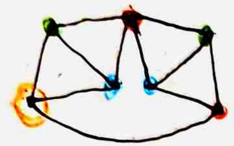
GRAPHS ON $\mathbb{R}$ ARE INTERESTING

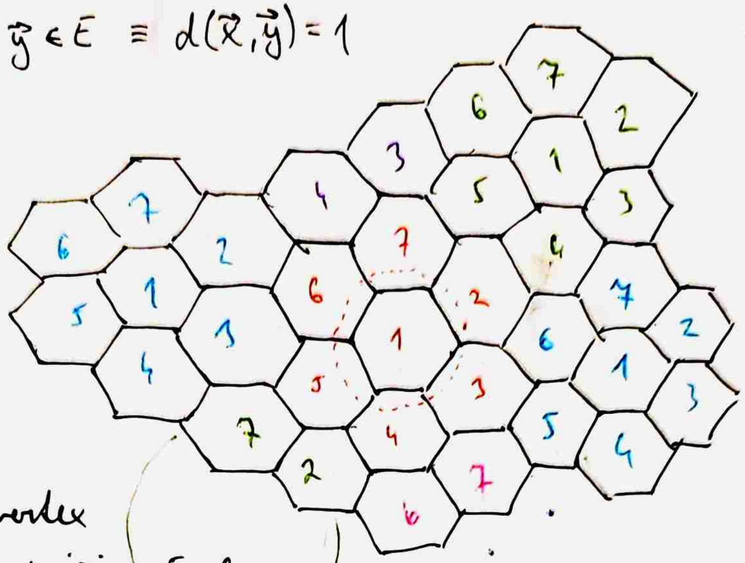
1 Chromatic number of the plane – Hadwiger, Nelson 1950s

$G = (\mathbb{R}^2, E)$ ... $\vec{x}\vec{y} \in E \equiv d(\vec{x}, \vec{y}) = 1$

UNITDISTANCE GRAPH

$4 \leq \chi(G) \leq 7$





2018: $\chi(G) \geq 5$

↳ computer found 1581-vertex unit distance graph requiring 5 colors

- de Brujin-Erdős theorem: $\chi(G) \leq k \iff \forall$ finite $H \subseteq G: \chi(H) \leq k$

  ↳ requires AC $\Rightarrow \chi(G)$ could be different in ZF × ZFC

2 Conditional chromatic number graph – Shelah, Soifer 2003

$G = (\mathbb{R}, E)$ ... $xy \in E \equiv |x - y| = \sqrt{2} + q \;,\; q \in \mathbb{Q}$

! $\chi(G)$ depends on axioms

- ZFC $\rightsquigarrow \chi(G) = 2$
- ZF+DC+LM $\rightsquigarrow \chi(G) > \omega$ ... Solovay's model, requires consistency of an inaccessible cardinal

  ↳ $\forall A \subseteq \mathbb{R}$ is Lebesgue measurable

LARGE CHROMATIC NUMBER & LARGE GIRTH

- girth := length of a shortest cycle
- odd girth := length of a shortest odd cycle

1958, Erdős: There exist graphs with arbitrarily large girth & chrom. number
↳ probabilistic method ↳ locally trees

1966, Lovász: The first explicit construction

GEOMETRIC GRAPHS

Def: G is a unit-distance graph in $\mathbb{R}^d$ $\equiv \exists f: V_G \hookrightarrow \mathbb{R}^d$ s.t. $uv \in E \Rightarrow d(f(u), f(v)) = 1$

diameter graph in $\mathbb{R}^d \equiv \exists f: V_G \hookrightarrow \mathbb{R}^d$ s.t. $uv \in E \Rightarrow d(f(u), f(v)) = \max_{x,y \in V} d(f(x), f(y))$

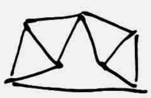
orthogonality graph in $S \subseteq \mathbb{R}^d \equiv \exists f: V_G \hookrightarrow S$ s.t. $uv \in E \Rightarrow f(u) \perp f(v)$

Borsuk conjecture: $\chi(G) \leq d+1$ for any diameter graph in $\mathbb{R}^d$ ... $d+1$ for regular simplex
↳ disproved, current record: $\chi(G) \geq (1.225 + o(1))^{\sqrt{d}}$ ... Raigorodskii
↳ existence of diameter graphs with $\chi(G)$ exponential in d ... open problem

Unit distance graphs: current record: $\chi(G) \geq (1.239 + o(1))^d$ ... Raigorodskii

? What if we also want high girth ?

Erdős 1975: There probably is no unit distance graph in $\mathbb{R}^2$ with $\chi = 4$ and girth ≥ 4



$\chi = 4$
girth $= 3$

- Nick Wormald 1978 ... found such a graph on 6448 vertices
- Robert Hochberg, Paul O'Donnell 1996 ... 23 vertices
- O'Donnell 1999 ... $\chi = 4$ and arbitrary girth exist

} unit distance in $\mathbb{R}^2$

Kupavskii 2012: $\forall g \, \exists c_g > 1$ s.t. there $\exists$ unit-distance graphs in $\mathbb{R}^d$ with girth $\geq g$ & $\chi(G) \geq (c_g + o(1))^d$

↳ Sagdeev & Raigorodskii (2017-2019) ... $c_g \geq 1 + 1/g^{2+o(1)}$

→ the paper we will be studying proves the first universal lower bound without any dependency on g

Orthogonal graphs

$S^{d-1} = \{a \in \mathbb{R}^d \mid \|a\| = 1\}$ ... unit distance sphere in $\mathbb{R}^d$

Raigorodskii

current record: $\exists$ orthogonality graphs in S^{d-1} with $\chi(G) \geq (1.154 + o(1))^d$

GEOMETRIC GRAPHS WITH EXPONENTIAL $\chi(G)$ AND ARBITRARY GIRTH

↳ by Matija Bucić & James Davies

Theorem [1]: There $\exists$ unit-distance graphs in $\mathbb{R}^d$ with $\chi(G) \geq (1.074 + o(1))^d$ that have arbitrarily large girth

Theorem [2]: There $\exists$ diameter graphs in $\mathbb{R}^d$ with $\chi(G) \geq (1.107 + o(1))^{\sqrt{d}}$ that have arbitrarily large girth

Theorem [3]: There $\exists$ orthogonality graphs in $\mathbb{S}^{d-1}$ with $\chi(G) \geq (1.074 + o(1))^d$ that have arbitrarily large girth

→ they first show slightly weaker bounds via a self contained proof

1.067 instead of 1.074 and 1.096 instead of 1.107

→ we will focus on this, the stronger bounds are "black box arguments"

- [3] implies [2] and [1] $(1.096 = 1.067^{\sqrt{2}}$ and $1.107 = 1.074^{\sqrt{2}})$

- $d := 8p$ where $p > 2$ prime (if d is not of this form, take $d' = 8p$ where p is the largest prime $< d/8$ ... Fact: the loss only impacts the $o(1)$ term)
- m ... large positive integer, $[m] = \{1, 2, \dots, m\}$

$$V' := \left\{ v \in \{\pm 1\}^{\frac{d}{2}} \;\middle|\; \sum_{i=1}^{d/2} v_i = 0 \;\; \& \;\; v_{\frac{d}{2}} = 1 \right\}$$

- for $v \in V'$ and $j \in [m]$ let $v(j) := \left(v \cdot \cos\frac{2\pi j}{m}, \; v \cdot \sin\frac{2\pi j}{m}\right) \in \mathbb{R}^d$

$$V := \{ v(j) \mid v \in V', \; j \in [m] \}$$

$V \subseteq \sqrt{\frac{d}{2}}\, \mathbb{S}^{d-1}$ ... $\|v(j)\|^2 = \sum_{i=1}^{d/2} v_i^2 \cos^2\left(\frac{2\pi j}{m}\right) + v_i^2 \sin^2\left(\frac{2\pi j}{m}\right) = \sum_{i=1}^{d/2} v_i^2 \cdot 1 = \frac{d}{2}$

$u \perp v \Rightarrow u(j) \perp v(i)$ ... $\sum_{i=1}^{d/2} u_i v_i = 0 \Rightarrow \sum_{i=1}^{d/2} u_i v_i \cos\left(\frac{2\pi j}{m}\right)\cos\left(\frac{2\pi i}{m}\right) + (\sin)$

$= \cos \cdot \cos \cdot \sum u_i v_i + (\sin) = 0$

- $G' := (V', \{uv \mid u \perp v\})$
- $G'^m :=$ blow-up of G' ... replace each vertex v with an ind. set $\{v(j) \mid j \in [m]\}$

  ... replace each edge with $K_{m,m}$

$\Rightarrow G'^m$ is an orthogonality graph in $\sqrt{\frac{d}{2}}\, \mathbb{S}^{d-1}$ with vertex set V

↳ after scaling by a factor of $\sqrt{\frac{2}{d}}$, we get orthogonality graph in $\mathbb{S}^{d-1}$

$\chi(G') = \chi(G'^m)$

Plan ① $\chi(G') = \chi(G'^{lm}) \geq (1.067+o(1))^d$

② we sample random graphs G_0 by removing some edges from G'^{lm}

③ using the probabilistic method we show that $\exists$ outcome where G_0 does not have a lot of short cycles

④ we remove 1 edge from each cycle and show that χ does not decrease

Lemma: $u,v \in V'$ ... $\langle u|v\rangle = 4(t-p)$ where $t = \# i : v_i = u_i = 1$

$$\boxed{d = 8p\ ,\ u,v \in \{\pm 1\}^{d/2}}$$

Proof: $\langle u|v\rangle = N_{1,1} + N_{-1,-1} - N_{1,-1} - N_{-1,1}$

- $N_{1,1} = t$, $N_{1,1} + N_{1,-1} = N_{1,\cdot} = \frac{d}{4} = 2p$ ... $\sum_{i=1}^{d/2} v_i = 0 \Rightarrow$ exactly half is 1

$\Rightarrow N_{1,-1} = 2p - t$, similarly $N_{1,1} + N_{-1,1} = 2p \Rightarrow N_{-1,1} = 2p - t$

$\Rightarrow N_{1,1} + N_{1,-1} + N_{-1,1} + N_{-1,-1} = 4p \quad \Rightarrow N_{-1,-1} = t$

$\Rightarrow \langle u|v\rangle = t + t - 2p + t - 2p + t = 4(t-p)$ ∎

Theorem: $\chi(G') \geq (1.067 + o(1))^d$

Proof: If we partition V' into $\chi(G') =: k$ sets $A_1, \dots, A_k$, then each A_i contains no orthogonal pair of vectors ... they would share an edge

claim: If A contains no orthogonal pair of vectors, then $|A| \leq 2\binom{4p}{p-1}$

$$\Rightarrow \chi(G') \geq \frac{|V'|}{2\binom{4p}{p-1}} = \frac{\binom{4p}{2p}}{4\binom{4p}{p-1}} \overset{\text{STIRLING}}{=} (27/16 + o(1))^p = (27/16 + o(1))^{d/8} \geq (1.067 + o(1))^d$$

proof: Define polynomials $f_v : \{\pm 1\}^{4p} \to \mathbb{Z}_p$ for each $v \in V'$ as

$$f_v(\vec{x}) := \prod_{i=1}^{p-1} (\langle x|v\rangle - i) \bmod p \quad \text{... poly in } x_1, \dots, x_{4p}$$

$u,v \in V'$, $f_v(u) \not\equiv 0 \pmod p \iff \langle u|v\rangle \equiv 0 \pmod p$

- $\Leftarrow$ $f_v(u) = (p-1)(p-2)\cdots(1) \not\equiv 0 \pmod p$
- $\Rightarrow$ if $\langle u|v\rangle \not\equiv 0 \pmod p$, then $\exists$ term $= p \Rightarrow f_v(u) \equiv 0 \pmod p$

$f_v(u) \not\equiv 0 \pmod p \iff u = v \ \vee\ u \perp v$

↳ Lemma: $\langle u|v\rangle = 4(t-p)$ ... $t \geq 1 \because u_{4p} = v_{4p} = 1$ & both u,v have exactly $2p$ ones

↳ $p > 0$ ... $4(t-p) \equiv 0 \pmod p \iff t - p \equiv 0 \pmod p \iff t = 2p \ \vee\ t = p$ from $(*)$

FINISHING THE PROOF

- for every $f_v(x)$ define $\overline{f_v}: \{\pm1\}^{4p} \to \mathbb{Z}_p$ by replacing each occurrence of x_i^r by $x_i^{r \bmod 2}$

$x_i^r \equiv x_i^{r \bmod 2} \pmod p$ for $x_i \in \{\pm 1\}$ $\Rightarrow$ $f_v(u) = \overline{f_v}(u)$ for $u \in V'$

↳ $x_i = 1$ … $1^r = 1^0 = 1^1 = 1$, $x_i = -1$ … $(p-1)(p-1) = p^2 - 2p + 1 \equiv 1 \pmod p$

$\{\overline{f_v} \mid v \in A\}$ are linearly independent over $\mathbb{Z}_p$

↳ $\overline{f_v}(v) \not\equiv 0$ but $\overline{f_v}(u) \equiv 0$ for $v \neq u \in A$ since A contains no orthogonal pairs

↳ suppose e.g. $f_w = f_v + f_u$ … $f_w(w) \not\equiv 0$, but $f_v(w) + f_u(w) \equiv 0$

→ the polynomials $\overline{f_v}$ are spanned by the set of monomials

$X := \{x_{i_1} x_{i_2} \cdots x_{i_k} \mid 1 \le i_1 < i_2 < \dots < i_k \le 4p,\ k \le p-1\}$

→ $|X| = \binom{4p}{0} + \binom{4p}{1} + \dots + \binom{4p}{p-1} \le 2\binom{4p}{p-1}$ … fact

→ $\{\overline{f_v} \mid v \in A\}$ is linearly independent, and spanned by $2\binom{4p}{p-1}$ elements

$\Rightarrow$ $|A| = |\{\overline{f_v} \mid v \in A\}| \le |\text{basis}| \le |X| \le 2\binom{4p}{p-1}$ ■

ESTIMATING #SHORT CYCLES

→ let g be given, we want a graph with girth $\ge g$

$k := \chi(G')$, $m := 2^{8pg} \cdot (4k)^{4g} \cdot (k-1)^2$

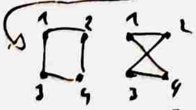
→ recall that G'^m is an orthogonality graph in S^{d-1}, $\chi(G'^m) = k$

$n = m \cdot 2^{4p} \ge |V(G'^m)|$ … simple upper bound on $V(G'^m)$

→ sample $G_0 \sim G'^m(q)$ … independently randomly take each edge with prob $q := \frac{n^{1/(2g)}}{n}$

$$\mathbb{E}[\#\text{cycles of length} = l \text{ in } G_0] \le \binom{n}{l} \cdot l! \cdot q^l \le \frac{n^l}{l!} \cdot l! \cdot q^l = (nq)^l = n^{l/(2g)}$$

- choose an l-element subset
- $l!$ permutations, but the rotation / direction do not matter
- prob of including all l edges



$(1,2,3,4) \simeq (4,1,2,3) \simeq (4,3,2,1)$

$$\mathbb{E}[\#\text{cycles of length} < g] \le \sum_{l=3}^{g-1} n^{l/(2g)} \le \sum_{l=0}^{g-1} (n^{1/(2g)})^l \overset{*}{\le} (n^{1/(2g)})^g = n^{\frac{1}{2}} = \sqrt{n}$$

* $n^{1/(2g)} \ge 2$ … $n = 2^{4p} \cdot m$ and $m \ge 2^{8pg} \rightsquigarrow (2^{8pg})^{\frac{1}{2g}} = 2^{4p} > 2$

Markov inequality: $P[X > t \cdot \mathbb{E}X] < \frac{1}{t}$ … finitely many outcomes

$P[\#\text{short cycles} > 2\sqrt{n}] < \frac{1}{2}$

ARGUING THAT χ DOES NOT DECREASE

$m_0 := \frac{m}{k-1} = 2^{8hg} \cdot (4k)^{4g} \cdot (k-1)^2$ $\qquad \rightarrow E(U,W) = $ edges between U and W

notation: $u, w \in V(G')$, U, W = vertices of G'^m corresponding to u resp. w

claim: $P\left[\exists uw \in E(G') \; \exists U, W \text{ of size } m_0 \text{ s.t. } |E(U,W)| \leq 2\sqrt{m}\right] < \frac{1}{2}$ $\qquad P_*$

$\Rightarrow P\left[\# \text{short cycles} \leq 2\sqrt{m}\right] > \frac{1}{2}$ $\quad$ & $\quad P\left[\forall uw \in E \; \forall U, W : |E(U,W)| > 2\sqrt{m}\right] > \frac{1}{2}$

$\Rightarrow \exists$ outcome G_0 in which both are satisfied

- $G := G_0$ after removing an arbitrary edge from each cycle

Theorem: $\chi(G) = \chi(G')$ ... giving Theorem $\boxed{3}$ with base $= 1.067$

Proof: Suppose $\exists$ proper coloring $\chi : V(G) \rightarrow [k-1]$ by $k-1 < k = \chi(G')$ colors

$\rightarrow$ define a coloring χ' of G' by coloring v using a majority color of the vertices of G corresponding to v

$\rightarrow \chi(G') > k-1 \Rightarrow \exists uw \in E(G')$ s.t. $\chi'(u) = \chi'(w)$

$\Rightarrow \exists U, W$ of size $\geq \frac{m}{k-1} = m_0$ with the same color

$\rightarrow$ from definition of G there $\exists$ $U-W$ edge $\lightning$ χ is not proper $\blacksquare$

PROOF OF CLAIM

$\#$ choices for U, W (over all $uw \in E(G')$) $\leq |E(G')| \cdot \binom{m}{m_0}^2$ $\qquad$ union bound

$P\left[|E(U,W)| \leq 2\sqrt{m}\right] = P\left[\begin{matrix}\forall F \subseteq U \times W \\ |F| = 2\sqrt{m}\end{matrix} : \begin{matrix}\text{there are no} \\ \text{edges from } F^c \text{ in } G\end{matrix}\right] \leq \binom{m_0^2}{2\sqrt{m}} (1-q)^{m_0^2 - 2\sqrt{m}}$

$\Rightarrow P_* \leq |E(G')| \cdot \binom{m}{m_0}^2 \cdot \binom{m_0^2}{2\sqrt{m}} (1-q)^{m_0^2 - 2\sqrt{m}} \leq 2^{8h} \cdot 2^{2m} \cdot 2^{4\sqrt{m}\log_2(m_0)} \cdot 2^{1 - q m_0^2} < \frac{1}{2}$

$|V(G')| \leq 2^{4h} \Rightarrow |E| \leq |V|^2 \leq 2^{8h}$ $\qquad$ $\sum_{i=1}^{m} \binom{m}{i} = 2^m \Rightarrow \binom{m}{m_0} \leq 2^m$

$\binom{N}{K} \leq N^K \Rightarrow \binom{m_0^2}{2\sqrt{m}} \leq (m_0^2)^{2\sqrt{m}} = (m_0)^{4\sqrt{m}} = 2^{\log_2(m_0) \cdot 4\sqrt{m}}$

$(1-q)^{m_0^2} \leq 2^{-q m_0^2}$ ... $1 - q \leq e^{-q} \leq 2^{-q}$ $\qquad$ $\boxtimes$ Bernoulli's ineq: $(1-x)^r \geq 1 - rx$

$(1-q)^{-2\sqrt{m}} \leq 2 \iff (1-q)^{2\sqrt{m}} \geq \frac{1}{2}$ ... $\overset{\boxtimes}{\geq} 1 - 2q\sqrt{m} = 1 - 2m^{\frac{1}{2g} - 1 + \frac{1}{2}} = 1 - 2m^{\frac{1-g}{2g}} = 1 - \text{almost } 0 \geq \frac{1}{2}$ $\qquad$ negative, enormous

$8h + 2m + 4\sqrt{m}\log_2(m_0) + 1 - q m_0^2 < -1 \iff q m_0^2 > 8h + 2m + 4\sqrt{m}\log_2(m_0) + 2$

$q m_0^2 = m^{\frac{1}{2g} - 1} \cdot \frac{m^2}{(k-1)^2} = \left(m\, 2^{4h}\right)^{\frac{1}{2g} - 1} \cdot \frac{m^2}{(k-1)^2} = \frac{m^{1 + 1/2g}}{2^{4h}(k-1)^2} \cdot 2^{\frac{4h}{2g}} \geq \frac{m^{1 + \frac{1}{2g}}}{2^{4h} \cdot k^2} > 4m \geq$

want: $m^{\frac{1}{2g}} > 4 \cdot 2^{4h} \cdot k^2$ ... we have $m^{\frac{1}{2g}} = 2^{4h} (4k)^2 (k-1)^{\frac{1}{g}} \geq 4^2\, 2^{4h} \cdot k^2$ ✓ $\blacksquare$

$\boxed{3} \Rightarrow \boxed{1}$. ORTHOGONALITY GRAPH $\Rightarrow$ UNIT-DISTANCE GRAPH

→ recall that $V \subseteq \sqrt{\frac{d}{2}}\, S^{d-1}$, so $\|v(j)\| = \sqrt{\frac{d}{2}} = \sqrt{4p}$

Pythagoras: $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\langle u|v\rangle$ ⊛

↳ ∀ orthogonal pair of vectors $u(i) \perp v(j) \in V$ satisfy

⊛ $\|u(i) - v(j)\| = \sqrt{4p + 4p - 2\cdot 0} = \sqrt{8p}$

⇒ after scaling everything down by a factor of $\frac{1}{\sqrt{8p}}$ we get a unit-distance graph

$\boxed{3} \Rightarrow \boxed{2}$: ORTHOGONALITY $\Rightarrow$ DIAMETER GRAPH

→ we know (⊛) that all endpoints of edges are at the same distance, but this may not be the maximal distance

↳ what if $\langle u|v\rangle < 0$ in ⊛ ?

- replace each $v \in V$ by $v \otimes v :=$ flattened $vv^T = \begin{bmatrix} v_1 \cdot \vec{v} \\ v_2 \cdot \vec{v} \\ \vdots \\ v_d \cdot \vec{v} \end{bmatrix} = \begin{bmatrix} v_1 v_1 \\ v_1 v_2 \\ \vdots \\ v_1 v_d \\ v_2 v_1 \\ \vdots \\ v_d v_d \end{bmatrix} \in \mathbb{R}^{d^2}$

$$\langle u \otimes u \,|\, v \otimes v\rangle = \sum_{i,j} u_i u_j v_i v_j = \sum_i u_i v_i \sum_j u_j v_j = \langle u|v\rangle^2 \geq 0$$

$\Rightarrow \|u \otimes u - v \otimes v\|$ is max. $\Leftrightarrow \langle u|v\rangle = 0 \Leftrightarrow u \perp v$

⇒ all edges are at max distance

! this would give $\chi(G) \geq (1.067 + o(1))^{\sqrt{d}}$

→ but we can save some dimensions

👁 even though $v \otimes v \in \mathbb{R}^{d^2}$, the set $\{v \otimes v \mid v \in V\}$ belongs to a subspace of $\mathbb{R}^{d^2}$ with dimension $d + \binom{d}{2}$ ... for $i \neq j$ is $v_i \cdot v_j$ repeated twice in $v \otimes v$

→ so only $d + \binom{d}{2}$ coordinates are being actively used

→ now $\chi(G) \geq (1.067 + o(1))^d$ diameter graph in $\mathbb{R}^{d + \binom{d}{2}} \approx \mathbb{R}^{\frac{d^2}{2}}$ $\Big|\ \frac{d^2}{2} = x,\ d = \sqrt{2x}$

$\chi(G) \geq (1.067 + o(1))^{\sqrt{2x}}$ diameter graph in $\mathbb{R}^x$

↳ $1.067^{\sqrt{2}} \approx 1.096$

$d + \binom{d}{2} = \frac{d^2 + d}{2} \approx \frac{d^2}{2}$

⇒ $\chi(G) \geq (1.096 + o(1))^{\sqrt{d}}$ diameter graph in $\mathbb{R}^d$

BLACK-BOX ARGUMENTS FOR THE BETTER BOUND

Fact (Raigorodskii): There $\exists$ orthogonality graphs in $\mathbb{S}^{d-1}$ with $\chi(G) \geq (1.1547 + o(1))^d$.

Proposition: If G is an orth. graph in $\mathbb{S}^{d-1}$, then $\forall m$ is G^m an orth. graph in $\mathbb{S}^{2d-1}$

$\hookrightarrow$ blow-up

Proof: Let $V = V(G) \subseteq \mathbb{S}^{d-1}$. For $\forall v \in V$ define

- $v' := (v, 0) \in \mathbb{S}^{d-1} \times \mathbb{R}^d \subseteq \mathbb{R}^{2d}$
- $v'' := (0, v) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \subseteq \mathbb{R}^{2d}$

If $u \perp v$, then $\operatorname{span}_{\mathbb{R}}\{u', u''\} \perp \operatorname{span}_{\mathbb{R}}\{v', v''\}$ are orth. planes in $\mathbb{R}^{2d}$

$\hookrightarrow \langle (a\cdot\vec{u}, b\cdot\vec{u}) | (c\cdot\vec{v}, d\cdot\vec{v}) \rangle = a\cdot c \langle u|v\rangle + b\cdot d\langle u|v\rangle = 0$

$\Rightarrow \forall v \in V$ choose distinct unit-vectors $v_1, \dots, v_m \in \operatorname{span}\{v', v''\}$ ... $v_i \in \mathbb{S}^{2d-1}$
so that they are also distinct between different v

$\rightarrow u \perp v \Rightarrow u_i \perp v_j$ $\quad \Rightarrow G^m$ is an orth. graph in $\mathbb{S}^{2d-1}$ ∎

Fact (Xuding Zhu): $\forall$ graph G $\forall g$ $\exists m$ s.t. G^m contains a subgraph $G_0 \subseteq G^m$ s.t.
G_0 has girth $\geq g$ & $\chi(G_0) = \chi(G^m) = \chi(G)$

Theorem $\boxed{3}$: There $\exists$ orthogonality graphs in $\mathbb{S}^{d-1}$ with $\chi(G) \geq (1.074 + o(1))^d$
that have arbitrarily large girth.

Proof: Take G in $\mathbb{S}^{d-1}$ with $\chi(G) \geq (1.1547 + o(1))^d$, let g be given

$\Rightarrow$ find m and $G_0 \subseteq G^m$ from fact $\boxed{2}$

$\rightarrow$ we know that G_0 is an orth. graph in $\mathbb{S}^{2d-1}$

$$\chi(G_0) \geq (1.1547 + o(1))^d \text{ in } \mathbb{S}^{2d-1}$$
$$\chi(G_0) \geq (1.1547 + o(1))^{\frac{x}{2}} \text{ in } \mathbb{S}^{x}$$
$$\chi(G_0) \geq (\underbrace{\sqrt{1.1547}}_{1.074} + o(1))^d \text{ in } \mathbb{S}^{d}$$

$2d-1 = x$
$d = \frac{x+1}{2} \approx \frac{x}{2}$

∎

Remark: They also provide an explicit proof of Fact $\boxed{2}$, from what I can tell previous proofs were probabilistic