Well-quasi-ordering infinite trees by homomorphisms

Midsummer Combinatorial Workshop XXXI

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July 31, 2026

Well-quasi-orderings

Definition (Quasi-order)

A binary relation $\preceq$ on a class Q is a quasi-order if it is reflexive and transitive.

- $x \equiv y$ iff $x \preceq y$ and $y \preceq x$. . . . x and y are equivalent
- $x \prec y$ iff $x \preceq y$ and $x \not\equiv y$.
- x is minimal if there is no $y \prec x$.

Definition (Well-quasi-order)

Q is wqo if in every infinite sequence $x_0, x_1, x_2, \ldots$ of elements of Q, there are $i < j$ such that $x_i \preceq x_j$.

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Proposition

The following conditions are equivalent to Q being wqo:

1. *Every infinite sequence $x_0, x_1, x_2, \ldots$ of elements of Q contains an infinite non-decreasing subsequence $x_{i_0} \preceq x_{i_1} \preceq x_{i_2} \preceq \cdots$ for some indices $i_0 < i_1 < i_2 < \cdots$.*
2. *Q admits no infinite decreasing sequences $x_0 \succ x_1 \succ x_2 \succ \cdots$ and no infinite antichains.*
3. *Every nonempty subset of Q contains at least one but only finitely many non-equivalent minimal elements.*

Well-quasi-orderings

Definition

A property φ of Q is monotone if whenever $\varphi(x)$ holds and $y \preceq x$, then $\varphi(y)$ holds as well.
It is finitely testable if there exists a finite obstruction set $\mathcal{F}$ such that for all $x \in Q$ we have

$$\varphi(x) \text{ holds} \iff (\nexists z \in \mathcal{F}) : z \preceq x.$$

Example (Wagner's Theorem)

Planarity is a finitely testable property of finite graphs with respect to the minor relation. Its minimal obstructions are K_5 and $K_{3,3}$.

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Example (Wagner's Theorem)

Planarity is a finitely testable property of finite graphs with respect to the minor relation. Its minimal obstructions are K_5 and $K_{3,3}$.

Proposition

If Q is wqo, then every monotone property of Q is finitely testable.

Proof sketch: Define $\mathcal{F}$ as the (nonempty finite) set of non-equivalent minimal elements of the class

$$\{x \in Q \mid \varphi(x) \text{ does not hold}\}.$$

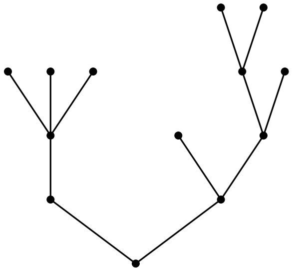
Order-theoretic trees

Definition

A tree is a partially ordered set $(T, <_T)$ such that for every $x \in T$, its downset

$$(\leftarrow, x) := \{y \in T \mid y <_T x\}$$

is a finite chain, and there is a unique minimal element called the root of T.



$$y <_T x$$

$$\Updownarrow$$

y lies on the unique path from x to the root.

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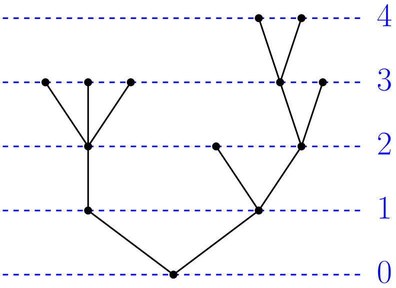

Terminology:

- $|x|_T := |(\leftarrow, x)|$ is the level of x in T.
- $\mathrm{H}(T) := \sup\{|x|_T + 1 \mid x \in T\} \leq \omega$ is the height of T.

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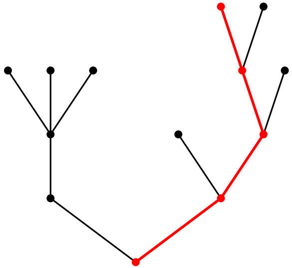

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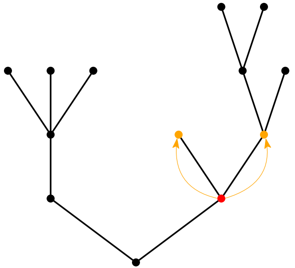

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- A branch of T is an inclusion-maximal chain $B \subseteq T$.
- x is an immediate successor of y if $y <_T x$ and $|x|_T = |y|_T + 1$.
- A node with no immediate successors is a leaf.

Containment relations of trees

Definition

A homomorphism from S to T is a map $\varphi\colon S \to T$ such that $x <_S y \implies \varphi(x) <_T \varphi(y)$. It is

1. level-preserving if $|x|_S = |\varphi(x)|_T$ holds for all $x \in S$,
2. leaf-preserving if whenever x is a leaf of S, then $\varphi(x)$ is a leaf of T,

Homomorphism order:

- $S \preceq_h T \iff$ there exists a homomorphism from S to T.

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- $S \preceq_h T \iff$ there exists a homomorphism from S to T.

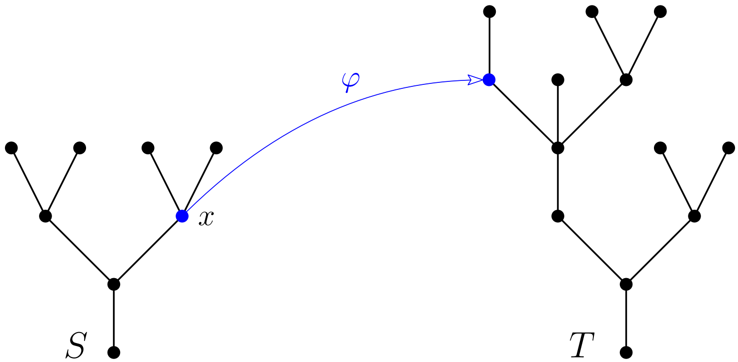
👁 $S \preceq_h T \iff$ there exists a level-preserving homomorphism from S to T.

When restricted to trees without infinite branches, then also

👁 $S \preceq_h T \iff$ there exists a leaf-preserving homomorphism from S to T.

Proof of the equivalence

Homomorphism $\implies$ level-preserving homomorphism



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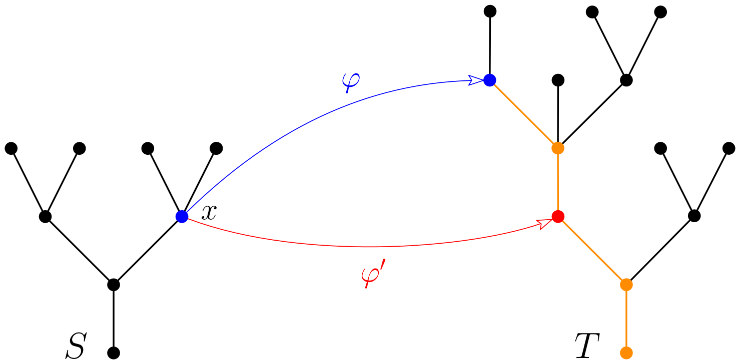
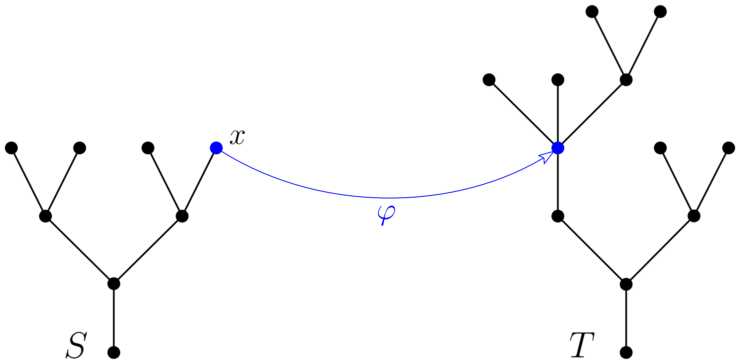


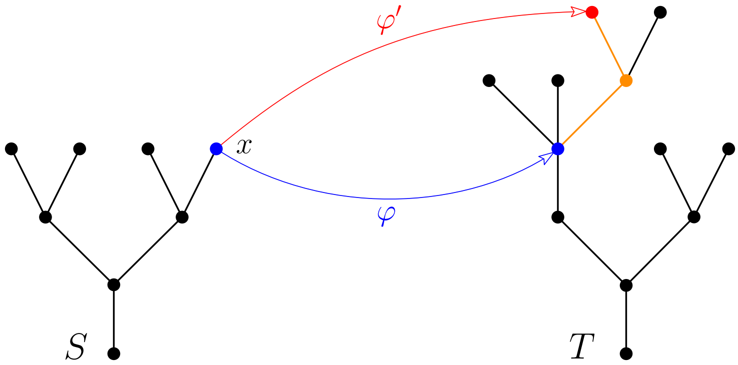
Proof of the equivalence

Homomorphism $\implies$ leaf-preserving homomorphism



Proof of the equivalence

Homomorphism $\Longrightarrow$ leaf-preserving homomorphism



Nash-Williams' theorem about infinite trees

Theorem (Kruskal's Tree Theorem, 1960)

The class of all finite trees is wqo by the homeomorphic embedding relation.

In order to extend this result to infinite trees, Nash-Williams introduced the stronger notion of better-quasi-orderings or bqo. Most wqos encountered in practice are, in fact, bqos.

- Well-ordered $\implies$ bqo $\implies$ wqo.

Theorem (Nash-Williams, 1965)

The class of all finite or infinite trees is bqo by the homeomorphic embedding relation.

Nash-Williams' theorem about infinite trees

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Theorem (Nash-Williams, 1965)

The class of all finite or infinite trees is bqo by the homeomorphic embedding relation.

Corollary

The class of all finite or infinite trees is wqo by the homomorphism relation.

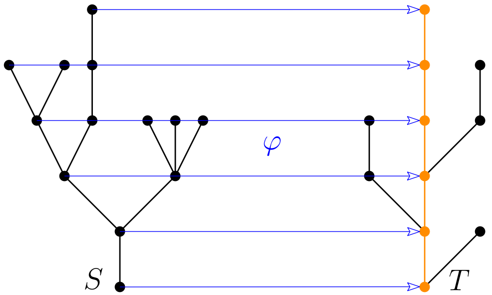
The proof of Nash-Williams' theorem relies on the heavy machinery of better-quasi-orderings.
I found a proof of this corollary that uses only well-quasi-orderings and the axiom of choice.

Initial observation

Observation

Let S and T be arbitrary trees.

1. *If* $\mathrm{H}(S) < \omega$ *and* $\mathrm{H}(S) \leq \mathrm{H}(T)$, *then* $S \preceq_h T$.



Proof:

- T contains a branch B of length at least $\mathrm{H}(S)$
- Define a level-preserving homomorphism φ from S in T by

$$x \mapsto \text{unique element of } B \text{ at height } |x|_S.$$

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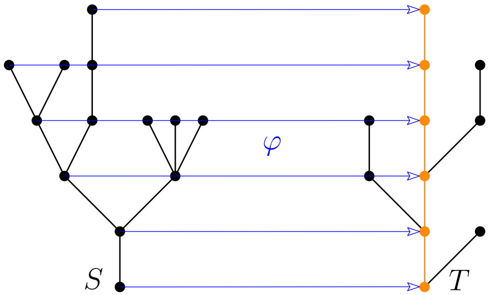
1. *If* $\mathrm{H}(S) < \omega$ *and* $\mathrm{H}(S) \leq \mathrm{H}(T)$, *then* $S \preceq_h T$.
2. *If* $\mathrm{H}(S) \leq \omega$, *and* T *contains an infinite branch, then* $S \preceq_h T$.



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- T contains a branch B of length at least $\mathrm{H}(S)$
- Define a level-preserving homomorphism φ from S in T by

$$x \mapsto \text{unique element of } B \text{ at height } |x|_S.$$

Important:

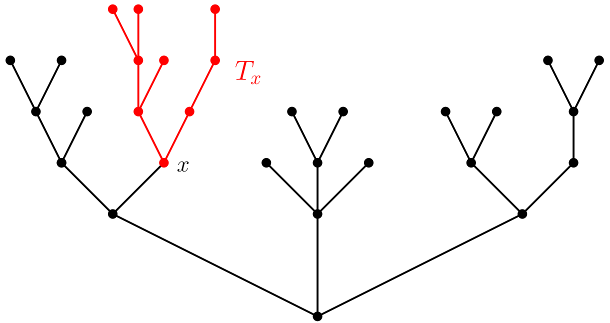
From now on, by "tree" we mean a tree without an infinite branch.

Branch trees

Definition

A subtree of T is any $S \subseteq T$ with the induced order, that is also a tree (has a root).

- $T_x := \{y \in T \mid x \leq_T y\}$ is the subtree of T sprouting from $x \in T$.



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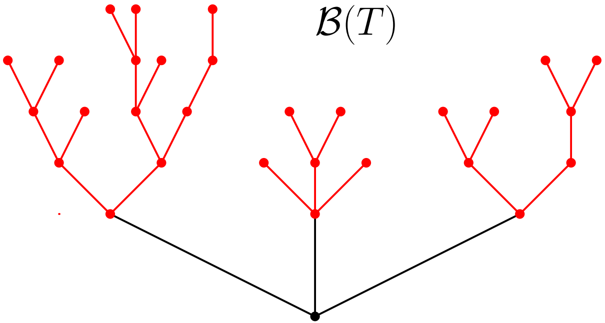
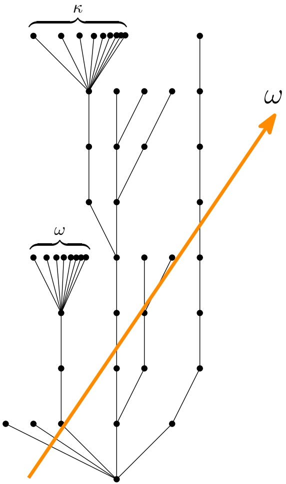
Branch trees of T are the subtrees sprouting from the immediate successors of the root.



Bushes

Definition

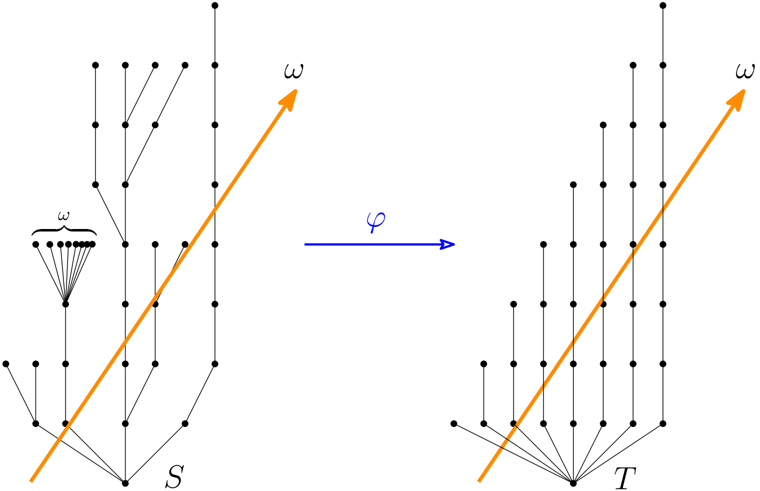
T is a bush if $\mathrm{H}(T) = \omega$ but $H(B) < \omega$ for all branch trees $B \in \mathcal{B}(T)$.



Bushes

Lemma

If S and T are *bushes**, then* $S \preceq_h T$.

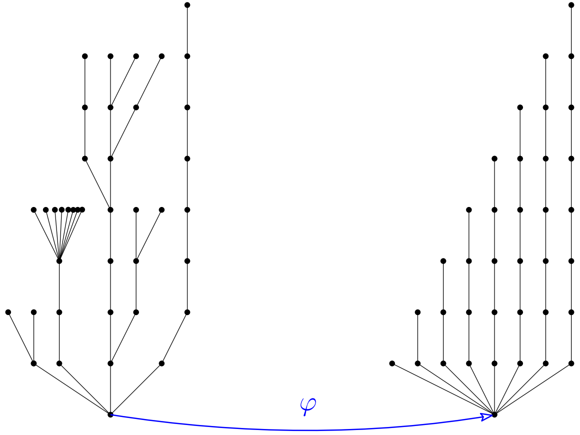


Bushes

Lemma

If S and T are bushes, then $S \preceq_h T$.

Proof: Map the root to the root and for every $B \in \mathcal{B}(S)$ find $B' \in \mathcal{B}(T)$ with $\mathrm{H}(B) \leq \mathrm{H}(B')$.



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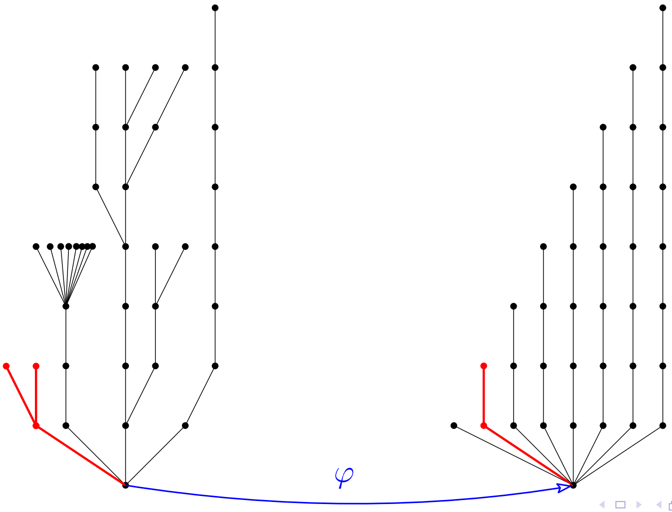

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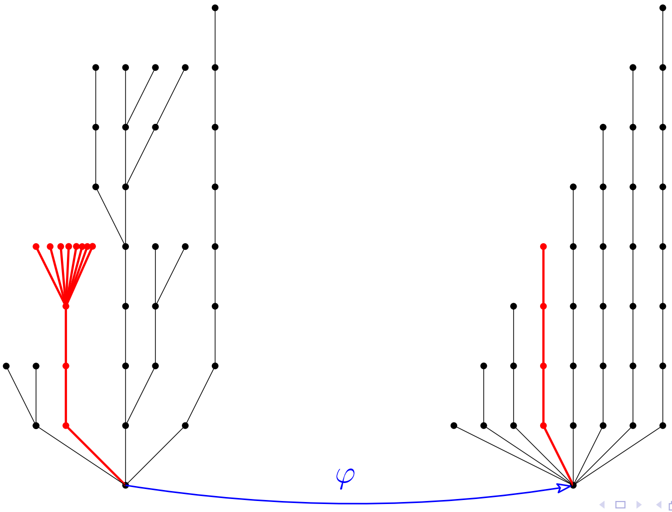
If S and T are bushes, then $S \preceq_h T$.



Bushes

Lemma

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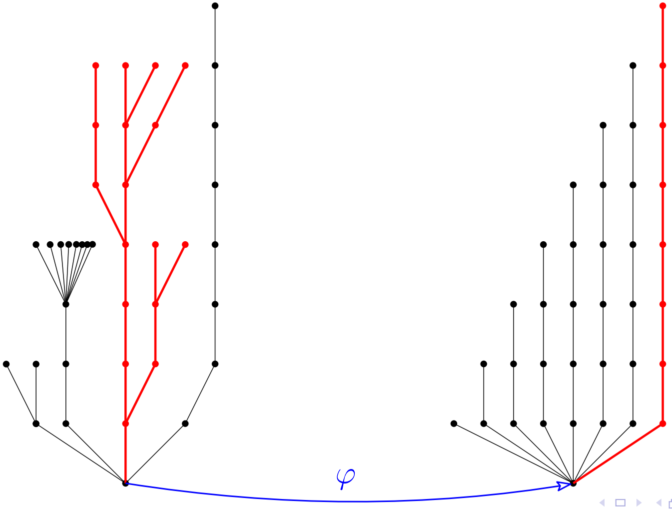
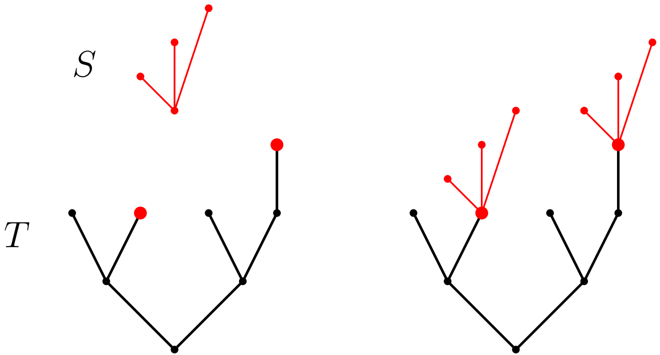
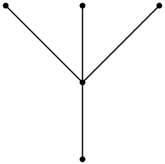


Replacing two red leaves of T with S.



Type of a tree

Type 0 $\iff$ trees with finite height.
Type 1 $\Leftrightarrow$ bushes.



$$\mathrm{tp} = 0$$

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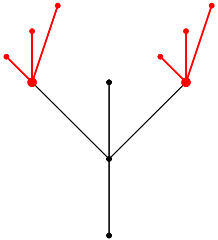

$$\mathrm{tp} = 2$$

Type of a tree

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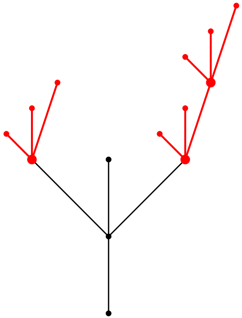
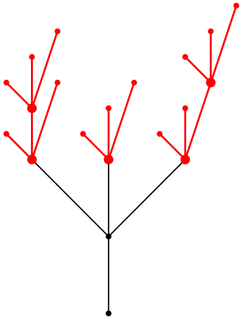


$$\mathrm{tp} = 3$$

Type of a tree

Type 0 $\iff$ trees with finite height.
Type 1 $\Leftrightarrow$ bushes.



$\mathrm{tp} = 3$

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$$T_1 \qquad T_2 \qquad \cdots \qquad T_n \qquad \cdots$$

Type of a tree

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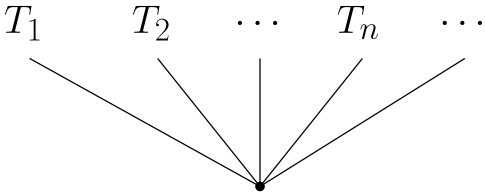

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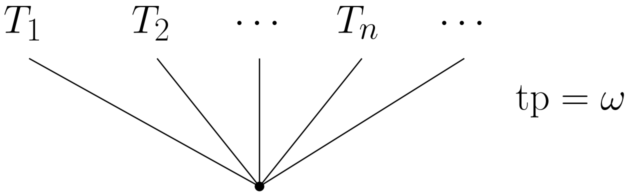

Type of a tree

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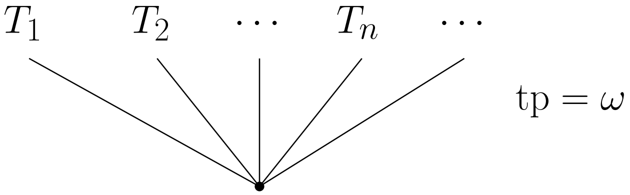
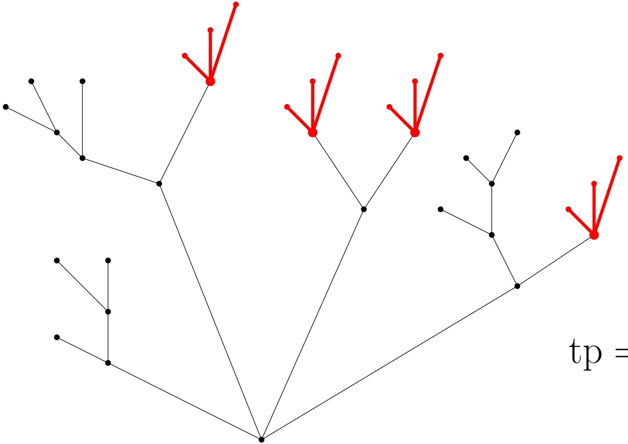


- α is a bushy ordinal if $\alpha = 1$ or if α is limit.
- T is an α-bush if $\mathrm{tp}(T) \not< \alpha$ but $\mathrm{tp}(B) < \alpha$ for every branch-tree $B \in \mathcal{B}(T)$.

Type α $\iff$ α-bushes

Type of a tree

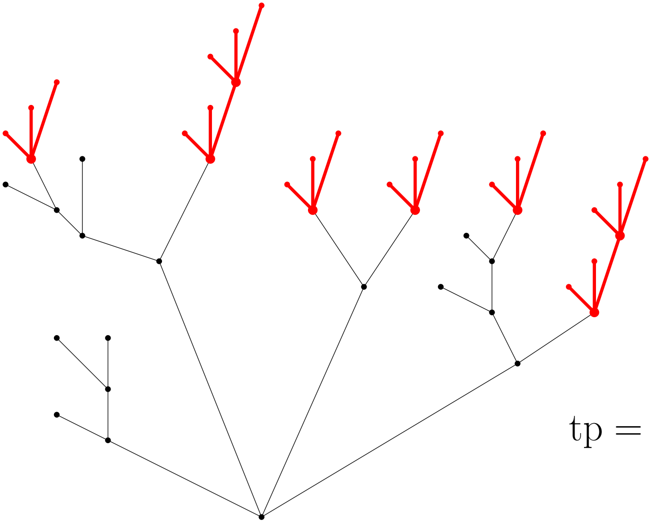
Type $\alpha \iff \alpha$-bushes



$$\mathrm{tp} = \alpha + n$$

Type of a tree

Type $\alpha \iff$ α-bushes



$$\mathrm{tp} = \alpha + n + 1$$

Basic properties

If S is a subtree of T, then $\mathrm{tp}(S) \leq \mathrm{tp}(T)$.

If S was obtained from T by deleting some of its leaves, then $\mathrm{tp}(S) = \mathrm{tp}(T)$.

If every branch-tree of T has its type defined, then T also has its type defined.

Basic properties

If S is a subtree of T, then $\mathrm{tp}(S) \leq \mathrm{tp}(T)$.

If S was obtained from T by deleting some of its leaves, then $\mathrm{tp}(S) = \mathrm{tp}(T)$.

If every branch-tree of T has its type defined, then T also has its type defined.

Lemma

Every tree T has its type defined.

Proof:

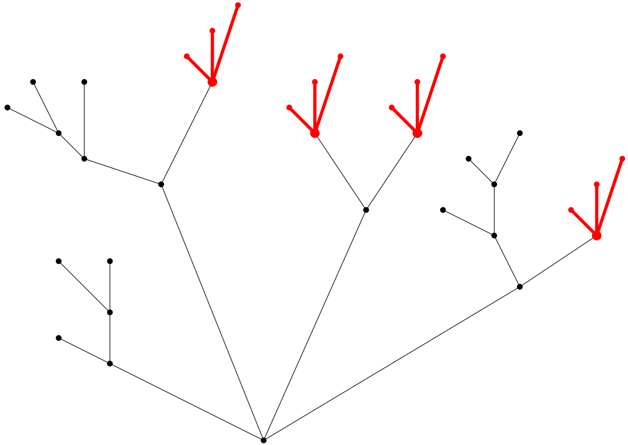
- Call a node $x \in T$ bad if $\mathrm{tp}(T_x)$ is undefined.
- 👁 $\mathrm{tp}(T)$ is undefined $\iff$ the root of T is bad.
- 👁 If x is bad, then it has an immediate successor that is also bad.
- This allows us to construct an infinite branch if $\mathrm{tp}(T)$ is undefined, a contradiction.

Collapse of a tree

Definition (Collapse)

Let α be a bushy ordinal. The α-collapse of a tree T is the subtree

$$\mathrm{col}_\alpha(T) := \{x \in T \mid \mathrm{tp}(T_x) \geq \alpha\}.$$



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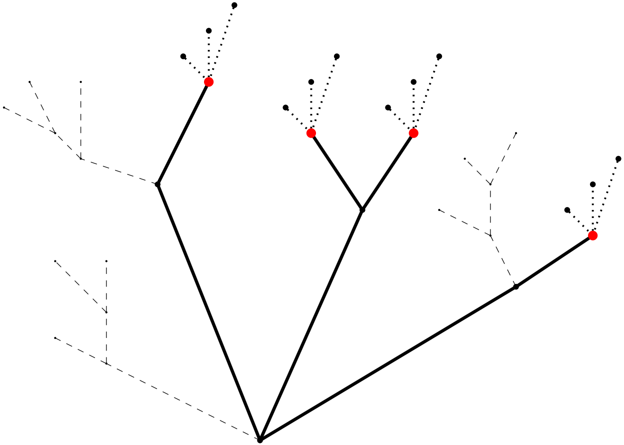

If x is a leaf of $\mathrm{col}_\alpha(T)$, then T_x is an α-bush.

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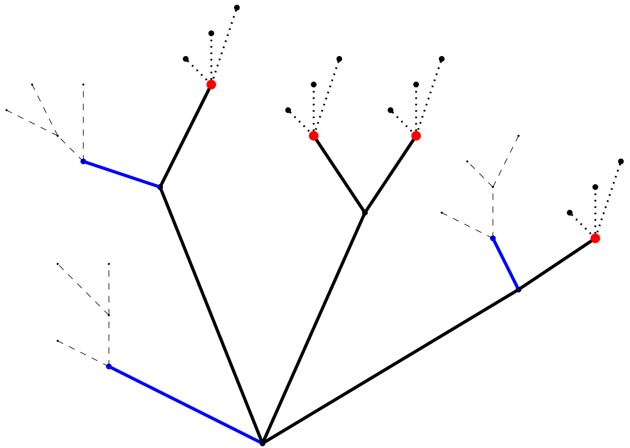

Add roots of short deleted sprouts.

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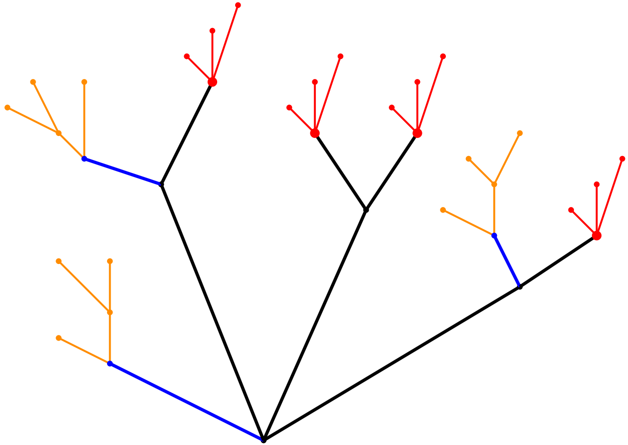

We can now obtain T by replacing some leaves with trees of type $\leq \alpha$.

Key lemma

Lemma

Let S and T be arbitrary trees and let α be a bushy ordinal.

1. *If* $\mathrm{tp}(S) < \mathrm{tp}(T)$, *then* $S \preceq_h T$.
2. *If* S *and* T *are both* α*-bushes, then* $S \preceq_h T$.
3. *If* $\mathrm{col}_\alpha(S) \preceq_h \mathrm{col}_\alpha(T)$, *then* $S \preceq_h T$.

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3. *If $\mathrm{col}_\alpha(S) \preceq_h \mathrm{col}_\alpha(T)$, then $S \preceq_h T$.*

We will show (3) assuming (1) & (2).

$$\mathrm{col}_\alpha(S) \preceq_h \mathrm{col}_\alpha(T) \implies S \preceq_h T$$

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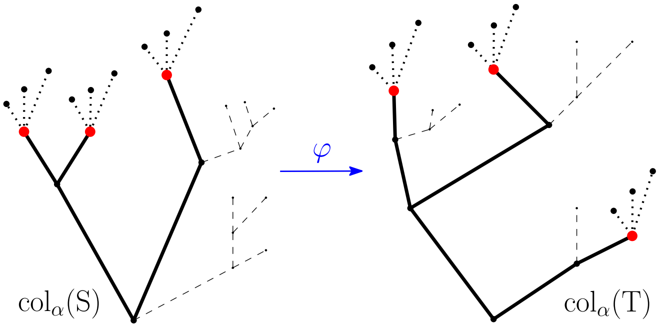

Let $\varphi\colon \mathrm{col}_\alpha(S) \to \mathrm{col}_\alpha(T)$ be a leaf-preserving homomorphism.

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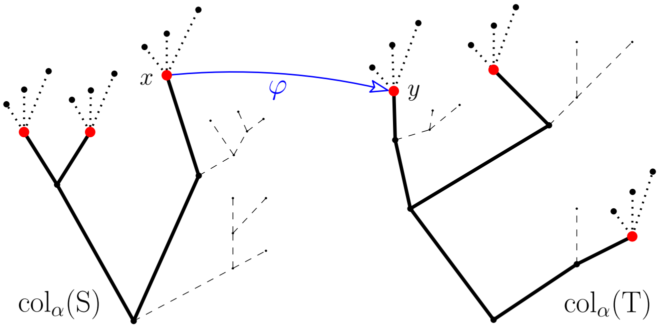
2. *If S and T are both α-bushes, then* $S \preceq_h T$.
3. *If* $\mathrm{col}_\alpha(S) \preceq_h \mathrm{col}_\alpha(T)$, *then* $S \preceq_h T$.



Let $\varphi\colon \mathrm{col}_\alpha(S) \to \mathrm{col}_\alpha(T)$ be a leaf-preserving homomorphism.

If $x \in S$ is a leaf, then S_x and $T_{\varphi(x)}$ are α-bushes, so $S_x \preceq_h T_{\varphi(x)}$.

Key lemma

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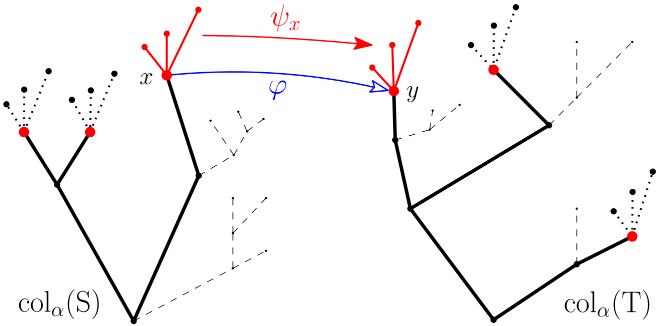

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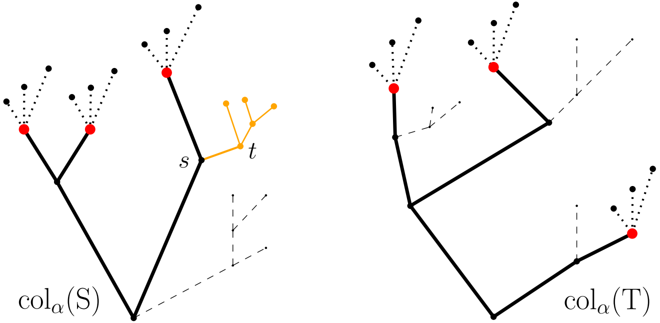

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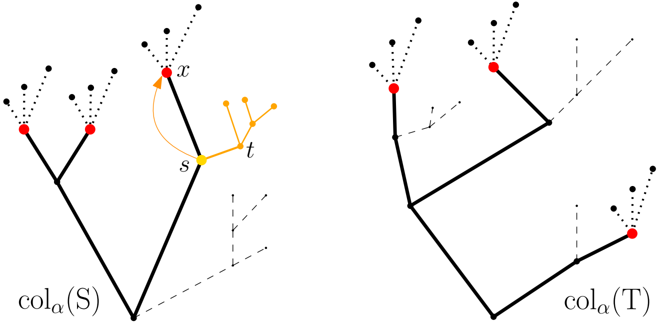

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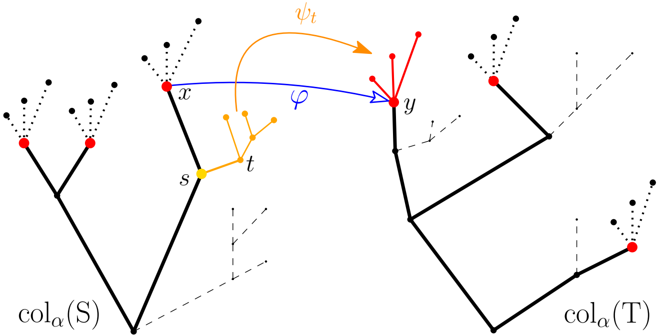

Let $\varphi\colon \mathrm{col}_\alpha(S) \to \mathrm{col}_\alpha(T)$ be a leaf-preserving homomorphism.

If $x \in S$ is a leaf, then S_x and $T_{\varphi(x)}$ are α-bushes, so $S_x \preceq_h T_{\varphi(x)}$.

If $t \in S$ is the root of a short deleted sprout growing from s, find a leaf x above s and a homomorphism $S_t \to T_{\varphi(x)}$.

Finishing the proof

Lemma (★)

For every ordinal α, the class $\mathcal{T}(\alpha)$ of all trees with type $\leq \alpha$ is wqo by the homomorphism relation.

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Corollary

Trees without infinite branches are wqo by the homomorphism relation.

Proof:

- Let $T_0, T_1, T_2, \ldots$ be a sequence of trees without infinite branches.
- Consider the sequence of their types $\alpha_0, \alpha_1, \alpha_2 \ldots$ and let $\alpha := \sup_i \alpha_i$.
- Then $T_i \in \mathcal{T}(\alpha)$ for all $i \in \omega$.
- Lemma ★: there are $i < j$ such that $T_i \preceq_h T_j$.

Finishing the proof

Lemma (★)

For every ordinal α, the class $\mathcal{T}(\alpha)$ of all trees with type $\leq \alpha$ is wqo by the homomorphism relation.

Proof of Lemma ★. By transfinite induction on α.

- Let $T_0, T_1, T_2, \ldots$ be a sequence of trees of type at most α.
- If $\alpha = 0$, consider $\mathrm{H}(T_0), \mathrm{H}(T_1), \mathrm{H}(T_2), \ldots$ Since ω is well-ordered, there are $i < j$ such that $\mathrm{H}(T_i) \leq \mathrm{H}(T_j)$ and thus $T_i \preceq_h T_j$.

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- Suppose $\alpha > 0$ and assume that the claim holds for all $\beta < \alpha$.
- If there are $i < j$ such that $\mathrm{tp}(T_i) < \mathrm{tp}(T_j)$, then $T_i \preceq_h T_j$ by the Key Lemma.

Finishing the proof

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- Suppose $\alpha > 0$ and assume that the claim holds for all $\beta < \alpha$.
- If there are $i < j$ such that $\mathrm{tp}(T_i) < \mathrm{tp}(T_j)$, then $T_i \preceq_h T_j$ by the Key Lemma.
- Hence assume $\mathrm{tp}(T_0) \geq \mathrm{tp}(T_1) \geq \mathrm{tp}(T_2) \geq \cdots$.
- If there is an infinite subset of indices $A \subseteq \omega$ such that $\mathrm{tp}(T_i) < \alpha$ for all $i \in A$, then

$$\beta := \sup\{\mathrm{tp}(T_i) \mid i \in A\} = \mathrm{tp}(T_{\min A}) < \alpha.$$

- Thus $T_i \in \mathcal{T}(\beta)$ for all $i \in A$.
- Induction hypothesis for β: there are $i < j$ in A such that $T_i \preceq_h T_j$.

Finishing the proof

Lemma (★)

For every ordinal α, the class $\mathcal{T}(\alpha)$ of all trees with type $\leq \alpha$ is wqo by the homomorphism relation.

Proof of Lemma ★. By transfinite induction on α.

- Let $T_0, T_1, T_2, \ldots$ be a sequence of trees of type at most α.
- Hence assume $\mathrm{tp}(T_0) \geq \mathrm{tp}(T_1) \geq \mathrm{tp}(T_2) \geq \cdots$.
- Thus WLOG $\mathrm{tp}(T_i) = \alpha$ for all $i \in \omega$.
- If α is a bushy ordinal, then $T_i \preceq_h T_j$ for all $i < j$ from the Key Lemma.
- Therefore assume that $\alpha = \gamma + n$ is a successor ordinal, where γ is bushy and $n \geq 1$.
- Consider the sequence of collapsed trees $T_0', T_1', T_2', \ldots$ where $T_i' = \mathrm{col}_\gamma(T_i)$.
- For all $i \in \omega$ we have $T_i' \in \mathcal{T}(\gamma + n - 1)$.
- Induction hypothesis: there are $i < j$ such that $T_i' \preceq_h T_j'$ and by the Key Lemma also $T_i \preceq_h T_j$.

Thank you!