# Well-quasi-ordering certain classes of infinite graphs

Bachelor Thesis Defense

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# Well-quasi-orderings

## Definition (Quasi-order)

A binary relation $\preceq$ on a class $Q$ is a quasi-order if it is reflexive and transitive.

- $x \equiv y$ iff $x \preceq y$ and $y \preceq x$. . . . $x$ and $y$ are equivalent
- $x \prec y$ iff $x \preceq y$ and $x \not\equiv y$.
- $x$ is minimal if there is no $y \prec x$.

## Definition (Well-quasi-order)

$Q$ is wqo if in every infinite sequence $x_1, x_2, x_3, \ldots$ of elements of $Q$, there are $i < j$ such that $x_i \preceq x_j$.

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## Proposition

*The following conditions are equivalent to $Q$ being wqo:*

1. *Every infinite sequence $x_1, x_2, x_3, \ldots$ of elements of $Q$ contains an infinite non-decreasing subsequence $x_{i_1} \preceq x_{i_2} \preceq x_{i_3} \preceq \cdots$ for some indices $i_1 < i_2 < i_3 < \cdots$.*
2. *$Q$ admits no infinite decreasing sequences $x_1 \succ x_2 \succ x_3 \succ \cdots$ and no infinite antichains.*
3. *Every nonempty subset of $Q$ contains at least one but only finitely many non-equivalent minimal elements.*

# Well-quasi-orderings

## Definition

A property $\varphi$ of $Q$ is monotone if whenever $\varphi(x)$ holds and $y \preceq x$, then $\varphi(y)$ holds as well.
It is finitely testable if there exists a finite obstruction set $\mathcal{F}$ such that for all $x \in Q$ we have

$$\varphi(x) \text{ holds} \iff (\nexists z \in \mathcal{F}) : z \preceq x.$$

## Example (Wagner's Theorem)

Planarity is a finitely testable property of finite graphs with respect to the minor relation. Its minimal obstructions are $K_5$ and $K_{3,3}$.

## Proposition

*If $Q$ is wqo, then every monotone property of $Q$ is finitely testable.*

**Proof sketch:** Define $\mathcal{F}$ as the (nonempty finite) set of non-equivalent minimal elements of the class

$$\{x \in Q \mid \varphi(x) \text{ does not hold}\}.$$

# The Graph Minor Theorem

**Proposition**

*If $Q$ is wqo, then every monotone property of $Q$ is finitely testable.*

**Theorem (Robertson, Seymour, 1983–2004)**

*The class of all finite graphs is wqo by the minor relation.*

**Corollary**

*For every surface, there exists a finite obstruction set for embeddability of finite graphs on that surface.*

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*For every surface, there exists a finite obstruction set for embeddability of finite graphs on that surface.*

**Theorem (Thomas, 1988)**

*The class of all finite or infinite graphs is not wqo by the minor relation.*

**Conjecture:** The class of all countable graphs is wqo by the minor relation.

**Theorem (Komjáth, 1995)**

*For every $\kappa > \omega$, there is an antichain of $2^\kappa$ many graphs of size $\kappa$ (with respect to the minor relation).*

# Better-quasi-orderings

**Theorem (Kruskal's Tree Theorem, 1960)**

*The class of all finite trees is wqo by the topological minor relation.*

In order to extend this result to infinite trees, Nash-Williams introduced the stronger notion of better-quasi-orderings or bqo. The wqos encountered in practice are, in fact, bqos.

- Well-ordered $\implies$ bqo $\implies$ wqo.

**Theorem (Nash-Williams, 1965)**

*The class of all finite or infinite trees is bqo by the topological minor relation.*

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**Theorem (Nash-Williams, 1965)**

*The class of all finite or infinite trees is bqo by the topological minor relation.*

**Theorem (Thomas, 1989)**

*Any class of finite or infinite graphs with bounded tree-width is bqo by the minor relation.*

**Corollary**

*Any class of finite or infinite graphs with bounded tree-depth is bqo by the minor relation.*

**Proof:** Bounded tree-depth implies bounded tree-width.

# Tree-depth & Ding's Theorem

**Theorem (Ding, 1992)**

*Any class of finite graphs with bounded tree-depth is wqo by the induced subgraph relation.*

- Tree-depth measures how close a given graph is to a star.

**Theorem (J.S.)**

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Tree-depth of finite graphs was studied by Nešetřil and Ossona de Mendez. I extended the notion to infinite graphs and showed that its basic properties hold in this more general setting. For example:

**Lemma**

- *If $H$ is a minor of $G$, then* $\mathrm{td}(H) \leq \mathrm{td}(G)$. ... *true even when* $\mathrm{td}(H) \geq \omega$
- *If $G$ has finite tree-depth, then* $\mathrm{tw}(G) < \mathrm{td}(G)$.

I also established a compactness theorem:

**Theorem (J.S.)**

*Let $G$ be an arbitrary graph. Then* $\mathrm{td}(G) < k$ *if and only if* $\mathrm{td}(H) < k$ *for every finite subgraph $H$ of $G$.*

# Bounded tree-depth classes

## Definition

The path-depth of a graph $G$ is $\max\{n \mid P_n \subseteq G\}$, or $\infty$ if the maximum does not exist.

## Example

- path-depth of a tree is $1 +$ its diameter.
- path-depth of $K_n$ is $n$.

## Proposition

*A class of graphs $\mathcal{G}$ has bounded tree-depth if and only if it has bounded path-depth.*

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## Theorem (J.S.)

*Any class of finite or infinite graphs with bounded path-depth is bqo by the induced subgraph relation.*

## Corollary

*Any class of finite or infinite trees with bounded diameter is bqo by the (induced) subgraph relation.*

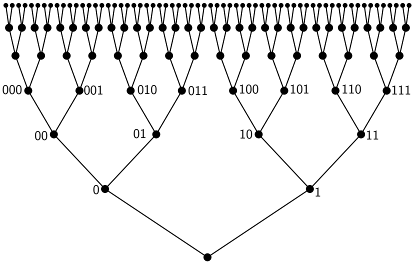
# Order-theoretic trees

## Definition (Rooted tree of height at most $\omega$)

An (order-theoretic) tree is a partially ordered set $(T, <_T)$ such that for every $x \in T$, its downset

$$(\leftarrow, x) := \{y \in T \mid y <_T x\}$$

is a finite chain, and there is a unique minimal element called the root of $T$.




$(2^{<7}, \sqsubseteq)$

**Terminology:**

- $x <_T y$ means that $x$ lies on the unique path from $y$ to the root.
- $|x|_T := |(\leftarrow, x)|$ is the height or level of $x$ in $T$.
- $x \wedge y$ is the meet of $x$ and $y$ (their closest common ancestor).
- A branch of $T$ is an inclusion-maximal chain $B \subseteq T$.

# Containment relations of trees

## Definition (Embedding)

An embedding of $S$ into $T$ is a mapping $\varphi\colon S \to T$ such that $x <_S y \implies \varphi(x) <_T \varphi(y)$. It is a

1. level-embedding if $|x|_S = |\varphi(x)|_T$ holds for all $x \in S$.
2. homeomorphic embedding if it is injective and $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ for all $x, y \in S$.

**Intuition:**

- Level-embeddings are homomorphisms of rooted trees (edges go to edges).
- Injective level-embeddings are subgraphs for rooted trees where root goes to root.
- Homeomorphic embeddings are topological minors for rooted trees.

## Proposition

*Let $S$ and $T$ be trees. Then the two following conditions are equivalent:*

1. *There exists an embedding of $S$ into $T$.*
2. *There exists a level-embedding of $S$ into $T$.*

# Tree-homomorphisms

**Theorem (Nash-Williams, 1965)**

*The class of all order-theoretic trees is wqo by the homeomorphic embedding relation.*

**Corollary**

*The class of all order-theoretic trees is wqo by the level-embedding relation.*

The proof of Nash-Williams' theorem relies on the heavy machinery of bqo theory and is highly intricate. I found a new, direct proof of this corollary that completely avoids bqo theory.

**Proof idea:**

- Observe that it is trivial for trees of finite height and for trees that contain infinite branches.
- Hence it is enough to focus on trees of infinite height but without an infinite branch.
- Define a hierarchy of such trees.
- Prove by transfinite induction along this hierarchy that for every ordinal $\alpha$, the class of all trees that appear before level $\alpha$ is wqo by level-embeddings.
- Show that every tree without an infinite branch appears in some level of the hierarchy.

# Generalizations of WQOs

**Definition (Shelah, 1982)**

$Q$ is $\kappa$-wqo (where $\kappa$ is a cardinal) if in every sequence $\langle q_\alpha \mid \alpha < \kappa \rangle$ there are $\alpha < \beta$ such that $q_\alpha \preceq q_\beta$.
The well-ordering number of $Q$ is the least cardinal $\kappa$ such that $Q$ is $\kappa$-wqo (if it exists).

**Example**

- Finite graphs are $\omega_1$-wqo by [your favorite graph containment relation].
- The class $\mathcal{G}^{\leq\kappa}$ of graphs of size $\leq \kappa$ is $(2^\kappa)^+$-wqo by [your favorite graph containment relation].

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## Theorem (Vopěnka, Pultr, Hedrlín, 1965)

*For every $\kappa \geq \omega$, the well-ordering number of $\mathcal{G}^{\leq \kappa}$ with the homomorphism relation is $(2^\kappa)^+$.*

## Theorem (Komjáth, 1995)

*For every $\kappa > \omega$, the well-ordering number of $\mathcal{G}^{\leq \kappa}$ with the minor relation is $(2^\kappa)^+$.*

## Corollary

*The class of all graphs is not $\kappa$-wqo by [your favorite graph containment relation] for any $\kappa$.*

# Results of Shelah

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## Theorem (Shelah, 1982)

*The well-ordering number of order-theoretic trees with injective level-embeddings is a (mildly strong) large cardinal. In particular, the existence of such a cardinal cannot be proved in* ZFC.

This suggests that ZFC likely cannot prove the existence of a cardinal number $\kappa$ such that graph-theoretic trees would be $\kappa$-wqo by the subgraph relation.

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This suggests that ZFC likely cannot prove the existence of a cardinal number $\kappa$ such that graph-theoretic trees would be $\kappa$-wqo by the subgraph relation.

## Definition

$Q$ is class-wqo if in every sequence $\langle q_\alpha \mid \alpha \in \mathrm{On} \rangle$ there are $\alpha < \beta$ such that $q_\alpha \preceq q_\beta$.

## Problem

*Is it possible to prove (without relying on any large cardinals) that graph-theoretic trees are class-wqo by the subgraph relation?*

# Vopěnka's principle

## Axiom (Vopěnka's principle)

*The category of graphs has no large discrete full subcategory.*

Vopěnka's principle is weaker than a huge cardinal but it implies a proper class of measurable cardinals.

## Theorem

*Denote by $\mathcal{G}$ the class of all graphs. The following statements are equivalent to Vopěnka's principle:*

1. $\mathcal{G}$ *is class-wqo by the homomorphism relation.*
2. $\mathcal{G}$ *is class-wqo by the subgraph relation.*
3. $\mathcal{G}$ *is class-wqo by the induced subgraph relation.*

## Theorem (Komjáth, 1995)

$\mathcal{G}$ *is not $\kappa$-wqo by the minor relation for any $\kappa$, no matter how large.*

## Problem

*Vopěnka's principle clearly implies that $\mathcal{G}$ is class-wqo by the minor (or topological minor) relation. Does the converse implication hold as well?*