A Proof of the 3/4-Conjecture for the Total Domination Game

Julien Portier and Leo V. Versteegen

Outline

- Total domination game
- Results
- Simplifications
- Types of leaves
- Terminology
- Phases

- Every vertex starts **white**, and it may be colored **black** only once it has been totally dominated. Once colored **black**, we cannot change its color back to **white**.

- A vertex is only allowed to be marked as **depleted** if it has not been played so far and has no **white** neighbours. In particular, all **depleted** vertices are unplayable.

- A vertex is only allowed to be **dependent** if it has ≤ 1 **white** neighbours.

- A vertex is not allowed to be marked both as **depleted** and **dependent**, and if it has been marked in any way, we cannot remove the mark.

  - $\beta(t)$ = the number of black vertices
  - $\delta(t)$ = the number of depleted vertices
  - $\lambda(t)$ = the number of leaves that have been played or marked as depleted
  - $\sigma(t)$ = the number of *unplayed* dependent vertices
  - $\nu(t)$ = the number of undominated neighbours of dependent vertices
  - $\chi(t) = \sigma(t) - \nu(t)$

Phase 1

- $\beta(t_1) \geq \frac{3}{2}T_1$.
- $\delta(t_1) \geq 2r_F + r_P$.
- $r_F + r_P \geq T_1/2$.
- $\lambda(t_1) \leq \delta(t_1) + T_1/2$.
- *No vertex has been marked as dependent.*
- *Each leaf of type B has been played or marked as depleted.*
- *There are at most* $L - r_P$ *undominated leaves left.*

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  - $\beta(t) =$ the number of black vertices
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Phase 2

- *No vertex has been marked as dependent.*
- $\lambda(t_2) - \lambda(t_1) \leq T_2$
- $\beta(t_2) - \beta(t_1) \geq 2T_2$
- $\delta(t_2) - \delta(t_1) \geq T_2/2$

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Phase 3

- $\beta(t_3) - \beta(t_2) \geq \frac{3}{2}T_3$.
- $\delta(t_3) - \delta(t_2) = m_1 + m_3 + q$.
- $\chi(t_3) \geq m_2$.
- $\lambda(t_3) - \lambda(t_2) = m_1 + m_2 + m_3 + q$.
- $T_3/2 = m_3 + p + q$.
- *No leaf has been marked as dependent.*
- *There are at most $L - p - r_P$ undominated leaves left.*

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Phase 4

- $\beta(t_4) - \beta(t_3) \geq 2T_4.$

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Phase 5

- $\beta(t_5) - \beta(t_4) \geq \frac{3}{2}T_5$
- $4(\delta(t_5) - \delta(t_4)) + 2(\chi(t_5) - \chi(t_4)) \geq T_5$

Phase 5

- $\beta(t_5) - \beta(t_4) \geq \frac{3}{2} T_5$
- $4(\delta(t_5) - \delta(t_4)) + 2(\chi(t_5) - \chi(t_4)) \geq T_5$
- D can play the next move so that

$$\beta(t+2) - \beta(t) \geq 3, \qquad \delta(t+2) - \delta(t) + \chi(t+2) - \chi(t) \geq 1.$$

- Or, D can play the next two moves so that

$$\beta(t+4) - \beta(t) \geq 6, \qquad \delta(t+4) - \delta(t) \geq 1.$$

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Phase 6

Regardless of what moves either player makes:

- $\beta(t_6) - \beta(t_5) \geq T_6.$
- $T_6 \leq \chi(t_5) + L - p - r_P.$