INTRODUCTION TO WQO THEORY

Def: Q is quasi-order $\equiv$ reflexive & transitive

- $x \equiv y$ if $x \leq y$ & $y \leq x$
- $x < y$ if $x \leq y$ & $x \not\equiv y$

identifying $\equiv$-equiv elements produces a partial order

- x is minimal if $\forall y \leq x$ we have $y \equiv x$

Def: Q is WQO $\equiv$ $\forall$ infinite sequence $x_1, x_2, x_3, \ldots$ $\exists\, i<j$ s.t. $x_i \leq x_j$

↳ good / bad sequence

Theorem: The following are equivalent

1. Q is WQO
2. $\forall$ sequence $x_1, x_2, x_3, \ldots$ contains an infinite nondecreasing subsequence $x_{i_1} \leq x_{i_2} \leq \ldots$
3. Q admits no infinite decreasing chains an no infinite (non-equiv) antichains
4. $\forall\, \emptyset \neq A \subseteq Q$ contains at least 1, but only finitely many min. elements

Proof: 2 $\Rightarrow$ 1, 1 $\Rightarrow$ 3, 3 $\Rightarrow$ 2 by Ramsey's theorem

3 $\Rightarrow$ 4 no infinite decreasing chains $\Rightarrow$ $\exists$ minimal element
no infinite antichains $\Rightarrow$ only finitely many

4 $\Rightarrow$ 3 by the same reasoning $\blacksquare$

Remark: If κ is a regular cardinal, and Q WQO, then $\forall$ Q-sequence of length κ contains a non-decreasing subsequence of the same length

Def: A property $\varphi(x)$ is monotone $\equiv \forall x: \varphi(x)$ & $y \leq x \Rightarrow \varphi(y)$

- finitely testable $\equiv$ $\exists$ finite set $\mathcal{F}$ s.t. $\varphi(x) \Leftrightarrow (\forall F \in \mathcal{F}): F \not\leq x$

Theorem: Q WQO $\Rightarrow$ $\forall$ monotone property is finitely testable.

Proof: Define $\overline{Q} = \{x \in Q \mid \neg\varphi(x)\}$ and let $\mathcal{F}$ be non-equiv min. elements of $\overline{Q}$ $F_1, F_2, \ldots, F_n$

- if $\varphi(x)$ then $\forall F_i \not\leq x$ ... if $F_j \leq x$ then $\varphi(F_j)$ $\because$ φ is monotone $\lightning$
- if $\forall F_i \not\leq x$ then $\varphi(x)$ ... if $\neg\varphi(x)$, take $Y := \{y \in \overline{Q} \mid y \leq x\}$
  → Y has a min. element $H \leq x$, and $H \equiv F_j$ for some j $\because$ H min. in $\overline{Q}$
  $\Rightarrow F_j \leq x$ $\lightning$ $\blacksquare$

Note: Graph minor thm $\Rightarrow$ $\forall$ surface $\exists$ finite obstruction set
Sphere ... $K_5, K_{3,3}$, Projective plane 35 obs., other than that not known

Preservation properties

- Q well-ordered $\Rightarrow$ Q WQO ... for example ω or $\{1,2,\dots,n\}$
- Q finite $\Rightarrow$ $(Q,=)$ WQO ... finite alphabets

Def: Product order of $P \times Q$: $(p,q) \leq (p',q')$ if $p \leq_P p'$ & $q \leq_Q q'$

Dickson's lemma: P, Q WQOs $\Rightarrow$ $P \times Q$ WQO.

Proof: $(p_n, q_n)_n$, take ∞-nondecreasing subsequence $p_{i_1} \leq p_{i_2} \leq \dots$

$\Rightarrow (p_{i_n}, q_{i_n})_n$, Q WQO ... $\exists \ell < k$ s.t. $q_{i_\ell} \leq q_{i_k}$ ∎

Def: $Q^{<\omega}$ ordered by $(a_i)_{i<n} \sqsubseteq (b_j)_{j<m}$ if $\exists$ increasing $f: n \to m$ s.t. $\forall i<n$: $a_i \leq b_{f(i)}$

- $(\Sigma,=)$ finite alphabet, $u \sqsubseteq v$ is word embedding
- $P(Q)$ ordered by $X \sqsubseteq_s Y$ if $\exists$ inj. $f: X \to Y$ s.t. $\forall x \in X$: $x \leq f(x)$.

Theorem (Higman, 1952): Q WQO $\Rightarrow$ $Q^{<\omega}$ WQO.

Proof: Construct a minimal bad sequence $w_0, w_1, w_2, \dots$ $\min |w_i|$

- write $w_i = a_i w_i'$
- Q WQO $\Rightarrow$ $\exists$ increasing $f: \omega \to \omega$ s.t. $a_{f(0)} \leq a_{f(1)} \leq a_{f(2)} \leq \dots$

claim: $w'_{f(0)}, w'_{f(1)}, w'_{f(2)}, \dots$ is good

↳ then $\exists i<j$: $w'_{f(i)} \sqsubseteq w'_{f(j)}$ so $w_{f(i)} \sqsubseteq w_{f(j)}$

Proof: Suppose not; then the sequence

$w_0, w_1, \dots, w_{f(0)-1}, w'_{f(0)}, w'_{f(1)}, \dots$ is bad

- if $w_j \sqsubseteq w'_{f(k)}$ for some $j \leq f(0) \leq f(k)$ then $w_j \sqsubseteq w_{f(k)}$ ↯

Hence it is bad, contradicting the minimality of $w_{f(0)}$ since $|w'_{f(0)}| < |w_{f(0)}|$

∎

Kruskal's tree theorem 1960

Def: Order-theoretic tree is a partial order $(T, \le)$ s.t. $\forall x \in T$:

$(\leftarrow, x) := \{y \in T \mid y < x\}$ is a finite chain

There is a unique minimal element ... root

Intuition: $y < x$ means "y lies on the unique path from x to the root"

Def: A homeomorphic embedding of two Q-labeled trees: $\varphi : S \to T$

i) $l_S(x) \le l_T(\varphi(x))$ ... respects labels   Write $S \trianglelefteq T$

ii) $x < y \Rightarrow \varphi(x) < \varphi(y)$

iii) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$

} topological minor for rooted trees

Theorem (Kruskal): Q WQO $\rightarrow$ $T_f(Q)$ = finite Q labeled trees WQO by $\trianglelefteq$

Corollary: Finite graph-trees are WQO by topological minors ... take $Q = \{\bullet\}$

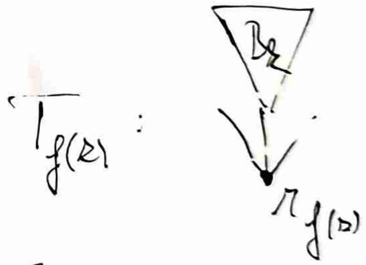
By Higman's lemma also forrests.

Proof: Construct a minimal bad sequence $T_0, T_1, T_2, \dots$ minimizing $|T_i|$

- r_i .... root of T_i
- B_i ... branch-trees of T_i

claim: $\mathcal{B} := \bigcup_{i<\omega} B_i$ is WQO by $\trianglelefteq$

proof: Suppose that $B_0, B_1, \dots$ is bad



- $\forall j$ select $f(j)$ s.t. $B_j \in \mathcal{B}_{f(j)}$
- let k minimize $f(k)$
- consider the sequence $T_0, T_1, \dots, T_{f(k)-1}, B_k, B_{k+1}, \dots$

☺ it is bad, contradicting the minimality of $T_{f(k)}$ since $|B_k| < |T_{f(k)}|$

↳ if $T_i \trianglelefteq B_j$ for some $i < f(k)$ and $j \ge k$ then $T_i \trianglelefteq T_{f(j)}$ & $i < f(k) \le f(j)$.

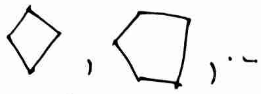
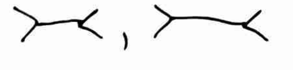
Because Q is WQO, we may assume that $l(r_0) \le l(r_1) \le \dots$

Since $\mathcal{B}$ is WQO, Higman's lemma gives $(\mathcal{B}, \sqsubseteq_s)$ WQO, so

$\exists\, i < j$ s.t. $B_i \sqsubseteq_s B_j$

Together, $T_i \trianglelefteq T_j$, contradicting badness of the minimal bad sequence ∎

Ding's theorem 1992

cycles  ,.. and forks F_n  are

induced subgraph $\leq_i$ and subgraph $\subseteq$ antichains

Theorem (Ding) If $\mathcal{G}$ is a $\subseteq$-closed class of finite graphs, then the following are equiv.:

1. $\mathcal{G}$ is WQO by $\subseteq$
2. $\mathcal{G}$ is WQO by $\leq_i$
3. $\mathcal{G}$ contains only finitely many C_n and F_n

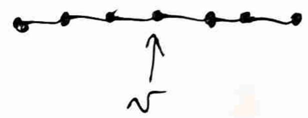
clearly: (2) $\Rightarrow$ (1) $\Rightarrow$ (3)

Theorem (Ding) If $\exists n$ s.t. no $G \in \mathcal{G}$ has $P_n \subseteq G$, then $\mathcal{G}$ is WQO by $\leq_i$.

Def: Tree-depth of a graph G is

$$td(G) := \begin{cases} 1, & \text{if } |G| = 1 \\ \max_i td(G_i) & \text{if } G \text{ is disconnected with components } G_1, \dots, G_m \\ (*)\ 1 + \min_{v \in G} td(G - v) & \text{otherwise} \end{cases}$$

$td(P_n) \approx \log_2(n)$ ... $td(P_n) = 1 + td(P_{\lceil n/2 \rceil})$



$\Rightarrow$ if $td(G) < \ell$ then $P_{2^\ell} \not\subseteq G$ because td is minor monotone

if $P_n \not\subseteq G$, then $td(G) < n$

↳ if $td(G) \geq n$, then by taking the best v in (*) we construct path $v_1, \dots, v_n \subseteq G$

Def: Induced subgraphs for Q-labeled graphs

$(G, \ell_G) \leq_i (H, \ell_H)$ if $\exists$ injection $f: V_G \to V_H$ s.t.

(1) $uv \in E_G \iff f(u)f(v) \in E_H$ (2) $\forall v: \ell_G(v) \leq \ell_H(f(v))$

Notation: $D_n(Q)$ ... finite Q-labeled graphs with tree-depth $\leq n$

Theorem: $\forall n$: Q WQO $\Rightarrow$ $D_n(Q)$ WQO by $\leq_i$.

Proof: Induction on n ... case $n=1$ holds from Higman's lemma

- let $n > 1$, sequence (G_i, ℓ_i) ... WLOG $\forall i: td(G_i) \geq 2$ ... there are ∞ many i
- $Q^+ := Q \times \{0,1\}$ ordered by $(q, e) \leq (q', e')$ if $q \leq q'$ & $e = e'$ is WQO
- $\forall i$ pick $v_i \in G_i$ s.t. $td(G_i - v_i) \leq n-1$ ... $H_i := G_i - v_i$
- since Q is WQO we may assume that $\ell_0(v_0) \leq \ell_1(v_1) \leq \dots$
- Q^+-label H_i by $\ell_i^+(u) := (\ell_i(u), e_i(u))$ where $e_i(u)$ encodes whether $uv_i \in E(G_i)$
- Induction hypothesis: $D_{n-1}(Q^+)$ is WQO $\Rightarrow \exists i<j$ s.t. $(H_i, \ell_i^+) \leq_i (H_j, \ell_j^+)$
- let $\varphi: V(H_i) \to V(H_j)$ witness this, then $\varphi \cup \{(v_i, v_j)\}$ witnesses $(G_i, \ell_i) \leq_i (G_j, \ell_j)$ ∎

Excursion to infinity

1954

Rado's counterexample: Higman's lemma fails for Q^ω

$Q = \{(i,j) \mid i<j<\omega\}$ ordered by $(i_1,j_1) \le (i_2,j_2)$ if $i_1 = i_2$ & $j_1 \le j_2$ or $j_1 < i_2$

→ easy to verify by Dickson's lemma that Q is WQO

	1	2	3	4	5	6	
0	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	...
1		$\boxed{(1,2)}$	(1,3)	(1,4)	(1,5)	(1,6)	...
2			(2,3)	(2,4)	(2,5)	(2,6)	...
3				(3,4)	(3,5)	(3,6)	...
4					$\boxed{(4,5)}$	(4,6)	

$P(Q)$ is not WQO by order-respecting maps

If $\sqsubseteq$, then

$\le$ something green ↯

Better-quasi-orderings ... Nash-Williams 1965

Q BQO $\Rightarrow$ $P(Q)$, $Q^{<On}$ BQO ... $Q^{<On}$ = transfinite sequences

Q BQO $\Rightarrow$ $T_\omega(Q)$ = finite or infinite Q-labeled trees BQO by $\le$

↳ thus the class of all graph-trees is WQO by topo-minors

Q BQO $\Rightarrow$ $D_n^{inf}(Q)$ = graphs of tree-depth $\le n$ BQO by $\le_i$

Some statements do not need BQO theory

Q WQO $\Rightarrow$ transfinite sequences $Q^{<On}$ with finite range WQO

Def: $S \le_h T$ if $\exists$ mapping $f: S \to T$ preserving immediate successors

↳ tree-homomorphism

Theorem (Smolik, 2026): One can prove without BQO theory that

T_ω = finite or infinite trees are WQO by tree-homomorphisms.