Def (Tree): Well-founded partial order $(T, <_T)$ s.t. $\forall x: (\leftarrow, x)$ is a chain (1)

$|T| \leq \omega$ — finite

unique minimal element ... root

finitely branching !

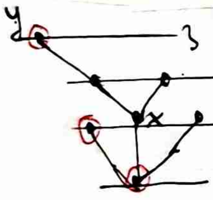
Def: Subtree $S \subseteq T$ with inherited order

Terminology

- for x: $y < x$ ... predecessor, $x < y$ ... successor, min successor = imm. successor
- leaf ... no imm. successors
- $|x|_T := |(\leftarrow, x)|$ ... height of x in T
- $T(n) := \{x \in T \mid |x_T| = n\}$ ... level n of T
- height of T ... $ht(T) :=$ least $\alpha \leq \omega$ s.t. $\forall x \in T: \alpha > |x|_T$
- branch in T ... $\subseteq$-max chain

- branching factor
- finitely branching

Examples:



$x < y \iff x$ lies on the unique path from y to the root

height $= 4$

$\hookrightarrow$ subtree:

Cantors tree

no leaves, finitely branching

height ω

Def: A tree S is a strong subtree of $T \equiv$

(1) S is a subtree of T ... S has a root

(2) S is level preserving ... $\exists$ level function $f: ht(S) \to ht(T)$

$(\forall n < ht(S)): S(n) \subseteq T(f(n))$

(3) S is ballanced $\begin{cases} ht(S) = \omega \Rightarrow S \text{ has no leave} \\ ht(S) < \omega \Rightarrow \text{all leaves of } S \text{ are on the same level} \end{cases}$

(4) $\forall x \in S$ that is not a leaf satisfies: $\forall y$ imm. succ of x in T $\exists y'$ imm. succ. of x in S $y \leq_T y'$

$\hookrightarrow$ branching factor

$\Rightarrow BF_S(x) = BF_T(x)$

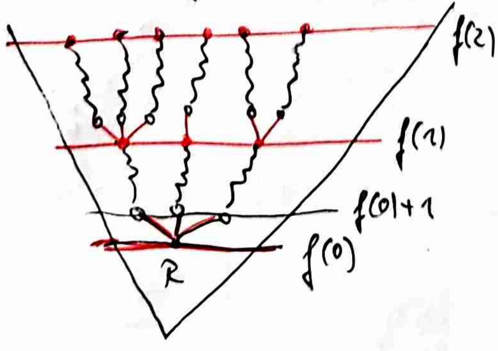
Notation: $STR_\alpha(T)$

$\alpha \leq \beta$

$S_\beta \in STR_\beta(T)$

$S_\alpha \in STR_\alpha(S_\beta)$

$\Rightarrow S_\alpha \in STR_\alpha(T)$



T … height ω, finitely branching, no leaves

If T is finitely colored, then $\exists$ monochromatic $S \in STR_\omega(T)$

→ assume root is red, and try to build S

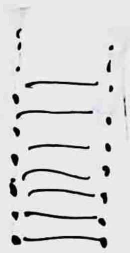
→ if fails, we can build a blue one

Def (Product tree): Sequence of trees $\Pi = (T_1, \dots, T_d)$

- product: $\prod_\pi := T_1 \times \dots \times T_2$
- level product: $\prod_\pi^{lv} := \{(x_1, \dots, x_d) \in \prod_\pi \mid |x_1|_{T_1} = \dots = |x_d|_{T_d}\}$

} two notions

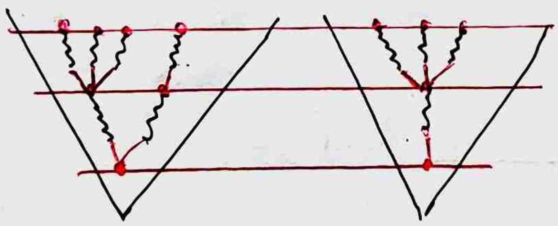
Example:

$\prod_{(\omega,\omega)}$  $\prod^{lv}_{(\omega,\omega)}$  ← easy - pigeonhole

↳ fails (counterexample) ← when colored

Def: A strong product subtree of height $\alpha \leq \omega$ of $\Pi = (T_1, \dots, T_d)$ is $S = (S_1, \dots, S_d)$ s.t. $\forall i$: $S_i \in STR_\alpha(T_i)$ & S_i share the level function

Notation: $STR_\alpha(\Pi)$



- $\Pi = (T_1, \dots, T_d)$ where T_i are finitely branching, height ω, no leaves

Theorem (Halpern–Läuchli, 1966): If $\prod_\Pi^{lv}$ is finitely colored, there $\exists$ homogeneous $S \in STR_\omega(\Pi)$ … $\prod_S^{lv}$ is monochromatic

Theorem (Milliken, 1979): $\forall n$: If $STR_n(\Pi)$ is finitely colored, there $\exists$ homogeneous $S \in STR_\omega(\Pi)$ … $STR_n(S)$ is monochromatic

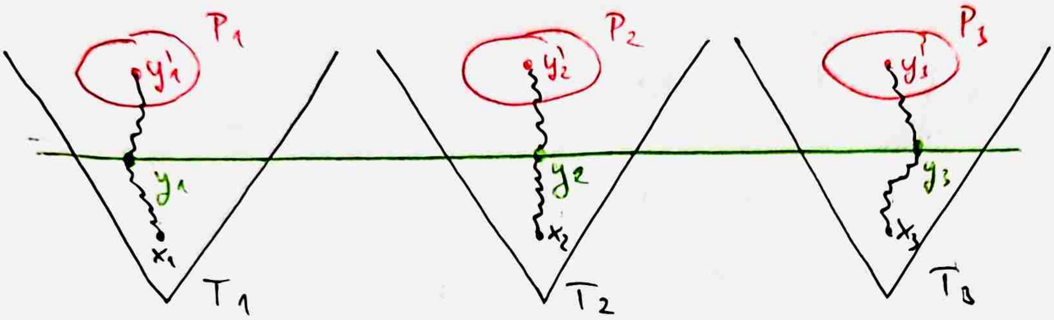
Milliken ⇒ Ramsey: $\Pi = (\omega)$ …

Halpern–Läuchli is Milliken for $n=1$

Notation: $y \in T$ extends $x \in T$ $\equiv$ $x \leq_T y$ ... y is above x

Def: Let $x \in T$, $k > |x|_T$. A subset $P \subseteq T$ is k-x-dense $\equiv$
P is strictly above level k & $\forall y \in T(k)$ extending x has an extension $y' \in P$

Def: Let $\bar{x} \in \prod_\pi^{lev}$, $k > |\bar{x}|_\pi$. A subset $P = P_1 \times \dots \times P_d \subseteq T_1 \times \dots \times T_d = \prod_\pi$ is a k-$\bar{x}$-dense matrix $\equiv$ each P_i is a k-x_i-dense subset of T_i.



Def: For $\pi = (T_1, \dots, T_d)$ finitely branching, height ω, no leaves, define

$\boxed{SS_d}$ $\forall$ finite coloring of $\prod_\pi^{lev}$: $\exists$ homogeneous $S \in STR_\omega(\pi)$

$\boxed{SD_d}$ $\forall$ finite coloring of $\prod_\pi$: $\exists \bar{x}\, \exists k\, \exists$ homogeneous k-$\bar{x}$-dense matrix $P = P_1 \times \dots \times P_d$

$\boxed{SD_d^{lev}}$ Moreover, P is a pancake — all P_i are contained in a single shared level
↳ so coloring $\prod_\pi^{lev}$ is enough ↳ $\exists n \forall i: P_i \subseteq T_i(n)$

$\boxed{DS_d}$ $\forall$ finite coloring of $\prod_\pi$: $\exists \bar{x}$ $\forall k > |\bar{x}|_\pi$ $\exists$ homogeneous k-$\bar{x}$-dense matrix P_k, and all P_k share the same color

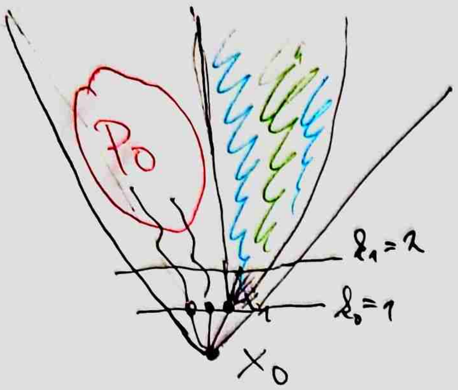
$\boxed{DS_d^{lev}}$ Moreover, each P_k is a pancake → so coloring $\prod_\pi^{lev}$ is enough

Battle plan:

1. SD_1
2. $SD_d \Rightarrow SD_d^{lev}$
3. $SD_d^{lev} \Rightarrow DS_d^{lev}$ ... resp. $SD_d \Rightarrow DS_d$
4. $DS_d^{lev} \Rightarrow SS_d$
5. SS_d & $DS_d^{lev} \Rightarrow SD_{d+1}$

Stevo Todorčević
Spencer Unger
Honza Hubička

SD_1



→ compactness: $H=(V,E)$

$V := \Pi\Pi_\pi$, $E := \{P \subseteq \Pi\Pi_\pi \mid \exists \bar{x} \exists \ell, P = \ell\text{-}\bar{x}\text{-dense m.}\}$

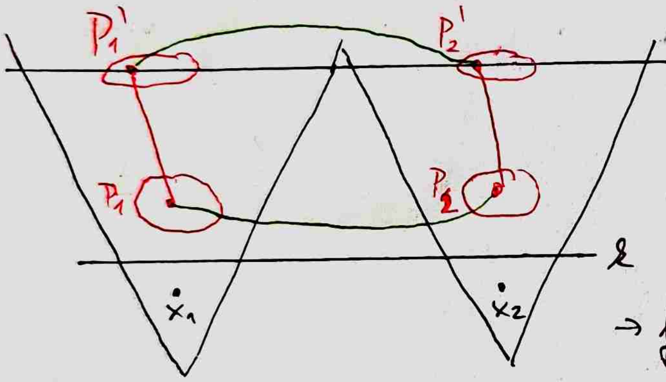
$SD_d \Rightarrow \chi(H) > r \Rightarrow \exists$ finite $W \subseteq V : \chi(H_W) > r$

↳ claim... N above everything in W

$SD_d \Rightarrow SD_d^{lev}$

claim: $\forall r \exists N \forall r$-coloring of $\Pi\Pi_\pi$ $\exists$ monochromatic ℓ-$\bar{x}$-dense matrix below level N

⊕ • we need only 1 extension in P for $\forall y \in T(\ell)$ ... but only bounded



N . let $\chi : \Pi\Pi_\pi \to r$ be given

• define $\chi' : \Pi\Pi_{\pi \restriction N} \to r$, $\chi'(\bar{x}) := \chi(f(\bar{x}))$

→ for $\forall \bar{x} \in \Pi\Pi_{\pi \restriction N}$ choose $f(\bar{x})$ above $\bar{x}$ at level N

→ find a χ'-monochr. ℓ-$\bar{x}$-dense matrix ⊕ P

$P' := \{f(\bar{y}) \mid \bar{y} \in P\}$ is χ-monochr. ℓ-$\bar{x}$-dense with the same color

$SD_d \Rightarrow PS_d$ resp. $SD_d^{lev} \Rightarrow PS_d^{lev}$

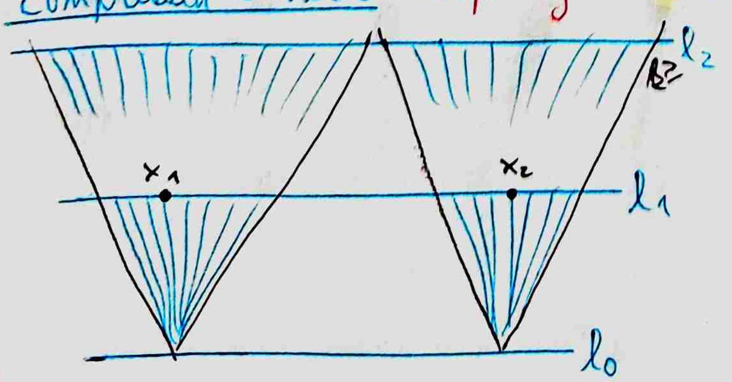
• assume PS_d fails for a given coloring $\chi : \Pi\Pi_\pi \to r$

$\Rightarrow \forall \bar{x} \exists \ell_{\bar{x}}$ s.t. there is no $\ell_{\bar{x}}$-$\bar{x}$-dense matrix

👁 50-$\bar{x}$-dense is stronger than 10-$\bar{x}$-dense

• define levels $l_0, l_1, \ldots$ as $l_0 := 0$, $l_{i+1} := \max\{\ell_{\bar{x}} \mid \bar{x} \in \Pi\Pi_{\pi \restriction l_i}\}$

compressed subtree ~ keep only $x \in T(l_i)$



← S has no ℓ-$\bar{x}$-dense matrix ↯ SD_1

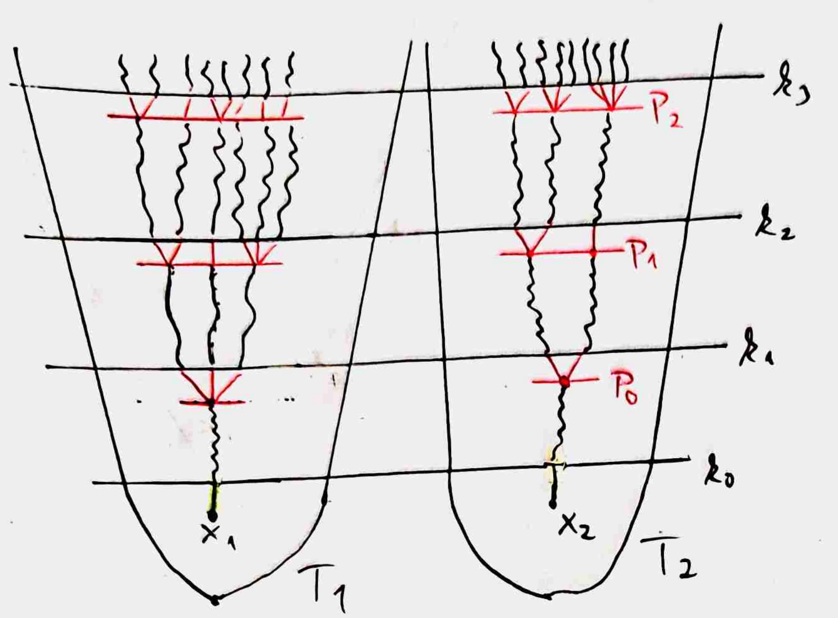
• suppose we have a ℓ-$\bar{x}$-dense m. in S

$\bar{x}$ ... level i in S, l_i of T

$\ell \geq i+1$ ... $\geq l_{i+1} \geq \ell_{\bar{x}}$ in T

→ it would be $\ell_{\bar{x}}$-$\bar{x}$-dense in T ↯

$DS_d^{lex} \Rightarrow SS_d$



$DS_d^{lex} \rightsquigarrow \bar{x}$

$k_0 = |x|_T + 1 \rightsquigarrow P_0$

$\Rightarrow$ choose root $\in P_0$

$k_{i+1} = |P_i|_T + 1 \rightsquigarrow P_{i+1}$

SS_d & $DS_d^{lex} \Rightarrow SD_{d+1}$

- $[T]$ = set of branches of T ... finitely branching, height ω, no leaves
- $D \subseteq T$ is dense above $t \in T$ $\equiv$ $\forall t' \geq t\ \exists s \geq t'$ s.t. $s \in D$

Lemma: $\forall g: [T] \to \omega\ \exists t^* \in T,\ \exists c^* \in \omega,\ \exists D \subseteq T$ dense above t^* s.t.
$\forall s \in D\ \exists b \in [T]$ s.t. $s \in b$ & $g(b) = c^*$

Proof: Try t_0 = root, $c_0 = 0$, D = nodes s above t_0 s.t. $\exists$ branch $b \ni s$, $g(b) = c_0$

$\to$ if fails, $\exists t_1 \geq t_0\ \forall s \geq t_1$, there is no branch of color 0 going through s

Try t_1, $c_1 = 1$, D = nodes s above t_1 s.t. $\exists$ branch $b \ni s$, $g(b) = c_1$

$\to$ if fails $\exists t_2 \geq t_1\ \forall s \geq t_2$, $\nexists$ branch of color 1 going through s

$\rightsquigarrow t_0, t_1, t_2, \ldots$ t_n ... no branch of color $0, 1, \ldots, n-1$ going through t_n

↳ determines a branch that has no color ∎

SS_d & $DS_d^{lev} \Rightarrow SD_{d+1}$

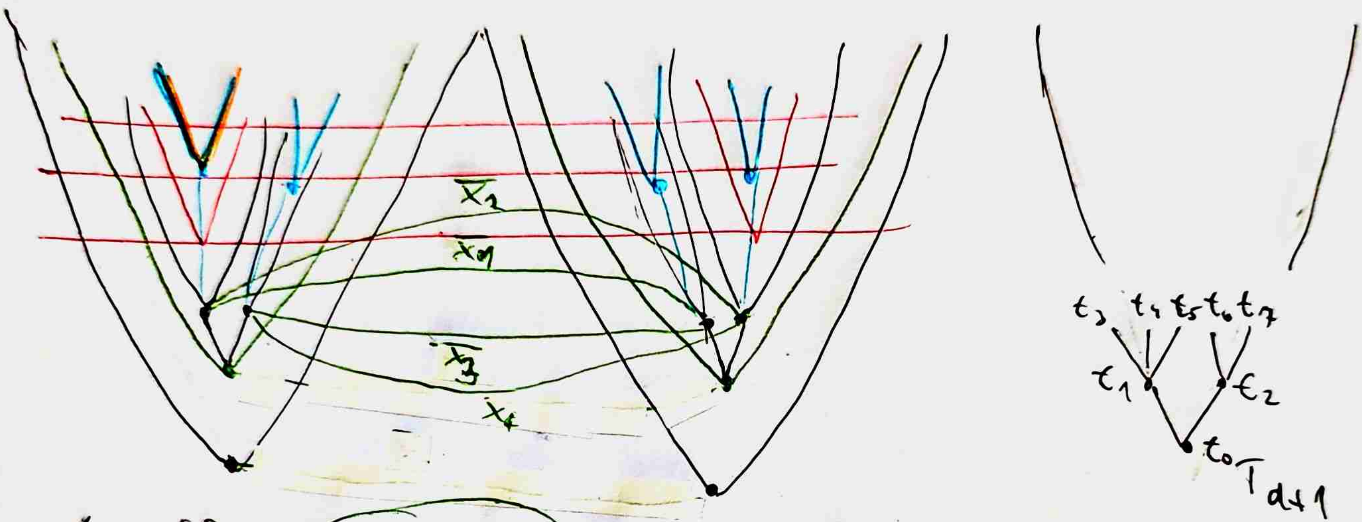
→ trees $T_1, \dots, T_d, T_{d+1}$, coloring $\chi : \prod_{(T_1,\dots,T_{d+1})} \to r$

• $\forall t \in T_{d+1}$ define $\chi_t : \prod_{(T_1,\dots,T_d)} \to r$, $\chi_t(\bar{x}) = \chi(\bar{x}^\frown t)$

Lemma: $\exists (S_1,\dots,S_d) \in STR_\omega(T_1,\dots,T_d)$ s.t. $\forall t \in T_{d+1}$:

$\forall \bar{x} \in \prod^{lev}_{(S_1,\dots,S_d)}$ with $|\bar{x}|_S \geq |t|_{T_{d+1}}$: $\prod^{lev}_{(S_1,\dots,S_d)} \upharpoonright \bar{x}$ is χ_t-monochr.

Proof: enumerate vertices of T_{d+1} level by level



• t_0 ... SS_d ⇝ strong subtree ... roots will never move

→ enumerate pairs of out-edges ⇝ $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$

• fixing for t_1 ⇝ SS_d for subtree above $\bar{x}_1$ ⇝ strong subtree

→ prune the rest of the tree ... make a strong subtree using levels of

→ repeat for $\bar{x}_2, \bar{x}_3, \bar{x}_4$

• fixing for t_2 ... repeat

⇒ finally, we have strong subtrees → roots will never move

• fix next layer ...

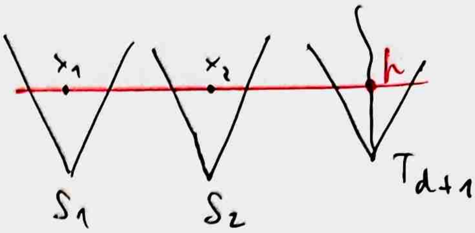
$\boxtimes$

$\Longrightarrow$ we get $S_1, \dots, S_d \in STR_\omega(T_1,\dots,T_d)$

SS_d & $DS_d^{lev} \Rightarrow SD_{d+1}$

• $\forall b \in [T_{d+1}]$ define $\chi^b : \prod^{lev}_{(S_1,\dots,S_d)} \to r$.

$$\chi^b(\bar x) := \chi(\bar x ^\frown p)$$



• DS_d^{lev} for $S_1,\dots,S_d$ and χ^b

↳ $\bar x_b$, c_b ⊛ $\forall k > |\bar x_b|_S$ $\exists$ k-$\bar x_b$-dense pancake P with χ^b-color c^b

• coloring of $[T_{d+1}]$ by $b \mapsto (\bar x_b, c_b)$ ... $\omega^d \cdot r \approx \omega$ colors

⊛

• Lemma: $\exists (\bar x^*, c^*)$, $\exists t^* \in T_{d+1}$ $\exists D \subseteq T_{d+1}$ dense above t^* ... $\forall s \in D\ \exists b \ni s$, $\chi^b \rightsquigarrow \bar x^*, c^*$

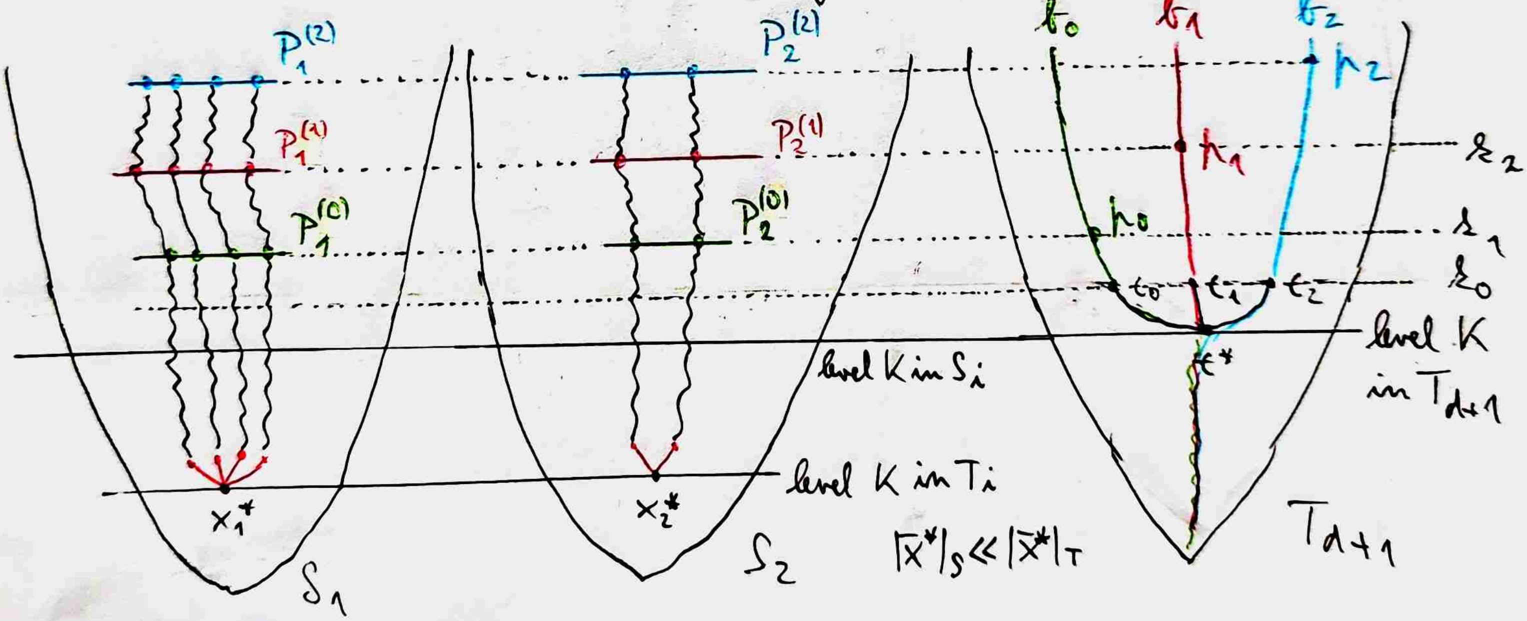
! move $\bar x^*$ up through S_i and t^* up through T_{d+1} so that $\underbrace{|\bar x^*|_T = |t^*|_{T_{d+1}}}_{K}$

• construct a $(K+1)$-$(\bar x^* {}^\frown t^*)$-dense matrix

• $t_0, t_1, \dots, t_n$ imm. succ. of t^*

👁 $\forall t_i$ $\exists$ branch $b \ni t_i$ s.t. $\chi^b \rightsquigarrow \bar x^*, c^*$ ⊛

↳ $\exists s \geq t_i$ in D, and $\forall$ branch going through s goes through t_i



→ $k_0 := K+1$ = level of $t_1 \dots t_n$ — theoretically $|\bar x^*|_S = |\bar x^*|_T = K$

→ $k_{j+1} := |P^{(j)}|_S$ ... so that $P^{(j+1)}$ is above $P^{(j)}$

• $\forall \bar y \in P_1^{(j)} \times \dots \times P^{(j)}$: $\chi^{b_j}(\bar y) = \chi_{p_j}(\bar y) = \chi(\bar y ^\frown p_j) = c^*$

→ since we keep only the extensions

• Lemma: $\forall i<j$: use lemma for $t = p_i$ to get $\chi_{p_i}(\bar y) = \chi(\bar y ^\frown p_i) = c^*$

$\Rightarrow P_1^{(n)} \times \dots \times P_d^{(n)} \times \{p_0, \dots, p_n\}$ is monochr. $(K+1)$-$(\bar x^* {}^\frown t^*)$-dense m.

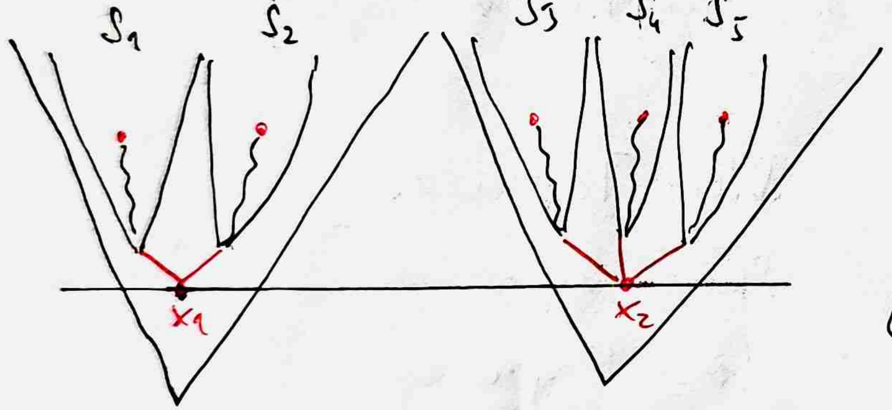
Halpern-Läuchli $\Rightarrow$ Milliken

- prove for $n=2$, for $n>2$ similar

$\to$ let $\chi : STR_2(\pi) \to r$, $\pi = (T_1, \dots, T_d)$

$\to$ we want to "clean up" $T_1, \dots, T_d$ so that $\chi(\vee)$ depends only on the root

$\to$ what we do for $\bar{x} \in \prod^{lv}_{\pi}$



imm. succ. of $x_1, \dots, x_d$

$\rightsquigarrow$ spanning trees $S_1, \dots, S_\ell$

$\bar{y} \in \prod^{lv}_{(S_1, \dots, S_\ell)} \sim T_{\bar{y}} \in STR_2(\pi)$ with root $\bar{x}$

$\hookrightarrow$ use this to define a coloring $\chi_{\bar{x}} : \prod^{lv}_{(S_1, \dots, S_\ell)} \to r$

Halpern-Läuchli: $\exists$ homogeneous $S_{\bar{x}} \in STR_\omega(S_1, \dots, S_\ell)$

Putting it together: enumerate $\bar{x} \in \prod^{lv}_{\pi}$ level by level

$\to$ for each level, find $S_{\bar{x}}$ for each $\bar{x}$ in the level and prune the rest of the tree to only keep the levels used by S

$\to$ when we are done, we go to the next level in what is left from the tree

$\to$ in the end, the color of each $T' \in STR_2(\pi')$ depends only on its root $\bar{x}$

$\Rightarrow$ final H-L: find a homogeneous strong subtree $\blacksquare$