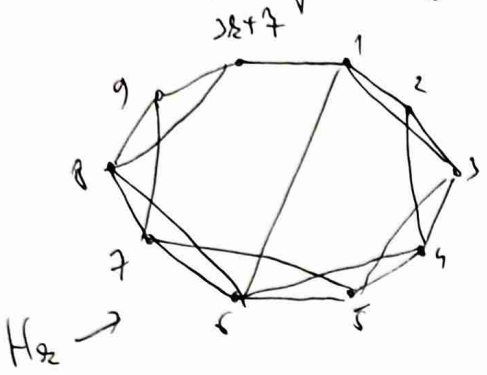
RIGID GRAPHS

→ graph / digraph

- G is rigid if the only homomorphism $G \to G$ is the identity



$\chi = 4$, but after deleting any vertex $\chi = 3$

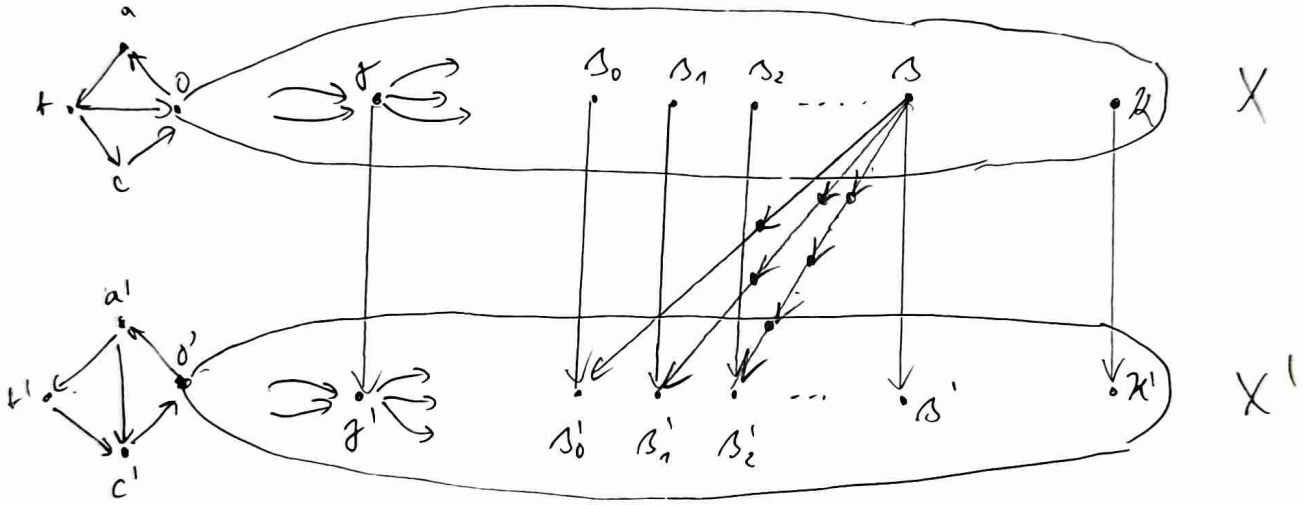
→ homomorphisms = automorphisms

$G \mapsto G$, $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 3$ since it is adj to 1, 2 deg 3

$\not\in \triangle$ not adj to deg 3

Theorem (Vopěnka, Pultr, Hedrlín, 65'): There ∃ rigid digraph on every ∞ vertex set.

Proof (Nešetřil, 2002): Let $X = 2^\kappa + 1$ for some cardinal κ



- $\forall \delta < \gamma \leq 2^\kappa$ add arcs $\delta \to \gamma$, $\delta' \to \gamma'$
- $\forall \gamma \leq 2^\kappa$ add arc $\gamma \to \gamma'$
- $\forall$ limit $\beta \leq 2^\kappa$ with $cf(\beta) = \omega$ choose cofinal $\langle \beta_n \mid n < \omega \rangle$ and add disjoint paths $\beta \to \beta_n'$ of length $2+n$

Let $f: G \to G$ be a homomorphism. $A := \{a, b, c, 0\}$, $A' = \{a', b', c', 0'\}$

- $A \overset{h}{\twoheadrightarrow} A'$ ... in a 3-cycle, $f(0) \in \{0, 0'\}$ ... in fact $f(0) = 0$, $f(0') = f(0')$
- Thus $f[X] \subseteq X$, $f[X'] \subseteq X'$ ... arcs from $0, 0'$
- $f(\gamma) = \gamma'$ ... $\gamma \to \gamma'$ are the only arcs $X \to X'$
- $\delta < \gamma \Rightarrow f(\delta) < f(\gamma)$ ... monotonicity
- for contradiction let α be the least ordinal s.t. $f(\alpha) \neq \alpha$
  ↳ $f(\alpha) > \alpha$ since if $f(\alpha) < \alpha$ then $f(f(\alpha)) < f(\alpha)$ ↯

• define $\gamma_0 := \alpha$, $\gamma_1 := f(\alpha)$, $\gamma_{n+1} := f(\gamma_n)$ ... $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$

• limit $\beta := \sup \gamma_n$ ... consider $\langle \beta_n \mid n < \omega \rangle$ ... $[\beta_n]$ ↺ $[\gamma_n]$

β limit → $f(\beta) := \lambda$ limit ... path of length 2 : $\beta \to \beta'_0$

Want: $\forall n$: $f(\beta_n) = \lambda_n$

- current construction probably does not guarantee that
  ↳ the cofinal sequences for distinct limit ordinals can intersect
- solution: augment each path $\beta \to \beta'_n$ with 2- and 5- cycles



Now: $\beta = \sup \beta_n = \sup \gamma_n$ ⇒ $\sup f(\beta_n) = \sup f(\gamma_n) = \sup \gamma_n = \beta$ } $f(\beta) = \beta$

Hence $f(\beta_n) = \beta$ for $\forall n$. $\sup \lambda_n = f(\beta)$

$\gamma_n < \beta_m \leq \gamma_{n+1}$ ⇒ $f(\gamma_n) < f(\beta_m)$

for some n, m $\gamma_{n+1} < \beta_m$ ↯ monotonicity ∎

Note: We take $\chi = \kappa + 1$ just in case $\mathrm{cf}(\kappa) = \omega$ and $\sup \gamma_n = \kappa$

THE INDICATOR TECHNIQUE

Replacement operation: G digraph, J graph with vertices $a, b \in V_J$



↳ always irreflexive

Def: J is a strong replacement graph if $\forall$ digraph G $\forall$ hom. $f: J \to G * J$, the image $f[J] \subseteq J_{xy}$ for some $xy \in E(G)$

Def: Indicator = rigid strong replacement graph

Lemma: J indicator. If G, H are digraphs without isolated vertices, then $\forall$ homomorphism $h: G * J \to H * J$ is equal to $f * J$ for some hom $f: G \to H$

Pf: No isolated vertices ⇒ $\forall$ vertex $\in$ some copy J_{xy}

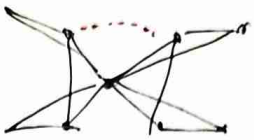
- J is strong ⇒ $\forall$ copy J_{xy} in G maps to a copy J_{uv} in H
- J is rigid ⇒ this is identity
- f is the homomorphism induced by this. ∎

Corollary: Let J be indicator.

① G rigid digraph $\Rightarrow G * J$ rigid graph

② $\langle G_i \mid i \in I \rangle$ incomparable family of digraphs $\Rightarrow \langle G_i * J \mid i \in J \rangle$ inc. fam. of graphs

Example: The graphs H_k are indicators

↳ Δ-connected , as  it is strong replacement graph

Corollary: $\forall \kappa \geq \omega$ $\exists$ incomparable family of rigid graphs $\langle G_\alpha \mid \alpha < 2^\kappa \rangle$ of size $|G_\alpha| = \kappa$.

Proof: Take the rigid digraph G of size κ. $\rightsquigarrow$ rigid graph H of size κ

- H rigid $\Rightarrow$ no two of the 2^κ possible orientations of H are isomorphic

$\Rightarrow$ They are incomp. fam. of rigid digraphs $\rightsquigarrow$ indicators: rigid graphs. ☐

BINARY I-SYSTEMS

Def: Binary I-system $G = \langle \bar{E}_i \mid i \in I \rangle$ on a set V is a digraph with $|I|$ types of arcs

Def: For a family of indicators $\mathcal{J} = \langle J_i \mid i \in I \rangle$ we define $G * \mathcal{J}$ as expected

Fact: $\forall n \in \omega$ there $\exists$ incomp. family of Δ-connected indicators $\langle J_i \mid i < n \rangle$

Theorem: If $\mathcal{J}$ is and G, H are binary I-systems without loops & isolated vertices, then $\forall$ hom. $h: G * \mathcal{J} \to H * \mathcal{J}$ is equal to $f * \mathcal{J}$ for some hom. $f: G \to H$

Proof: The same as for digraphs and for showing that H_k is strong.

VOPĚNKA'S PRINCIPLE:

VP: There $\nexists$ proper class of rigid graphs that is an incomparable family

Huge cardinal $\Rightarrow$ VP $\Rightarrow$ Super compact

$\Rightarrow$ proper class of measurable

Note: $\nexists$ proper class of measurable $\Rightarrow$ The homomorphism order of digraphs $(\mathbb{D}, \leq)$ is universal (embeds $\forall$ ordered class $(P, \leq)$).

Hedrlín, Kučera 1980

↳ VP ~ full discrete subcategory ~ would be an obstruction

Ways to study VP:

① axiom schema VS in ZFC } Hamkins 2016: $\mathrm{Con}(\mathrm{NBG} + \mathrm{VS}) \not\Rightarrow \mathrm{Con}(\mathrm{NBG} + \mathrm{VP})$,

② single axiom VP in NBG } but NBG + VP is conservative over ZFC + VS

③ inaccessible κ s.t. $(V_\kappa, \in) \models \mathrm{VP}$ where we interpret classes as subsets of V_κ. κ = Vopěnka cardinal

→ They have the same consequences about sets

WQOs:

Def: $(Q, \leq)$ is WQO if $\forall \langle q_i \mid i \in \omega \rangle \; \exists i < j : q_i \leq q_j$

Observation: TFAE:

1. Q is WQO
2. Q has no antichains & decreasing sequences of size ω
3. $\forall \langle q_i \mid i \in \omega \rangle \; \exists$ monotone $f : \omega \to \omega : i < j \Rightarrow f(i) \leq f(j)$

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

(3) $\Rightarrow$ (1) Ramsey $\omega \to (\omega)^2_3$

Thm (Graph minor theorem): Finite graphs are WQO by minors.

Thm (Thomas, 1988): Graphs of size 2^ω are not WQO by minors

Thm (Komjáth, 1995): $\forall \kappa > \omega \; \exists$ antichain of 2^κ many graphs of size κ. (w.r.t. minors)

Conjecture (Thomas): Countable graphs are WQO by minors

Def (Shelah, 1982): Q is κ-WQO if $\forall \langle q_\alpha \mid \alpha < \kappa \rangle \; \exists \alpha < \beta : q_\alpha \leq q_\beta$

The well-ordering number of Q is the least cardinal κ s.t. Q is κ-wqo.

Observation: If κ is weakly compact then TFAE

1. Q is κ-WQO
2. Q has no antichains & decreasing sequences of size κ
3. $\forall \langle q_\alpha \mid \alpha < \kappa \rangle \; \exists$ monotone $f : \kappa \to \kappa : \alpha < \beta \Rightarrow f(\alpha) \leq f(\beta)$

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

(3) $\Rightarrow$ (1) $\kappa \to (\kappa)^2_3$

Examples:

- finite graphs are ω_1-WQO by anything
- $\mathcal{G}^{\leq \kappa}$ = graphs of size $\leq \kappa$ are $(2^\kappa)^+$-WQO by anything
- Vopěnka, Pultr, Hedrlín: we cannot do better for homomorphisms
- Komjáth: we cannot do better for minors
- $\mathcal{G}$ = all graphs are not κ-WQO for any κ by anything interesting

Def: Q is class-WQO if $\forall \langle q_\alpha \mid \alpha \in On \rangle \; \exists \alpha < \beta : q_\alpha \leq q_\beta$

Theorem: $\mathcal{G}$ = class of all graphs. TFAE

1. Vopěnka's principle $\Leftrightarrow$ axiom of global choice
2. $\nexists \langle G_\alpha \mid \alpha \in On \rangle$ rigid graphs s.t. $\forall \alpha \neq \beta : G_\alpha \not\to G_\beta$ & $G_\beta \not\to G_\alpha$
3. $\mathcal{G}$ is class-WQO by homomorphisms
4. $\mathcal{G}$ is class-WQO by subgraphs
5. $\mathcal{G}$ is class-WQO by induced subgraphs

Bagaria, Wilson 2023: (2) $\Rightarrow$ (4)

Proof: Clearly (1) ⇒ (2), (5) ⇒ (4) ⇒ (3) ⇒ (2)

(2) ⇒ (1): We well-order any proper class of graphs $\mathcal{H}$ as On, using the axiom of global choice
- for $\alpha \in On$ let $\mathcal{H}_\alpha = \mathcal{H} \cap V_\alpha$. By AC, each $\mathcal{H}_\alpha$ can be well-ordered
- using global choice, chose a well-order $\prec_\alpha$ for each $\mathcal{H}_\alpha$
- well-order $\mathcal{H}$ by $H \prec G$ iff $rank(H) < rank(G)$ or $rank(H) = rank(G) = \alpha$ & $H \prec_\alpha G$
- every initial segment is a set $\Rightarrow (\mathcal{H}, \prec) \cong (Ord, <)$

(2) ⇒ (5): Let $\langle G_\alpha \mid \alpha \in On \rangle$ be a sequence of graphs, $G_\alpha = (V_\alpha, E_\alpha)$
- WLOG no two isomorphic
- Since $\forall \kappa$ there are only 2^κ many non-isom. graphs of size $\leq \kappa$, we can WLOG assume $\omega \leq |V_\alpha| < |V_\beta|$ for all $\alpha < \beta$
- view graph ~ symmetric digraph. Consider the binary relational system

$$G_\alpha^+ := (V_\alpha, E_\alpha, R_\alpha, K_\alpha, C_\alpha)$$

R_α ... rigid graph on V_α; K_α ... complete graph on V_α; $C_\alpha = K_\alpha - E_\alpha$ ... complement of E_α

- let $\mathcal{J}$ be a family of 4 incomparable Δ-connected indicators
- $H_\alpha = G_\alpha^+ * \mathcal{J}$ ... rigid due to R_α
- by (2) there $\exists \alpha \neq \beta$ and $h : H_\alpha \to H_\beta$ (and our choice of $\mathcal{J}$)
- let $f : G_\alpha^+ \to G_\beta^+$ by homomorphism satisfying $h = f \circ \mathcal{J}$
- f must be injective due to K_α $\Rightarrow \alpha < \beta$
- f preserves edges due to E_α
- f preserves non-edges due to C_α

} f witnesses $G_\alpha \subseteq_i G_\beta$

PROBLEMS:

Problem: Is VP equivalent to $\mathcal{G}$ being WQO by minors? (or topological minors ...)

Theorem (Shelah, 1982): The well-ordering number of the class of all trees with subgraphs is a (mildly) large cardinal $\aleph_\kappa$ ... least beautiful cardinal
↳ weakening of ω-Erdős, natural in context of DQOs → limit of weakly compact

Problem: What about graphs of bounded tw?

Thm (Thomas 1989): Bounded tw → WQO by minors

Thm (S): Bounded td ⇒ WQO by induced subgraphs
Bounded α ⇒ WQO by induced subgraphs

} infinite graphs

Problem: Is "trees are class-WQO by $\subseteq$" a theorem of NBG?
↳ it is much weaker than VP ... Shelah