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Proof of the 3/4-conjecture for the total domination game

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Introduction

Let $G = (V, E)$ be a graph without isolated vertices. A vertex v is *totally dominated* by a set $A \subseteq V$ if it has a neighbour in A. The total domination game on G is played by 2 players, Dominator and Staller, who alternate in selecting vertices such that each newly selected vertex increases the number of vertices that are totally dominated by the set of selected vertices A. The game stops when A totally dominates every vertex of G. Dominator's aim is to minimize $|A|$, while Staller wants to maximize it. The *game total domination number* $\gamma_{tg}(G)$ is the number of vertices in the resulting set when Dominator starts the game and both players play optimally. Henning, Klavžar, and Rall [1] proved that if G has no isolated vertices or edges and $|V| = n$, then $\gamma_{tg}(G) \leq \frac{4}{5}n$, and they conjectured that in fact $\gamma_{tg}(G) \leq \frac{3}{4}n$. Portier and Versteegen [2] confirmed this conjecture.

Main idea of the proof

We describe a strategy for Dominator (D) that achieves this. Note that if v has not been played and all neighbours of v are already totally dominated, then v is unplayable. Intuitively, D wants the sum of the numbers of totally dominated vertices and unplayable vertices to grow as quickly as possible. Whether it is easier to totally dominate many new vertices or to make many of them unplayable changes over the course of the game. For this reason, the strategy of D consists of a number of different *phases* that follow each other linearly and come with separate sets of instructions for D.

Describing the state of the game

As observers of the game, we will keep some pieces of data that will help us analyze it. Vertices have a color: they are either *white* or *black*, and we may mark them as *depleted* or *dependent*.

- Every vertex starts **white**, and it may be colored **black** only once it has been totally dominated. Once colored **black**, we cannot change its color back to **white**.
- A vertex is only allowed to be marked as **depleted** if it has not been played so far and has no **white** neighbours. In particular, all **depleted** vertices are unplayable.
- A vertex is only allowed to be **dependent** if it has ≤ 1 **white** neighbours.
- A vertex is not allowed to be marked both as **depleted** and **dependent**, and if it has been marked in any way, we cannot remove the mark.

We use the following notation, where t denotes a point in our analysis after t moves have been played, always assuming that D has played according to our instructions.

- $\beta(t)$ = the number of black vertices
- $\delta(t)$ = the number of depleted vertices

- $\lambda(t)$ = the number of leaves that have been played or marked as depleted
- $\sigma(t)$ = the number of *unplayed* dependent vertices
- $\nu(t)$ = the number of undominated neighbours of dependent vertices
- $\chi(t) = \sigma(t) - \nu(t)$

Observation 1 *β, δ, λ and χ are all non-decreasing in t.*

Simplifications

Suppose for contradiction that the 3/4-conjecture is false, and fix a counter-example G that is minimal with respect to the number of its edges. The *parent* $p(x)$ of a leaf x is its unique neighbour.

Lemma 2 *No parent in G has more than one leaf.*

Lemma 3 *Let u be a parent in G with a leaf w and another neighbour v. Then $G - uv$ has an isolated edge.*

Definition 4 *A leaf x has* type A *if no neighbour of $p(x)$ is a parent, and it has* type B *otherwise.*

Observation 5 *If x is a leaf of type A, then $p(x)$ has exactly one other neighbour z, and z is neither a leaf nor a parent. We say that z is the* grandparent *of both x and $p(x)$.*

Phases of the game

There will be 6 phases, and for each $i \in [6]$, t_i is the last move before the end of phase i, and we let $t_0 = 0$. Denote by $T_i := t_i - t_{i-1}$ the number of moves in phase i. Each phase ends once it is Dominator's turn and a certain configuration for white vertices is no longer possible.

1. No grandparent is adjacent to two or more white parents, and no parent is adjacent to one or more white parents.
2. No grandparent is adjacent to a white parent and at least three white vertices overall.

During phase 3, we will decide after each move of Staller whether or not to set a reaction flag. A necessary condition for the phase to end is that the reaction flag is not set.

3. There exists no parent such that both its leaf and grandparent are white, and there exists no grandparent that is adjacent to a white parent and a second white vertex.
4. No vertex has three or more white neighbours.
5. No vertex has two white neighbours.
6. The last phase ends once no undominated vertex is left.

Bibliography

[1] Michael A. Henning, Sandi Klavžar, and Douglas F. Rall. *The 4/5 upper bound on the game total domination number.* Combinatorica, 37(2), 223-251, 2017.

[2] Julien Portier and Leo V. Versteegen. *Proof of the 3/4-conjecture for the total domination game.* SIAM Journal on Discrete Mathematics, 39(1), 1-18, 2025.