

2-wise = ↗ pairwise

$$K^{*n}_{\Delta(k)} := \left\{ A_1 \uplus \cdots \uplus A_n \in K^{*n} \;\middle|\; \begin{array}{l}(A_1,\dots,A_n) \text{ is } k\text{-wise disjoint} \\ \text{i.e. every } k \text{ among } A_i \text{ have empty intersection}\end{array} \right\}$$

G-actions on deleted joins: → forming G-spaces

① $G = S_n$ = symmetric group of permutations

↳ acts by swapping coordinates ... generalises τ

→ we can do the same for any H subgroup of S_n..

② $G = \mathbb{Z}_n$

→ all φ_g, $g \in \mathbb{Z}_n$ are determined by φ_1 ... e.g. $\varphi_2 = \varphi_{1+1} = \varphi_1 \circ \varphi_1$

→ action φ_1 on $X^{*n}_{\Delta(k)}$ resp. $K^{*n}_{\Delta(k)}$

$$\varphi_1 : t_1x_1 \oplus t_2x_2 \oplus \cdots \oplus t_nx_n \longmapsto t_2x_2 \oplus \cdots \oplus t_nx_n \oplus t_1x_1$$

$$\varphi_1 : (v, j) \longmapsto (v, j+1 \mod n)$$

↳ vertex v in j-th copy of K

these are really just cyclic permutations ...we can view $\mathbb{Z}_n \subseteq S_n$

Exercise: This action is free ⟺ n is a prime

Fact: Let p be a prime, $d \geq 1$. Then there ∃ $\mathbb{Z}_p$-map

⊕⊕
$$(\mathbb{R}^d)^{*p}_{\Delta(p)} \xrightarrow{\mathbb{Z}_p} S^{(d+1)(p-1)-1}$$

↗ $\mathbb{Z}_2$-action = cyclic shifts from ②

↘ some suitable free $\mathbb{Z}_2$-action

THE TOPOLOGICAL TVERBERG THEOREM

Theorem (Topological Tverberg theorem): Let p be a prime, let $d \geq 1$, and put $N := (d+1)(p-1)$. Then for $\forall$ continuous map $f: \|\sigma^N\| \to \mathbb{R}^d$ there are p disjoint faces $A_1, \dots, A_p \in \sigma^N$ s.t.

$$f[\|A_1\|] \cap \dots \cap f[\|A_p\|] \neq \emptyset$$ ... images of the faces intersect

Note: Radon is Tverberg for $p = 2$

Conjecture: It in fact holds even when p is not a prime

Proof: Suppose for contradiction that $f: \|\sigma^N\| \to \mathbb{R}^d$ violates the theorem.

→ take the p-fold join of f, and restrict it to the p-fold 2-wise deleted join

$$f^{*p}: \|(\sigma^N)^{*p}_{\Delta(2)}\| \longrightarrow (\mathbb{R}^d)^{*p}_{\Delta(p)}$$ → need to check that this is legit

→ note that $\|(\sigma^N)^{*p}_{\Delta(2)}\| \subseteq \|(\sigma^N)^{*p}\| \subseteq \|\sigma^N\|^{*p}$

- we are mapping p-tuples of pair-wise disjoint faces $A_1, \dots, A_p$ of σ^N
- by our assumption of f, the images of these faces have empty intersection

$\Rightarrow$ if $x_1 \in \|A_1\|, \dots, x_p \in \|A_p\|$, then $f^{*p}(t_1x_1 \oplus \dots \oplus t_px_n) \in (\mathbb{R}^d)^{*p}_{\Delta(p)}$

$$f^{*p}: t_1x_1 \oplus \dots \oplus t_px_p \longmapsto t_1 f(x_1) \oplus \dots \oplus t_n f(x_n) \neq \frac{1}{p}z + \dots + \frac{1}{p}z$$

→ cannot all be the same

Note: $(\sigma^N)^{*p}_{\Delta(2)} \cong \left((\sigma^0)^{*(N+1)}\right)^{*p}_{\Delta(2)} \overset{⊛}{\cong} \left((\sigma^0)^{*p}_{\Delta(2)}\right)^{*(N+1)} \cong (D_p)^{*(N+1)}$

σ_0 = single point

↳ p copies of 1 point and $\Delta(2)$ does nothing ⇒ we get D_p = p discrete points

⊛ **Exercise**: $(K * L)^{*p}_{\Delta(2)} \cong K^{*p}_{\Delta(2)} * L^{*p}_{\Delta(2)}$

→ we have the cyclic shift action on $(\sigma^N)^{*p}_{\Delta(2)}$ and it carries over to $(D_p)^{*(N+1)}$

Hence: $$\|(D_p)^{*(N+1)}\| \xrightarrow{\mathbb{Z}_p} (\mathbb{R}^d)^{*p}_{\Delta(p)} \xrightarrow{\mathbb{Z}_p} S^{(d+1)(p-1)-1} = S^{N-1}$$

cyclic shift action "

$\mathbb{Z}_p$ map given by f^{*p}

$\mathbb{Z}_p$-map & free $\mathbb{Z}_p$-action on S^{N-1} given by Fact ⊛ ⊛

Fact: $\|(D_p)^{*(N+1)}\|$ is $(N-1)$-connected

we have a $\mathbb{Z}_p$ map from an $(N-1)$-connected $\mathbb{Z}_p$-space into a free $\mathbb{Z}_p$-space of dimension $N-1$ $\Rightarrow$ ↯ Dold's theorem ▨

Theorem (Tverberg's theorem): For any $d \geq 1$ and $r \geq 2$, any set of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into r disjoint subsets $A_1, \dots, A_r$ s.t. their convex hulls intersect

Tverberg partition

Note: Radon is Tverberg for $r=2$.

Affine reformulation: For $d \geq 1$, $r \geq 2$, put $N := (d+1)(r-1)$.
Then for $\forall$ affine map $f: \|\sigma^N\| \to \mathbb{R}^d$, there exist p disjoint faces $A_1, \dots, A_p \in \sigma^N$ s.t. $f[\|A_1\|] \cap \dots \cap f[\|A_p\|] \neq \emptyset$

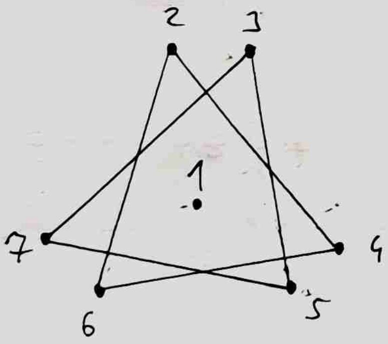
→ equivalent by the same argument as Radon's theorem

for $r =$ prime it follows from the topological Tverberg theorem because every affine map is continuous

SIERKSMA'S CONJECTURE

→ Tverbergs theorem says that a Tverberg partition exists, but how many are there?

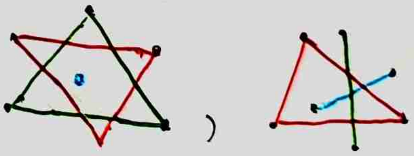
Example: $d=2$, $r=3$ .... $(d+1)(r-1)+1 = 7$



(1)(246)(357)
(1)(247)(356)
(1)(256)(347)
(1)(346)(257)

all Tverberg partitions ⇒ 4 in total

Note: If the points would be farther apart, the partitions would change:



, etc.

Fact: A similar configuration as in the example for general d and r has exactly $(r-1)!^d$ Tverberg partitions.

Conjecture (Sierksma): Every collection of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ has at least $(r-1)!^d$ Tverberg partitions.

Theorem (Vrećica, Živaljević): If p is a prime, then there are at least

$$\frac{1}{(p-1)!} \binom{p}{2}^{(d+1)(r-1)/2}$$

Tverberg partitions

→ the only known proof is topological and much harder than Tverberg's theorem