TOTAL DOMINATION GAME

THM [1]: $\forall G$ with minimum degree ≥ 2 we have $\gamma_{tg}(G) \leq \frac{5}{7}(n+2)$

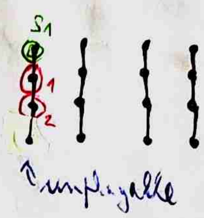
THM [2]: $\forall G$ without isolated vertices or edges: $\gamma_{tg}(G) \leq \frac{3}{4}n$

- isolated edges are not interesting: both ends need to be played

Def: $v \in V$ is a *leaf* $\equiv \deg(v) = 1$; its unique neighbour is a *parent*

$\forall$ parent has to be played

THM [2] is optimal: disjoint union of P_4



PROOF OF THM [2]

→ strategy for Dominator is divided into phases

→ a phase ends when its Dominator's turn and something holds

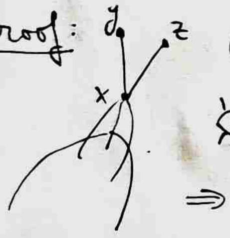
! we pretend that if v has no undominated neighbours, but some white neighbors, then D can play v ... D "playing" v is surely not better than playing an arbitrary legal move

SIMPLIFICATIONS

→ suppose THM [1] is false and fix a counterexample G, minimal in $|E(G)|$

Lemma [1]: No parent in G has more than 1 leaf

Proof:



G is minimal $\Rightarrow$ D has a strategy Δ for $G - z$ within $\frac{3}{4}n$ moves

both y and z can never be played in the same game

$\Rightarrow$ D can follow Δ for the game on G

→ if S plays z, we pretend he played y ∎

Lemma [2]: Let u be a parent in G with a leaf w and another neighbor v. Then $G - uv$ has an isolated edge.

Proof:

→ v is not a leaf (Lem [1]) $\Rightarrow$ $G - uv$ has no isolated vertex

→ if no isolated edge, D would have a strategy Δ for $G - uv$

$\Rightarrow$ we play using Δ

- what could go wrong: if S plays a vertex that is unplayable in $G - uv$

(1) S plays u ... but u has to be played in both games ∵ its a parent

(2) S plays v ... u not dominated yet, but all other neighbors of v are

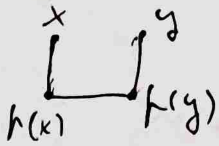
$\Rightarrow$ w was not played

→ D can pretend as if S played w ∎

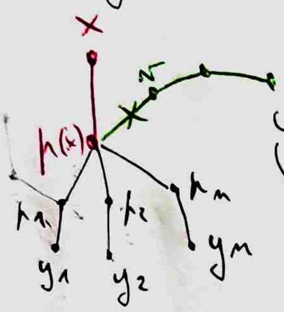
Def: Leaf TYPE A

…(not adjacent to any other parent)

TYPE B



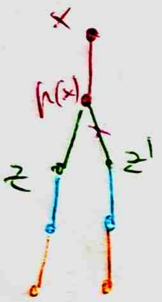
👁 x has type B. Then the component of G containing $p(x)$ contains exactly vertices: $x, p(x), p_1, \dots, p_m, y_1, \dots, y_n$



↳ leaf impossible due to Lem $\boxed{1}$

↳ if 3 edges then after deleting $p(x)v$, no isolated edge ↯ Lemma $\boxed{2}$

👁 x has type A. Then $p(x)$ has exactly 1 other neighbour z, and z is neither a leaf or a parent $\Rightarrow$ $z =$ <u>grandparent</u> of x and $p(x)$



← $p(x)$ has ~~two~~ another neighbr ∵ $x\,p(x)$ would be isolated

← $p(x)$ cannot have more than 1 leaf

← $p(x)$ has type A

! when we delete $p(x)z'$, there is no isolated edge ↯

<u>PHASES OF THE GAME</u> … 6 phases

t = point after t moves have been played

$t_0 := 0$

- for $i \in [6]$: $t_i :=$ the last move before the end of phase i
- $T_i := t_i - t_{i-1}$ … num moves in phase i

<u>Terminology</u>

- white × black vertex
- unplayable … all neigbours dominated
- depleted … not played & no white neighbors $\Rightarrow$ unplayable
- dependent … if ≤ 1 white neighbors

$\beta(t) =$ # black vertices, $\delta(t) =$ # depleted vertices

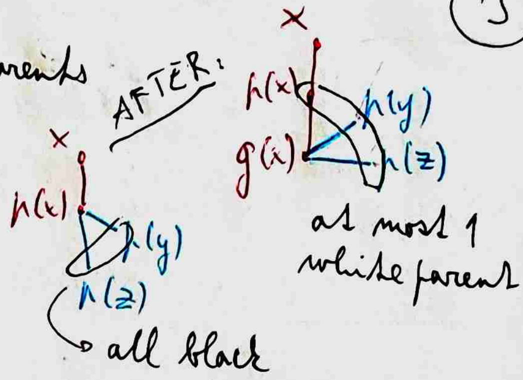
$\lambda(t) =$ # leaves that have been played or marked as depleted (unusable leaves)

$\sigma(t) =$ # unplayed dependent vertices, $\nu(t) =$ # undominated neighbors of dependent vertices

$\chi(t) = \sigma(t) - \nu(t) \geq 0$ 👁 $\beta, \lambda, \delta, \chi$ are non-decreasing in t

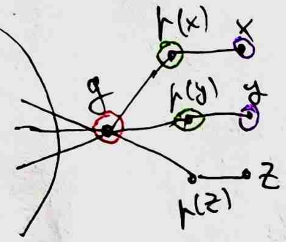
PHASE 1

- no grandparent is adjacent to ≥2 white parents
- no parent is adjacent to ≥1 white parents



NOTATION: L = set of leaves
P = set of parents
F = set of grandparents
R = the rest

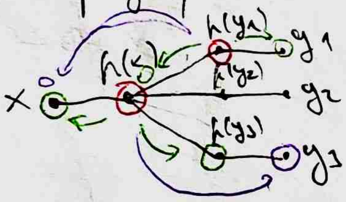
(1) D plays grandparents with ≥2 white neighbors in P



+2 black +2 depleted

m_F := # grandparent moves

(2) D plays parents with ≥1 white neighbors in P



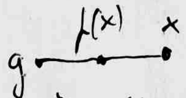
+2 black +1 depleted

m_P := # parent moves

- $b(t_1) \geq \frac{3}{2} T_1$ ... at least 3 black per 2 moves (color S dom. black)
- $\delta(t_1) \geq 2 m_F + m_P$
- $m_F + m_P \geq T_1/2$ ... D plays half of the moves
- $\lambda(t_1) \leq \delta(t_1) + T_1/2$ ... #wasted leaves ≤ depleted + S plays at most $\frac{T_1}{2}$ moves
  ↳ we do not play any leaves
- each leaf of type B is played or defended

(*) ↳ otherwise its parent white ⇒ his parent neighbor was played ↯

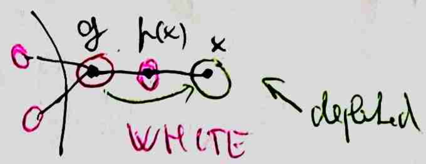
⇒ all leaves are like:



∵ other parents are black

(*) when S plays, color everything black and mark unplayed leaves whose parent is black as depleted

PHASE 2



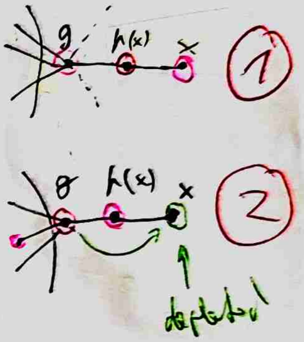
- no grandparent is adjacent to a white parent and at least 3 white vertices overall

→ D plays these vertices until there are no left

$\underline{+3 \text{ black}}$ + $\underline{1 \text{ deflated leaf}}$ (S handled like before) (deflate all leaves and color black)

- $\lambda(t_2) - \lambda(t_1) \leq T_2$ ↳ we deflate at most 1 vertex (and play at most 1)
- $\beta(t_2) - \beta(t_1) \geq 2T_2$
- $\delta(t_2) - \delta(t_1) \geq \frac{T_2}{2}$

PHASE 3 – we can set a reaction flag ∴ may not be set at the end



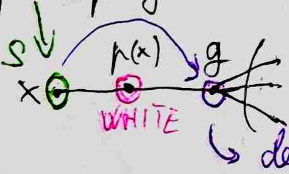
- no parent s.t. both its leaf & grandparent are white
- no grandparent adjacent to a white parent & a second white vertex

→ we color everything always black

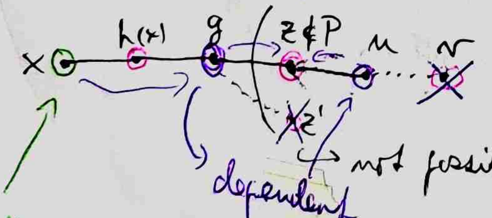
grandparent of a black leaf must be black

→ if no reaction flag: play 1 $\beta+2$ … p times
then 2 $\beta+2$, $\delta+1$ … q times

→ if S plays a leaf: 3 options (can only play leaf of type A)

1 

no white neighbors $\beta+1$, $\delta+1$ …. m_1 times ↗ PHASE 1

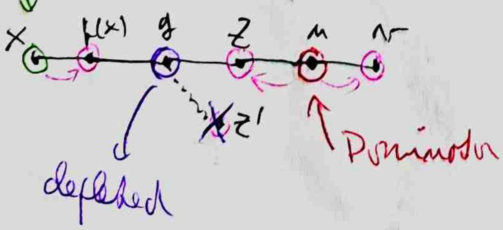
2 

not possible … PHASE 2

g has 1 other white neighbor z … z∉P
z∈L … else g is a parent and x has type B
⇒ ∃ u neighbor z … suppose u has no white neighbor v

Staller

$\beta+1$, $\chi+1$ … u, g unplayed (otherwise z black)
m_2 times → and they have in total only 1 uncolored neighbor: z

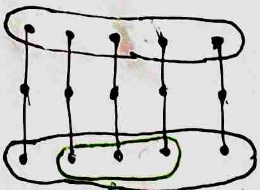
3 

$z \in P$, $z \in L$

$\beta+3$, $\delta+1$ … m_3 times

- $\beta(t_3) - \beta(t_2) \geq \frac{3}{2} T_3$, $\delta(t_3) - \delta(t_2) = m_1 + m_3 + q$
- $\chi(t_3) \geq m_2$, $T_3/2 = m_3 + p + q$ … # moves of D.
- $\lambda(t_3) - \lambda(t_2) = m_1 + m_2 + m_3 + q$

Lemma: $\chi(t_3) \geq |L| - m_1 - m_3 - q - T_2 - \lambda(t_1)$



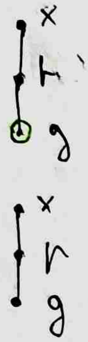
$I \subseteq L$ ... unplayed & not marked as depleted ... all type A

$\in P$ $\Rightarrow$ mark all $v \in (I \cup C)$ as dependent

$J \subseteq F$

$C \subseteq J$ ↳ unplayed & unmarked

__claim__: χ increases by at least $|I|$ $\overset{\shortparallel}{|L| - \lambda(t_3)}$



x & g unplayed $\Rightarrow$ p undominated; Phase 3 $\Rightarrow$ g no other white neighbor $\Big\}$ $\sigma + 2$, $\nu + 1$

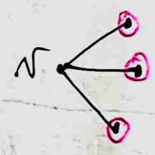
a) g played $\Rightarrow$ p dom. $\rightarrow$ $\sigma + 1$, $\nu + 0$

b) g marked – depleted ... only possible if x were played & dependent $\Rightarrow$ $\sigma + 1$, $\nu + 0$ (since p accounted for)

$$\chi(t_3) \geq m_2 + |I| = m_2 + |L| - \lambda(t_3) = |L| - m_1 - m_3 - q - \lambda(t_2)$$
$$\geq |L| - m_1 - m_3 - q - T_2 - \lambda(t_1) \qquad \blacksquare$$

PHASE 4

- no vertex has 3 or more white neighbors



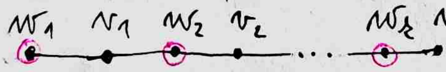
$\rightarrow$ we again color everything black always

- D plays these vertices $\Rightarrow \Delta(t_4) - \Delta(t_3) \geq 2T_4$

PHASE 5

- no vertex has 2 white neighbours

Def: k-walk: $w_1\ v_1\ w_2\ v_2 \dots w_k\ v_k$ — w_j distinct, v_i distinct

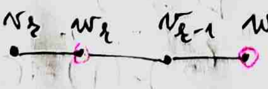


k-circuit: w_1 and v_k are connected

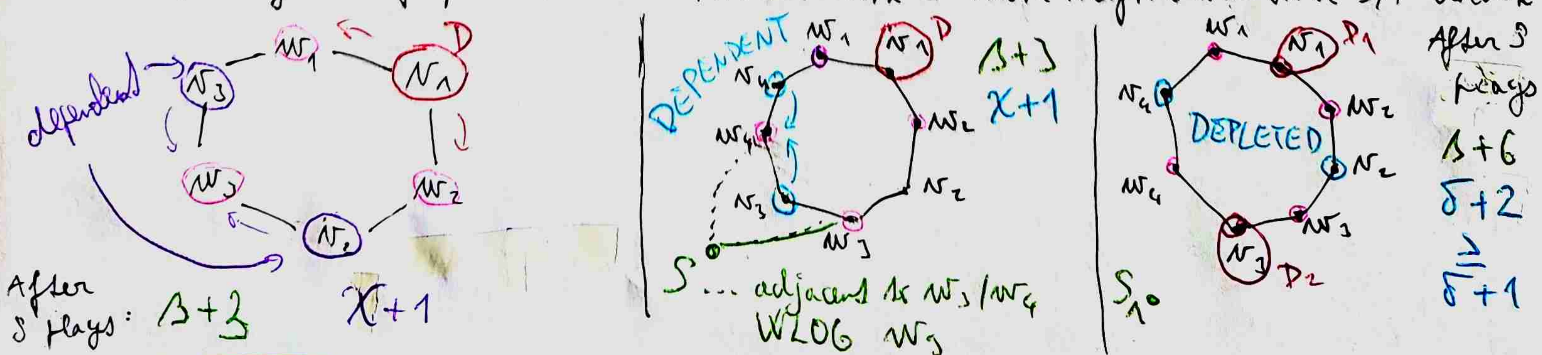
(!) no leaf can be part of any k-walk for $k > 1$

- w_1 leaf: $w_1\ v_1\ w_2\ v_2$ $\Rightarrow w_1$ / typ A ... w_2 white grandparent & P3; \ typ B ... w_2 must be a parent & P1

- v_k leaf: $v_k\ w_k\ v_{k-1}\ w_{k-1}$ $\Rightarrow v_k$ < typ D ↗ (recall); typ A ... v_{k-1} grandparent & P3



Fact: D can copy a part of his strategy for leaf-less graphs. Result: $\forall$ vertex with 2 white neighbors lies on a 3/4-circuit



Always:

① D can play the next move so that
$s(t+2)-s(t)\geq 3$ & $\delta(t+2)-\delta(t)+\chi(t+2)-\chi(t)\geq 1$

② or, D can play the next 2 moves so that
$s(t+4)-s(t)\geq 6$ & $\delta(t+4)-\delta(t)\geq 1$

$\Rightarrow s(t_5)-s(t_4)\geq \frac{3}{2}T_5$, $4\big(\delta(t_5)-\delta(t_4)\big)+2\big(\chi(t_5)-\chi(t_4)\big)\geq T_5$

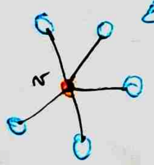
- the moves described for D satisfy this, and so do the moves hidden in "FACT"

PHASE 6 ... ends once the game is over

→ we color all vertices black immediately
! no vertex has 2 white neighbors now – mark all vertices that have a white neighbor as dependent
↳ white = undominated

claim: Regardless of moves: $\overbrace{s(t_6)-s(t_5)\geq T_6}^{\text{obviously}}$ & $T_6\leq \chi(t_5)+L-p-n_p$

- from PHASE 1: #undominated leaves $\leq L-p-n_p$
- if an undominated vertex is not a leaf:



← DEPENDENT → if one of them were played, v would be dominated

$\left.\begin{array}{l}\sigma=5\\ \nu=1\end{array}\right\}$ $\chi+4$ from this vertex

→ # such vertices $\leq \chi(t_5)$

BOUNDING #MOVES

$T:=t_6=T_1+\dots+T_6$. For contradiction we need to show $T\leq\frac{3}{4}n$

Two strategies: $s(T)\leq n$

- $\underset{\text{\#played}}{T}+\underbrace{\delta(T)+\chi(T)}_{\text{\#unplayed}}\leq n$

! $\chi=\sigma-\nu=\sigma$
↳ $\nu(T)=0$

→ combining everything together yields $8T\leq 6n$