Notation for trees

- $|x|_T \coloneqq |\{y \in T \mid y <_T x\}|$ ...height of x in T
- $T(n) \coloneqq \{x \in T \mid |x|_T = n\}$ ...n-th level of T
- $H(T) \coloneqq \min\{\alpha \mid \forall x \in T : \alpha > |x|_T\}$ ...height of T
- $T \restriction N \coloneqq \{x \in T \mid |x|_T \leq N\}$ ...restriction of T up to level N
- $STR_\alpha(T)$ ...set of all strong subtrees of T with height α

For a product tree $\pi = (T_1, \dots, T_d)$:

- $\pi \restriction N \coloneqq (T_1 \restriction N, \dots, T_d \restriction N)$ ...restriction of π up to level N
- $\prod_\pi \coloneqq T_1 \times \cdots \times T_d$ ...product of π
- $\prod_\pi^{lev} \coloneqq \left\{(x_1, \dots, x_d) \in \prod_\pi \,\middle|\, |x_1|_{T_1} = \cdots = |x_d|_{T_d}\right\}$ ...level product of π
- $\prod_\pi^{lev} \restriction \bar{x} \coloneqq \left\{\bar{x}' \in \prod_\pi^{lev} \,\middle|\, \forall i : x_i \leq_{T_i} x_i'\right\}$ ...subtree sprouting from $\bar{x}$
- $STR_\alpha(\pi)$ ...set of all strong product subtrees of T with height α

The Halpern–Läuchli theorem

Let $d, n \in \omega$, and let $\pi = (T_1, \dots, T_d)$ be a product tree of finitely branching trees of height ω with no leaves.

Theorem (Halpern–Läuchli, 1966). *If $\prod_\pi^{lev}$ is finitely colored, then there exists a homogeneous $\mathcal{S} \in STR_\omega(\pi)$. Meaning that $\prod_\mathcal{S}^{lev}$ is monochromatic.*

Theorem (Milliken, 1979). *If $STR_n(\pi)$ is finitely colored, then there exists a homogeneous $\mathcal{S} \in STR_\omega(\pi)$. Meaning that $STR_n(\mathcal{S})$ is monochromatic.*

Definitions for the proof

Definition (k-x-dense set). Let $x \in T$, $k > |x|_T$. A subset $P \subseteq T$ is *k-x-dense* if P is strictly above level k and $\forall y \in T(k)$ extending x has an extension $y' \in P$.

Definition (k-$\bar{x}$-dense matrix). Let $\bar{x} \in \prod_\pi^{lev}$, $k > |\bar{x}|_\pi$. A subset $\mathcal{P} = P_1 \times \cdots \times P_d \subseteq \prod_\pi$ is a *k-$\bar{x}$-dense matrix* if each P_i is a k-x_i-dense subset of T_i.

Definition. For $\pi = (T_1, \dots, T_d)$ as in the theorem, define statements:

SS_d $\forall$ finite coloring of $\prod_\pi^{lev}$ $\exists$ homogeneous $\mathcal{S} \in STR_\omega(\pi)$.

SD_d $\forall$ finite coloring of $\prod_\pi$ $\exists \bar{x}$ $\exists k$ s.t. $\exists$ monochromatic k-$\bar{x}$-dense matrix $\mathcal{P}$.

SD_d^{lev} Moreover, $\mathcal{P}$ is a *pancake* — all P_i are contained in a single shared level. Hence it is sufficient to color only $\prod_\pi^{lev}$.

DS_d $\forall$ finite coloring of $\prod_\pi$ $\exists \bar{x}$ s.t. $\forall k > |\bar{x}|_\pi$ $\exists$ monochromatic k-$\bar{x}$-dense matrix $\mathcal{P}_k$, and all $\mathcal{P}_k$ share the same color.

DS_d^{lev} Moreover, each $\mathcal{P}_k$ is a pancake. Hence it is sufficient to color only $\prod_\pi^{lev}$.